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On Sussmann theorem for orbits of sets of vector fields on Banach manifolds

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Abstract

The purpose of this paper is to give some generalizations, in the context of Banach manifolds, of Sussmann's results about the orbits of families of vector fields (Sussmann, 1973 [16]). Essentially, we define the notion of "l^1-orbits" for any family of vector fields on a Banach manifold, and we prove, under appropriate assumptions, that such an orbit is a weak Banach submanifold.
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1. Introduction

Let $\mathcal{X}$ be a family of local vector fields on a finite dimensional manifold M. According to the context of [16], the orbit of $\mathcal{X}$ through $x \in M$ is the set of points $\phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(x)$ where $\{X_1, \ldots, X_k\}$ is any finite family of vector fields in $\mathcal{X}$ and $\phi_t^{X_i}$ is the flow of X_i, $i = 1, \ldots, k$. One most important result of H.-J. Sussmann in [16] is that each such an orbit is an immersed submanifold of M. The proof of this result is founded on the two principal arguments:

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(i) Enlargement of $\mathcal{X}$ to the family $\hat{\mathcal{X}}$ of vector fields of type $(\phi_{t_p}^{X_p} \circ \cdots \circ \phi_{t_1}^{X_1})_*(X)$, for appropriate finite sets $\{X_1, \ldots, X_p, X\} \subset \mathcal{X}$ and each orbit of $\hat{\mathcal{X}}$ is also an orbit of $\mathcal{X}$.
(ii) The distribution $\hat{\mathcal{D}}$ generated by $\hat{\mathcal{X}}$ is integrable and each maximal integral manifold of $\hat{\mathcal{D}}$ is an orbit of $\hat{\mathcal{X}}$ and so also is an orbit of $\mathcal{X}$.

As the dimension of M is finite, the fundamental argument for the proof of this last property is that $\hat{\mathcal{D}}$ is finite dimensional.

For a generalization of such a result to Banach manifolds, we can enlarge any family $\mathcal{X}$ in the same way as (i), but in (ii), the argument of finite dimension of the distribution $\hat{\mathcal{D}}$ is, of course, no more valid. Naturally, we can hope that there exist some conditions under which analog arguments work for some "characteristic type" of families of (local) vector fields on Banach manifolds. So, given a set $\mathcal{X}$ of local vector fields on a Banach manifold M, after having enlarged $\mathcal{X}$ to a family $\hat{\mathcal{X}}$ of vector fields (in the same way as (i)), we can look for the orbits of $\hat{\mathcal{X}}$. It is natural to consider the set of points of type

$$y = \phi_{t_n}^{X_n} \circ \cdots \circ \phi_{t_1}^{X_1}(x) \quad \text{or} \quad y = \lim_{k \to \infty} \phi_{t_k}^{X_k} \circ \cdots \circ \phi_{t_1}^{X_1}(x) \tag{1}$$

as an orbit through x for any finite or countable family $\{X_k,\ k \in A\}$ of vector fields in $\hat{\mathcal{X}}$. Note that, if we restrict us to finite sets A, the binary relation defined by

$$y \backsim x \quad \text{if and only if} \quad y = \phi_{t_n}^{X_n} \circ \cdots \circ \phi_{t_1}^{X_1}(x),$$

is an equivalence relation. Moreover, in this case, there exists a piecewise smooth curve which joins x to y and whose each connected part is tangent to X_i or $-X_i$ for some $i = 1, \ldots, n$.

Unfortunately, in the previous general case, the associated binary relation clearly associated to (1) is not any more a relation of equivalence. The $\mathcal{X}$-orbit of x will be the set of such points y under some conditions so that the associated binary relation is an equivalence relation.

Given a family $\xi \subset \mathcal{X}(M)$, a ξ-**piecewise smooth curve** is a piecewise smooth curve $\gamma : [a, b] \to M$ such that each smooth part is tangent to X or $-X$ for some $X \in \xi$. In the context of (1), for such a point y, there exists a family $\gamma_k : [0, T_k] \to M$ of $\mathcal{X}$-piecewise smooth curves such that the sequences of ends $x_k = \gamma_k(T_k)$ converge to y. When the sequence T_k converges to some $T \in \mathbb{R}$ we have a continuous curve $\gamma : [0, T] \to M$ such that $\gamma(0) = x$ and $\gamma(T) = y$. For such a curve γ, there exists a countable partition $t = (t_\alpha)_{\alpha \in A}$ of $[0, T]$ such that, the restriction of γ to $]t_\alpha, t_{\alpha+1}[$ is an integral curve of X or $-X$, for some $X \in \mathcal{X}$. In particular the family $(\tau_\alpha = t_{\alpha+1} - t_\alpha)_{\alpha \in A}$ belongs to $l^1(\mathbb{N})$. Such a curve will be called an l^1-**curve of** $\mathcal{X}$. The precise definition of an orbit of $\mathcal{X}$ (see Section 2.1) is based on this notion of l^1-curve but for the family $\hat{\mathcal{X}}$. Of course, we need some sufficient conditions under which l^1-curves exist. It is easy to see that condition of "local boundedness" is a natural necessary condition, but, for the local existence, we need more: the local boundedness of the s-jets of vector fields of $\mathcal{X}$, for sufficiently large $s > 0$ (see Section 2.2). Under such assumptions, we can prove the existence of l^1-curves which are the integral curves of a vector field of type (see Theorem 1):

$$Z(x, t, u) = \sum_{\alpha \in A} u_\alpha(t) X_\alpha(x)$$

where:

A is a finite, countable or eventually uncountable set of indexes;
$\xi = \{X_\alpha\}_{\alpha\in A}$ are defined on a same open set and their s-jets are locally uniformly bounded (see Definition 2.3);
$u = (u_\alpha)_{\alpha\in A}$ is a bounded integrable map from some interval I to $l^1(A)$.

In fact, in this context, we get a **flow** $\Phi^\xi_u(t, \,)$ of such a vector field Z.

Let $\xi = \{X_\alpha\}_{\alpha\in A}$ be a set which satisfies a local boundedness condition for the s-jets for sufficiently large $s > 0$. The existence of l^1-curves which are integral curves of some $X \in \xi$ (or $-X$) on any subinterval $]t_\alpha, t_{\alpha+1}[$ associated to a countable partition of an interval I is obtained by application of the previous result to $u = \Gamma^\tau = (\Gamma^\tau_\alpha)$ where Γ^τ_α is the indicatrix function of $]t_\alpha, t_{\alpha+1}[$. Denote by $\Phi^\xi_\tau(t, \,)$ the associated flow, given any $x \in M$, for $T = \|(t_{\alpha+1} - t_\alpha)_{\alpha\in A}\|_1$, $\tau \to \psi^x(\tau) = \Phi^\xi_\tau(T, x)$ is a map from a neighborhood of $0 \in l^1(A)$ into M such that $\psi^x(0) = x$ and, of class C^{s-2}, if the condition of local boundness of s-jets of elements of $\mathcal{X}$, are satisfied (see Theorem 2).

Recall that our purpose is to prove, under appropriate assumptions, that each $\mathcal{X}$-orbit is a (weak) submanifold of M as integral manifold of some distribution. According to the proof of Sussmann's result, we first enlarge $\mathcal{X}$ into the set $\hat{\mathcal{X}}$ given by

$$\hat{\mathcal{X}} = \Big\{Z = \Phi_*(\nu Y),\ Y \in \mathcal{X},\ \Phi = \phi^{X_p}_{t_p} \circ \cdots \circ \phi^{X_1}_{t_1} \text{ for } X_1, \dots, X_p \in \mathcal{X} \\ \text{and appropriate } \nu \in \mathbb{R}\Big\}$$

(see Section 3.1). From this set $\hat{\mathcal{X}}$, we associate an appropriate pseudo-group $\mathcal{G}_\mathcal{X}$ of local diffeomorphisms, which is generated by flows of type ϕ^X_t with $X \in \mathcal{X}$ and diffeomorphisms of type $\Phi^\xi_u(\|\tau\|_1, .)$ (as we has seen previously) or its inverse for $\xi \subset \hat{\mathcal{X}}$. From this pseudogroup we get a coherent and precise definition of an **orbit of** $\mathcal{X}$ or $\mathcal{X}$ **orbit** in short. Note that, under this definition, $\mathcal{X}$ and $\hat{\mathcal{X}}$ **have the same orbits**, and moreover, if y is in the orbit of x, there is an l^1 curve which joins x to y and whose smooth parts are tangent to vector fields of $\hat{\mathcal{X}}$. Note that the binary relation associated to $\mathcal{G}_\mathcal{X}$ is then an equivalence relation. So, if y belongs to the $\mathcal{X}$-orbit of x, either we have an $\mathcal{X}$-piecewise smooth curve which joins x to y or there exists a sequence γ_k of $\mathcal{X}$-smooth piecewise curves whose origin is x (for all curves) and whose sequence of ends converges to y (see Proposition 3.4 for a complete description of an $\mathcal{X}$ orbit).

On the other hand, for any $x \in M$, under appropriate assumptions, we can associate the vector space $\hat{\mathcal{D}}_x = l^1(\hat{\mathcal{X}})_x$ which is the set of all absolutely summable families $\sum_{Y\in\hat{\mathcal{X}}} \tau_Y Y(x)$. In fact, in the same way, we can also associate the vector space $\mathcal{D}_x = l^1(\mathcal{X})_x$ generated by $\mathcal{X}$. Of course $\mathcal{D}_x \subset \hat{\mathcal{D}}_x$ (see Section 3.3) and we endox these vector spaces with a natural structure of Banach space. So we get weak distributions $\mathcal{D}$ and $\hat{\mathcal{D}}$ on M such that $\hat{\mathcal{D}}$ is invariant by any flow of vector fields in $\mathcal{X}$ and which is "minimal" for such a property (see Remark 3.7). Now, we need some conditions on $\hat{\mathcal{X}}$ which makes $\hat{\mathcal{D}}$ integrable. We will give two types of sufficient conditions.

For the first one (called (H) in Section 4.2), we assume that, for any $x \in M$, the Banach structure on $\hat{\mathcal{D}}_x$ is isomorphic to some $l^1(A)$ and there exists a family $\{X_\alpha\}_{\alpha\in A}$ of vector fields defined around x, which are "locally uniformly bounded at order s" and such that $\{X_\alpha(x)\}_{\alpha\in A}$ is an unconditional symmetric basis of $\hat{\mathcal{D}}_x$. Under this assumption, $\hat{\mathcal{D}}$ is lower trivial (see Section 3.1) but we cannot prove directly that $\hat{\mathcal{D}}$ is $\mathcal{X}^-_{\hat{\mathcal{D}}}$-invariant; in particular, **we cannot use directly Theorem 1 of [13]**. So we first prove that the map ψ^x, previously defined, gives rise to a local integral manifold of $\hat{\mathcal{D}}$ through x of class C^s for $s \geqslant 2$. This leads us to prove that $\hat{\mathcal{D}}$ is $\mathcal{X}^-_{\hat{\mathcal{D}}}$-invariant and so we can now apply Theorem 1 of [13] and we finally get a smooth integral manifold of $\hat{\mathcal{D}}$. When $\hat{\mathcal{D}}$ is closed, we then obtain that each l^1-orbit has a structure of weak Banach manifold. *Note that the assumption* (H) *is always satisfied when* $\hat{\mathcal{D}}$ *is finite dimensional* (*see Remark* 4.3). *So this result can be seen as a generalization of the proof Sussmann used in* [16].

The second sufficient conditions (called (H′) in Section 4.3) impose that $\hat{\mathcal{D}}$ is "upper trivial" (see Section 4.3) and also some local involutivity conditions on $\hat{\mathcal{X}}$. Under these conditions, by using a result of integrability from [13], we can show that $\hat{\mathcal{D}}$ is integrable and, when $\hat{\mathcal{D}}$ is closed, each maximal integral manifold is an $\mathcal{X}$-orbit (Theorem 5). Moreover, if we consider the family $\mathcal{X}^k$ defined by induction by

$$\mathcal{X}^1 = \mathcal{X} \quad \text{and} \quad \mathcal{X}^k = \mathcal{X}^{k-1} \cup \big\{[X,Y],\ X \in \mathcal{X},\ Y \in \mathcal{X}^{k-1}\big\} \quad \text{for } k \geqslant 2$$

we can associate, as previously, a weak distribution $\mathcal{D}^k = l^1(\mathcal{X}^k)$. When such a distribution satisfies the conditions (H′) and is closed, we have $\mathcal{D}^k = \hat{\mathcal{D}}$ and so we get another sufficient conditions under which each $\mathcal{X}$-orbit is a weak manifold modelled on some $l^1(A)$. For the case where $\mathcal{X}$ is a finite family of global vector fields we get a new proof of the result of accessibility in [15] (see Example 4.5). Moreover, when $\mathcal{X}$ is a countable family of global vector fields, the reader can find an application of these results in [14].

All these results can be naturally applied in the context of control theory on Banach manifolds (Theorem 7 and Theorem 8). These last theorems can be considered as a generalization of Sussmann's accessibility results of [16] in finite dimension.

The paper is organized as follows. In the next section, we study the problem of existence of l^1-curves. For any set $\mathcal{X}$ of vector fields which has the "local boundedness of the s-jets of vector fields", we give sufficient conditions for the existence of l^1-curves (Theorem 1) and we apply this result to get l^1-curves tangent to $X \in \mathcal{X}$ or $-X$, on each subinterval associated to a countable partition. We also construct the map ψ^x mentioned previously (Theorem 2).

The notion of orbit of $\mathcal{X}$ or $\mathcal{X}$-orbit, in short, is precisely defined in Section 3. In Section 3.1, we construct the announced enlargement $\hat{\mathcal{X}}$ of $\mathcal{X}$, the associated pseudogroup $\mathcal{G}_{\mathcal{X}}$ and we give a precise definition of an $\mathcal{X}$-orbit. The following subsection is devoted to all definitions and properties of distributions which will be used later.

Then the characteristic distributions $\mathcal{D}$ and $\hat{\mathcal{D}}$ generated by $\mathcal{X}$ and $\hat{\mathcal{X}}$ respectively, are defined in Section 3.3. Finally, the main results of structure of weak Banach manifolds on $\mathcal{X}$-orbits are given and proved in Section 4. In Section 4.2 under conditions (H) the corresponding result is given in Theorem 3. Under conditions (H′), the main results are given in Theorem 5. Section 5 is devoted to some applications: on one hand we obtain a new criterion of integrability of l^1-distributions in Theorem 6 (see Remark 5.1). On the other hand, we give general results on accessibility sets as applications of the previous results on $\mathcal{X}$-orbits (Theorem 7 and Theorem 8). The last section is devoted to the proof of Theorem 2.

2. On l^1-integral curve of a uniformly locally bounded set of vector fields

2.1. Problem of existence of l^1-integral curve

Let M be a smooth connected Banach manifold modelled on a Banach space E. A **local vector field** X on M is a smooth section of the tangent bundle TM defined on an open set of M (denoted by $\mathrm{Dom}(X)$). Denote by $\mathcal{X}(M)$ the set of all local vector fields on M. Such a vector field $X \in \mathcal{X}(M)$ has a flow ϕ_t^X which is defined on a maximal open set Ω_X of $\mathbb{R} \times M$.

In this whole work, A, B and Λ will denote a finite or a countable, eventually uncountable, ordered set of indexes. For such a countable set we shall often identify this one with $\mathbb{N}$ as ordered set of indexes.

Consider a subset $\mathcal{X}$ of $\mathcal{X}(M)$. As we have seen in the introduction, a curve $\gamma : [a,b] \to M$ is called an l^1-integral curve of $\mathcal{X}$, if there exists a sequence $t = (t_\alpha)_{\alpha \in A}$, where A is a finite or countable set such that:

- $t_0 = a$ and $t_{\alpha-1} \leqslant t_\alpha \leqslant b$ for $\alpha \in A$;
- $t_n = b$ if A is finite ($A \equiv \{1, \dots, n\}$) or $\lim_{\alpha\to\infty} t_\alpha = b$ (when A is countable);
- the restriction of γ to each subinterval $]t_{\alpha-1}, t_\alpha[$ is an integral curve of X_α or $-X_\alpha$ for some $X_\alpha \in \mathcal{X}$.

For such a curve γ, the point $x_0 = \gamma(a)$ (resp. $x_1 = \gamma(b)$) is called the origin (resp. the end) of γ and we say that x_0 is joined to x_1 by an l^1-integral curve of $\mathcal{X}$.

It is clear that for any finite set $A = \{0, \dots, n\}$ any l^1-integral curve is smooth by parts and, if we set $\tau_0 = a$ and $\tau_\alpha = t_\alpha - t_{\alpha-1}$ for $\alpha = 1, \dots, n$, then there exist vector fields $X_1, \dots, X_n$ in $\mathcal{X}$ such that for $\alpha = 1, \dots, n$, we have

$$\gamma(s) = \phi^{X_\alpha}_{s-t_{\alpha-1}} \circ \phi^{X_{\alpha-1}}_{\alpha-1} \circ \dots \circ \phi^{X_1}_{\tau_1}\big(\gamma(a)\big) \quad \text{for } s \in [t_{\alpha-1}, t_\alpha[\text{ and } \alpha = 1, \dots, n. \tag{2}$$

Given a countable set $A \equiv \mathbb{N}$, and an l^1-integral curve γ of $\mathcal{X}$, there exists a sequence of vector fields $\{X_\alpha,\ \alpha \in A\}$ in $\mathcal{X}$ such that (2) is true for all $\alpha \in A$. In particular, we must have

$$\lim_{\alpha\to\infty} \phi^{X_\alpha}_{t_\alpha} \circ \dots \circ \phi^{X_1}_{t_1}\big(\gamma(a)\big) = b. \tag{3}$$

We can cover such a curve by a finite number of charts (V_i, ϕ_i), $i = 1, \dots, r$ so that any $\gamma(]t_{\alpha-1}, t_\alpha[)$ is contained in one domain V_i. Note that there exists one of these domains which contains all $\gamma(]t_{\alpha-1}, t_\alpha[)$ for $\alpha \geqslant \alpha_0$ for some $\alpha_0 \in \mathbb{N}^*$ and we can assume that V_r has this property. Now, on each V_i, a norm $\| \ \|_{\phi_i}$ can be defined on each fibre T_xM, for $x \in V_i$ by $\|u\|_{\phi_i} = \|T_x\phi_i(u)\|$ where $\| \ \|$ is a norm on E. From (3), for any $\alpha \in A$, if $\gamma(]t_{\alpha-1}, t_\alpha[) \subset V_i$, we must have

$$\sup\big\{\big\|X_\alpha\big(\gamma(t)\big)\big\|_{\phi_i},\ t \in]t_{\alpha-1}, t_\alpha[\big\} \text{ is finite.}$$

On the other hand, consider any countable set $A \equiv \mathbb{N}$ and any subset $\{X_\alpha,\ \alpha \in A\}$ of $\mathcal{X}$ such that $\mathrm{Dom}(X_\alpha)$ contains V and

$$\sup\big\{\big\|X_\alpha(x)\big\|_{\phi_i},\ x \in V,\ \alpha \in A\big\} \text{ is finite.}$$

Let be $\tau = (\tau_\alpha) \in l^1(A)$ such that $\tau_\alpha > 0$ for any $\alpha \in A$. Set $t_0 = 0$ and $t_\alpha = \sum_{i=1}^{\alpha} \tau_i$ for $\alpha \in A$ and $T = \lim_{\alpha\to\infty} t_\alpha$. We set $\gamma(0) = x \in V$. If the flow $\phi^{X_1}_t(x)$ is defined for $t \geqslant \tau_1$, we set $\gamma(t) = \phi^{X_1}(t)$ for $t \in [t_0, t_1]$. By induction, suppose that we have defined $\gamma : [0, t_\alpha] \to V$ such that $\gamma : [t_i, t_{i+1}] \to V$ is defined by $\gamma(t) = \phi^{X_i}_{t-t_i}(\gamma(t_i))$ for all $i = 1, \dots, \alpha$. Then if the flow $\phi^{X_{\alpha+1}}_t(\gamma(t_\alpha))$ is defined for $t \geqslant \tau_{\alpha+1}$ then we put $\gamma(t) = \phi^{X_{\alpha+1}}_{t-t_\alpha}(\gamma(t_\alpha))$ for $t \in [t_\alpha, t_{\alpha+1}]$. So, when we can construct γ at each step, we get an l^1-integral curve of $\mathcal{X}$. Consequently, for the existence of l^1-integral curve associated to a countable subset $\{X_\alpha,\ \alpha \in A\}$ of vector fields of $\mathcal{X}$, we have to produce sufficient conditions under which sequences of compositions

$$\phi^{X_\alpha}_{\tau_\alpha} \circ \dots \circ \phi^{X_i}_{\tau_i} \circ \dots \circ \phi^{X_1}_{\tau_1},$$

converge when $\alpha \to \infty$ and the limit defines a local diffeomorphism. **These conditions are assumptions of uniform local boundness on the jets of vector fields** (see next subsection).

Remark 2.1. Consider a subset $\{X_\alpha,\ \alpha \in A\}$ of $\mathcal{X}$ with the previous assumptions and $\tau = (\tau_\alpha) \in l^1(A)$. Recall that, for any local vector field X, for $\nu \neq 0$, we have $\phi_t^X(x) = \phi_{\nu t}^{X/\nu}(x)$, when the second member is defined. It follows that given any $\nu > 0$ if an l^1-integral curve γ of $\{X_\alpha,\ \alpha \in A\}$ is defined on $[0, T]$ as before, we can also define an l^1-integral curve $\bar{\gamma}$ of $\{\frac{1}{\nu}X_\alpha,\ \alpha \in A\}$ in an obvious way on $[0, \nu T]$ and we have $\bar{\gamma}(t)) = \gamma(\nu t)$ for any $t \in [0, T]$.

2.2. *Set of vector fields uniformly locally bounded at order s*

Let $\Pi : TM \to M$ be the tangent bundle of M, with typical fibre E. Local vector fields on M are local sections of this bundle. Given $X \in \mathcal{X}(M)$, the s-order jet of X at $x \in M$ is denoted by $J^s(X)(x)$. The set $J^s(TM)$ of s-order jets of local vector fields on M is a Banach bundle $\Pi^s : J^s(TM) \to M$ of typical fibre $E \times \mathcal{L}(E, E) \times \mathcal{L}^2(E, E) \times \cdots \times \mathcal{L}^s(E, E)$ where $\mathcal{L}^k(E, E)$, $2 \leqslant k \leqslant s$ is the Banach space of symmetric k-linear maps from E^k into E endowed with the usual norm (see for instance [7] or [17]). The typical fibre $E \times \mathcal{L}(E, E) \times \mathcal{L}^2(E, E) \times \cdots \times \mathcal{L}^s(E, E)$ of $J^s(TM)$ is a Banach space for the norm $\| \ \|_s$ which is the sum of the norm on E, the canonical norms on $\mathcal{L}(E, E)$ and on $\mathcal{L}^k(E, E)$ for $2 \leqslant k \leqslant s$.

Consider a chart (V, ϕ) on M centered at x. On V there exists a trivialization (ϕ, Φ) of $[\Pi^s]^{-1}(V)$ on $\phi(V) \times J^s(E)$ where $J^s(E) = E \times \mathcal{L}(E, E) \times \mathcal{L}^2(E, E) \times \cdots \times \mathcal{L}^s(E, E)$ is the typical fibre. On V, we have

$$\forall y \in V, \quad \Phi\big[J^s(X)(y)\big] = J^s(\phi_* X)\big(\phi(y)\big).$$

So, on $[\Pi^s]^{-1}(V)$, we have a norm $\| \ \|_\phi$ characterized by

$$\big\| J^s(X)(y)\big\|_\phi = \big\| J^s(\phi_* X)\big(\phi(y)\big)\big\|_s.$$

Lemma 2.2. *(See [11].) Let V' be an open neighborhood of x having the same properties and (ϕ', Φ') the associated trivialization. Denote by $\|J^s(X)(y)\|_{\phi'} = \|J^s(\phi'_* X)(\phi'(y))\|_s$ the associated norm on $[\Pi^s]^{-1}(V')$. Then there exist a neighborhood $W \subset V \cap V'$ of x and a constant $C > 0$ such that*

$$\forall y \in W, \quad \big\| J^s[X](y)\big\|_{\phi'} \leqslant C \big\| J^s[X](y)\big\|_\phi.$$

Definition 2.3. Let $\mathcal{X}$ be a set of local vector fields on M. Given $x \in M$, we say that $\mathcal{X}$ satisfies the condition (LB(s)) at x (Locally Bounded at order s), if there exist a chart (V_x, ϕ) centered at x and a constant $k > 0$ such that:

For any $X \in \mathcal{X}$, whose domain dom(X) contains V_x, we have

$$\sup\big\{\big\| J^s[X](y)\big\|_\phi,\ X \in \mathcal{X},\ y \in V_x\big\} \leqslant k. \tag{4}$$

Remark 2.4. It follows from Lemma 2.2 that the property (4) does not depend neither on the choice of the norm on E, nor on the choice of the chart.

Examples 2.5.

(1) Let E and F be two Banach spaces and $T : F \to E$ a continuous operator. Given any finite or countable subset $\{a_\alpha,\ \alpha \in A\}$ uniformly bounded of F (i.e. $\|a_\alpha\| \leqslant k$ for any $\alpha \in A$) the assignment $x \mapsto X_\alpha(x) = x + T(a_\alpha)$ is a vector field on E and $\{X_\alpha,\ \alpha \in A\}$ satisfies the condition LB(s) at any $x \in E$ and for any $s \in \mathbb{N}^*$.

(2) Let $L(F, E)$ be the set of continuous operators between the Banach spaces F and E. Given a smooth map $\Phi : E \to L(F, E)$, we denote by Φ_x the continuous operator associated to $x \in E$. By smoothness of Φ, for any $x \in E$ and $s \in \mathbb{N}^*$, we can find an open neighborhood U of $x \in E$ such that the jet of order s of Φ is bounded on U (in the sense of Lemma 2.2). Then, for any finite or countable subset $\{a_\alpha,\ \alpha \in A\}$ uniformly bounded of F, denote by X_α the vector field on E defined by $X_\alpha(x) = \Phi_x(a_\alpha)$. The set $\{X_\alpha,\ \alpha \in A\}$ satisfies the condition (LB s) at any $x \in E$ and for any $s \in \mathbb{N}^*$.

(3) Let $\mathcal{X} = \{X_1, \dots, X_n\}$ be a finite family of (global) vector fields on a Banach manifold M. Then $\mathcal{X}$ satisfies the condition (LB s), for any $s \in \mathbb{N}$.

2.3. *Sufficient conditions for the existence of l^1-integral curves*

Notations 2.6.

- $B(x, r)$ (resp. $B_f(x, r)$) denotes the open (resp. closed) ball centered at $x \in E$ of radius r in the Banach space E.
- Given any Banach space L, if $f : \mathbb{R} \times E \times L \to E$ is a smooth map, we denote by $D_2 f$ (resp. $D_3 f$) the partial derivative relative to E (resp. L).
- Let $\mathbb{R}^A$ will be the set of families $(u_\alpha)_{\alpha\in A}$ of absolutely summable real numbers where A is countable or eventually uncountable set of indexes or the set of finite real sequences $u = (u_1, \dots, u_n)$ if $A = \{1, \dots, n\}$. We endow $\mathbb{R}^A$ with the norm

$$\|u\|_1 = \sum_{\alpha\in A} |u_\alpha|.$$

  It is well known that $(\mathbb{R}^A, \|\ \|_1)$ is a Banach space.
- Given any interval J in $\mathbb{R}$ we denote by $L^1_b(J)$ the set of functions $u : J \to R^A$ of class L^1 which are bounded. On $L^1_b(J)$ we define:
  - $\|u\|_1 = \int_J \sum_{\alpha\in A} |u_\alpha(t)|\, dt = \int_J \|u(t)\|_1\, dt$.
  - $\|u\|_\infty = \sup\{\sum_{\alpha\in A} |u_\alpha(t)|,\ t \in J\} = \sup\{\|u(t)\|_1,\ t \in J\}$.

Given a finite, countable or uncountable ordered set of indexes A, let

$$\xi = \{X_\alpha,\ \alpha \in A,\ X_\alpha \in \mathcal{X}(M)\}$$

be a set of vector fields on M such that $\bigcap_{\alpha\in A} \mathrm{Dom}(X_\alpha)$ contains an open set V of a chart (V, ϕ) centered at x such that the condition (LB$(s+2)$) at x is satisfies for some $s \in \mathbb{N}$. After restricting V if necessary, we can suppose that there exists $k > 0$ such that

$$\sup\{\|J^{s+2}(X)(y)\|_\phi,\ X \in \xi,\ y \in V\} \leqslant k.$$

Without loss of generality, we can suppose that V is an open set of the Banach space E. To the previous set of vector fields ξ, we can associate maps Z of type:

$$Z : J \times V \times L^1_b(J) \to E,$$
$$(t, x, u) \mapsto Z(t, x, u) = \sum_{i \in J} u_i(t) X_i(x).$$

It is easy to see that this map Z is of class C^{s+1} relatively to the second variable.

Given such a map Z, let J' be a subinterval of J and $(t_0, x, u) \in J' \times V \times L^1_b(I)$. A map $f : J' \to V$ is an **integral curve** of Z, with initial condition $f(t_0) = x$ if

$$\forall t \in I', \quad f(t) = x + \int_{t_0}^{t} Z\big(s, f(s), u\big)\, ds. \tag{5}$$

The following theorem gives the existence of a local flow for Z:

Theorem 1. *Consider a fixed u in $L^1_b(J)$, and we set $c = \|u\|_\infty$. Let (t_0, x_0, r, T', T_0) be an element of $J \times V \times {\mathbb{R}^*_+}^3$ such that*

$$\big]t_0 - T', t_0 + T'\big[\subset J \quad \text{and} \quad B_f(x_0, 2r) \subset V.$$

Moreover denote by

$$I_0 = [t_0 - T_0, t_0 + T_0] \quad \text{and} \quad B_0 = B(x_0, r - kcT_0).$$

If $T_0 < \min(\frac{r}{kc}, T')$, then there exists a flow $\Phi_u : I_0 \times B_0 \to V$, with the following properties:

1. *For all x in B_0, each curve $\Phi_u(.,x) : I_0 \to V$ is the unique integral curve of Z, with initial conditions $\Phi_u(t_0, x) = x$.*
2. *For all $t \in I_0$, there exists an open connected neighborhood U_0 of x_0, contained in B_0 such that the map $\Phi_u(t, .) : U_0 \to \Phi_u(t, U_0)$ is a C^s-diffeomorphism. Moreover, if $D_2\Phi_u(t, .)$ and $D_2^2\Phi_u(t, .)$, denote the first and second derivative relative to the second variable, we have*

$$\forall x \in U_0, \quad D_2\Phi_u(t, x) = Id_E + \int_0^t D_2 Z\big(s, \Phi_u(s, x), u\big) \circ D_2\Phi_u(s, x)\, ds,$$

$$D_2^2\Phi_u(t, x) = \int_0^t \Big(D_2^2 Z\big(s, \Phi_u(s, x), u\big) \circ \big(D_2\Phi_u(s, x), D_2\Phi_u(s, x)\big) + D_2 Z\big(s, \Phi_u(s, x), u\big) \circ D_2^2\Phi_u(s, x)\Big)\, ds.$$

This result is certainly well known for specialists. The reader can find a complete proof in [11].

Let Φ and Ψ be two local diffeomorphisms on M which are defined on the domains Ω_Φ and Ω_Ψ respectively. When $\Psi(\Omega_\Psi) \cap \Omega_\Phi \neq \emptyset$, we can define the composition $\Phi \circ \Psi$ which is a local diffeomorphism defined on $\Psi^{-1}[\Psi(\Omega_\Psi) \cap \Omega_\Phi]$. In this situation we will say that $\Phi \circ \Psi$ is well defined. More generally, we can consider any finite composition $\Phi_n \circ \cdots \circ \Phi_1$ of local diffeomorphisms $\Phi_1, \ldots, \Phi_n$ when successive compositions $\Phi_i \circ (\Phi_{i-1} \circ \cdots \circ \Phi_1)$ are well defined for $i = 2, \ldots, n$. So, for a finite set $A = \{1, \ldots, n\}$, and a finite set $\xi = \{X_\alpha\}_{\alpha \in A}$ of vector fields with the associate flows $\{\phi^{X_\alpha}_{t_\alpha}\}_{\alpha \in A}$, it is clear that, for $\tau = (\tau_1, \ldots, \tau_n)$, under appropriate assumptions, the composition $\phi^\xi_\tau = \phi^{X_n}_{\tau_n} \circ \cdots \circ \phi^{X_1}_{\tau_1}$ is defined. When A is a countable or eventually uncountable ordered set of indexes we have the following result:

Theorem 2. *Let $\xi = \{X_\alpha\}_{\alpha\in A}$ be a set of local vector fields such that* $\mathrm{Dom}(X_\alpha)$ *contains V for all $\alpha \in A$. Let be $x_0 \in V$ and $r > 0$ such that $B_f(x_0, 2r)$ is contained in V and we assume that ξ satisfies the condition* (LB($s + 2$)) *at x_0 where the relation* (4) *is true for all $y \in V$ and for the integer $s + 2$.*

Then, there exists an open connected neighborhood U_0 of x_0, such that:

1. *Fix any $\tau = (\tau_\alpha)_{\alpha\in A} \in \mathbb{R}^A$ with $\|\tau\|_1 \leqslant \frac{r}{k}$. Let B be any countable subset of A which contains all the indexes α such that $\tau_\alpha \neq 0$. Identifying the set B with $\mathbb{N}$ (as ordered sets), we denote by $\{\tau_m,\ m \in B\}$ the associated subsequence of $\{\tau_\alpha,\ \alpha \in A\}$. Then for any $x \in U_0$ we have*:
   (a) $\phi^\xi_\tau(x) = \lim_{m\to\infty} \phi^{X_m}_{\tau_m} \circ \cdots \circ \phi^{X_1}_{\tau_1}(x)$ *exists.*
   (b) $\hat{\phi}^\xi_\tau(x) = \lim_{m\to\infty} \phi^{X_1}_{-\tau_1} \circ \cdots \circ \phi^{X_m}_{-\tau_m}(x)$ *exists.*
   (c) *The map $\phi^\xi_\tau : x \mapsto \phi^\xi_\tau(x)$ is a local C^s-diffeomorphism whose inverse mapping is $\hat{\phi}^\xi_\tau : x \mapsto \hat{\phi}^\xi_\tau(x)$.*
2. *The map Ψ^x defined in the following way*:

$$\Psi^x : B\left(0, \frac{r}{k}\right) \to V,$$
$$\tau \mapsto \Psi^x(\tau) = \phi^\xi_\tau(x) \text{ is of class } C^s.$$

   When the point x will be fixed we simply denote Ψ instead of Ψ^x.

The proof of this theorem is long and technical, so it will be given in Section 6.

Remark 2.7.

1. Denote by Φ^ξ_τ (resp. $\hat{\Phi}^\xi_\tau$) the flow given in Theorem 1 associated to ξ and $u = \Gamma^\tau$ (resp. $\hat{u} = \hat{\Gamma}^\tau$) (see Section 6). On the associated neighborhood U, we have

$$\hat{\Phi}^\xi_\tau(t, z) = \Phi^\xi_\tau\big(\|\tau\|_1 - t, \Phi^\xi_\tau\big(-\|\tau\|_1, z\big)\big),$$
$$\phi^\xi_\tau(z) = \Phi^\xi_\tau\big(\|\tau\|_1, z\big),$$
$$\hat{\phi}^\xi_\tau(z) = \big[\phi^\xi_\tau\big]^{-1}(z) = \hat{\Phi}^\xi_\tau\big(\|\tau\|_1, z\big) = \Phi^\xi_\tau\big(-\|\tau\|_1, z\big).$$

2. In fact, both limits $\phi^\xi_\tau(x)$ and $\hat{\phi}^\xi_\tau(x)$ do not depend on the choice of the set B but only depend on the countable set $A_\tau = \{\alpha \in A$ such that $\tau_\alpha \neq 0\}$. Moreover, the set A_τ is independent of $x \in U_0$.
3. To each τ the associated set $A_\tau = \{\alpha \in A$ such that $\tau_\alpha \neq 0\}$ can be written $A_\tau = \{\alpha_k,\ k \in \mathbb{N}\}$ or $A_\tau = \{\alpha_k,\ k = 1, \dots, n\}$. Consider the associated subdivision $\{t_{\alpha_k}\}_{k\in\mathbb{N}}$ of the interval $[0, T]$ defined by

$$t_0 = 0 \leqslant t_1 = |\tau_{\alpha_1}| \leqslant \cdots \leqslant t_i = \sum_{k=1}^{i} |\tau_{\alpha_k}| \leqslant \cdots \leqslant T = \sum_{\alpha\in A_\tau} |\tau_\alpha|.$$

Fix some $x \in U_0$ and let $(x_k)_{\alpha_k \in A_\tau}$ be the sequence defined by

$$x_0 = x, \quad \text{and} \quad \text{for } \alpha_k \in A_\tau, \quad x_k = \phi^{X_{\alpha_k}}_{\tau_{\alpha_k}}(x_{k-1}) = \phi^{\xi}_{\tau}(x_{k-1}).$$

Then the curve $\gamma : [0, T] \to M$ defined by $\gamma(s) = \phi^{X_{\alpha_k}}_{s-t_{k-1}}(x_{k-1}) = \Phi^{\xi}_{\tau}(s, x)$ for $s \in [t_{k-1}, t_k[$ is an l^1-curve which joins x to $\Psi^{\xi}_{\tau}(x)$. On the other hand, to $\hat{\phi}^{\xi}_{\tau}$ we can associate the curve $\hat{\gamma} : [0, T] \to M$ defined by $\hat{\gamma}(s) = \gamma(T - s)$. So $\hat{\gamma}$ joins $\gamma(0) = \phi^{\xi}_{\tau}(x)$ to x. We also call such a curve, the l^1-curve associated to $\hat{\phi}^{\xi}_{\tau}$.

3. The orbits of $\mathcal{X}$ or $\mathcal{X}$-orbits

3.1. *Definition of an orbit of $\mathcal{X}$*

In this section, we consider a fixed set $\mathcal{X}$ of vector fields on M with the following properties:

(Hi) $M = \bigcup_{x \in \mathcal{X}} \mathrm{Dom}(X)$;
(Hii) there exists $s \geqslant 0$ with the following property: for any $x \in M$ there exists a chart (V_x, ϕ) centered at x such that for the set $\mathcal{X}_x$ of vector fields $X \in \mathcal{X}$ whose $\mathrm{Dom}(X)$ contains x we have

$$\sup\{ \| J^{s+2}[X](x) \|_{\phi},\ X \in \mathcal{X} \} < \infty.$$

The announced enlargement $\hat{\mathcal{X}}$ of $\mathcal{X}$ is obtained from the following lemma:

Lemma 3.1. *Let (V_x, ϕ) be a chart centered at x and a constant k such that*

$$\sup\{ \| J^{s+2}[X](x) \|_{\phi},\ X \in \mathcal{X} \} \leqslant k.$$

Let $\hat{\mathcal{X}}_x$ be the set of local vector fields of type $Y = (\phi^{X_p}_{t_p} \circ \cdots \circ \phi^{X_1}_{t_1})_*(\nu.X)$, for any $\nu > 0$, where $X_1, \ldots, X_p, X$ belongs to $\mathcal{X}$, whose domain contains x and such that

$$\| J^{s+2}[Y](x) \|_{\phi} \leqslant k. \tag{6}$$

We set

$$\hat{\mathcal{X}} = \bigcup_{x \in M} \hat{\mathcal{X}}_x.$$

(i) *$\hat{\mathcal{X}}$ contains $\mathcal{X}$ and satisfies the conditions* (Hi) *and* (Hii).
(ii) *Let $\hat{\hat{\mathcal{X}}}$ be the set of vector fields obtained from $\hat{\mathcal{X}}$ in the same way as $\hat{\mathcal{X}}$ from $\mathcal{X}$. Then, we have $\hat{\hat{\mathcal{X}}} = \hat{X}$.*

Remark 3.2. According to Remark 2.1, the flow of any vector field $Y = (\phi^{X_p}_{t_p} \circ \cdots \circ \phi^{X_1}_{t_1})_*(\nu.X)$ can be written

$$\phi^{Y}_{\tau} = \phi^{X_1}_{-t_1} \circ \cdots \circ \phi^{X_p}_{-t_p} \circ \phi^{X}_{\tau/\nu} \circ \phi^{X_p}_{t_p} \circ \cdots \circ \phi^{X_1}_{t_1}. \tag{7}$$

Proof. Let be $x \in M$. For any $X \in \mathcal{X}_x$, we have $(\phi_0^X)_* X = X$ and, by construction, the vector fields in $\hat{\mathcal{X}}_x$ satisfies the condition (6) with the same constant k, so $\mathcal{X}_x$ is contained in $\hat{\mathcal{X}}_x$. If follows that $\hat{\mathcal{X}}$ satisfies (Hi). The condition (Hii) follows from the definition of $\hat{\mathcal{X}}_x$.

By construction, $\hat{\hat{\mathcal{X}}}_x$ is the set of vector fields $Z = (\phi_{t_p}^{Y_p} \circ \cdots \circ \phi_{t_1}^{Y_1})_*(\nu.Y)$ where $Y_1, \ldots, Y_p, Y$ belongs to $\hat{\mathcal{X}}$ for some $\nu > 0$. As we have $Y = (\phi_{t_q}^{X_q} \circ \cdots \circ \phi_{t_1}^{X_1})_*(\nu'.X)$, from Remark 3.2 we get

$$Z = \left(\phi_{s_m}^{X_m} \circ \cdots \circ \phi_{s_1}^{X_1}\right)_* \left(\nu\nu' X\right)$$

for appropriate vector fields $X_1, \ldots, X_m, X$ in $\mathcal{X}$ and appropriate real values $s_1, \ldots, s_m$.

Now, on the considered chart (V_x, ϕ), we have $\| J^{s+2}[Y](x)\|_\phi \leqslant k$, $X \in \mathcal{X}$. So we also have $\| J^{s+2}[Z](x)\| \leqslant k$. We conclude that Z belongs to $\hat{\mathcal{X}}$. □

Let $\mathcal{G}_\mathcal{X}$ **be the pseudogroup** of local diffeomorphisms Ψ which are defined in the following way:

$\Psi = \phi_n \circ \cdots \circ \phi_k \circ \cdots \circ \phi_1$ when these compositions are well defined and where ϕ_k is a local diffeomorphism of one of the following type:

(i) $\phi_{\tau_k}^X$ for some $X \in \mathcal{X}$ and $\tau_k \in \mathbb{R}$.

(ii) $\phi_{\tau_k}^{\xi_k}$ or $[\phi_{\tau_k}^{\xi_k}]^{-1}$ as defined in Theorem 2, where $\xi_k = \{X_\alpha,\ \alpha \in A_k\}$ is a finite or countable subset of $\hat{\mathcal{X}}$ and $\tau_k \in \mathbb{R}^{A_k}$.

Comments 3.3.

1. From (7) any flow ϕ_τ^Y for $Y \in \hat{\mathcal{X}}$ belongs to $\mathcal{G}_\mathcal{X}$.
2. Let be $\Psi = \phi_n \circ \cdots \circ \phi_1 \in \mathcal{G}_\mathcal{X}$. By construction of Ψ, to each ϕ_k is associated a family $\xi_k = \{X_\alpha,\ \alpha \in A_k\}$ which is a finite or countable subset of $\hat{\mathcal{X}}$ and $\tau_k \in \mathbb{R}^{A_k}$, we have a real positive number $\sum_{k=1}^n \|\tau_k\|_1 < \infty$ associated to Ψ. If ϕ_k is of type (ii), according to Remark 2.7.1, denote by Φ_k the flow associated to each ξ_k with $u = \Gamma^{\tau_k}$ or $u = \hat{\Gamma}^{\tau_k}$ if $\phi_k = \phi_{\tau_k}^{\xi_k}$ or $\phi_k = [\phi_{\tau_k}^{\xi_k}]^{-1}$ respectively. If ϕ_k is of type (i) ξ_k is reduced to some $X_k \in \mathcal{X}$ and we have $\Phi_k(t, y) = \phi_t^{X_k}(y)$.
   Take any pair $(x, y) \in M^2$ such that $y = \Psi(x)$. We set $t_0 = 0$ and $t_k = \sum_{i=1}^k \|\tau_i\|_1$ for $k = 1, \ldots, n$. Consider the sequence (x_k) defined by $x_0 = x$ and $x_k = \Phi_k(\tau_k, x_{k-1})$. So for each k, we can consider the l^1-curve $\gamma_k : [t_{k-1}, t_k] \to M$ defined by $\gamma_k(t) = \Phi_k(t - t_{k-1}, x_{k-1})$ (see Remark 2.7.3). By construction, we have $\gamma(t_k) = x_k$ and $y = x_n$. So if $T = \sum_{k=1}^n \|\tau_k\|_1$ we get a sequence of l^1-curve $\gamma = [0, T] \to M$, defined by $\gamma_{|[t_{k-1}, t_k[} = \gamma_k$, such that $\gamma(0) = x$ and $\gamma(T) = y$.
3. Given a family $\xi \subset \mathcal{X}(M)$, recall that a ξ-**piecewise smooth curve** is a piecewise smooth curve $\gamma : [a, b] \to M$ such that each smooth part is tangent to X or $-X$ for some $X \in \xi$.
   When $y = \phi_\tau^Y(x)$ for $Y \in \hat{\mathcal{X}}$, from (7), we can clearly associate a ξ-piecewise smooth curve which joins x to y.
   Now, consider any $\xi = \{X_\alpha,\ \alpha \in A\} \subset \hat{\mathcal{X}}$ and τ small enough such that ϕ_τ^ξ is defined and consider $y = \phi_\tau^\xi(x)$. If $A = \{1, \ldots, n\}$ is finite, from the previous argument, there exists a family $\xi_n \subset \mathcal{X}$ and an associate ξ_n-piecewise smooth curve γ_n' which joins x to y. On the other hand, if A is countable, to each $k \in A$, we can associate a family $\xi_k \subset \mathcal{X}$ and a ξ_k-piecewise smooth curve γ_k' which joins $x = x_0$ to x_k (as defined in Remark 2.7.3). So we

get a sequence of $\mathcal{X}$-piecewise smooth curves whose origin is x_0 (for all curves) and whose sequence of ends converges to y. Note that, for Theorem 2, the same result is true for any pair $(z, \Phi^{\xi}_{\tau}(z))$ for any z in some neighbourhood U of x we have

$$\phi^{\xi}_{\tau}(z) = \lim_{m\to\infty} \phi^{X_m}_{-\tau_m} \circ \cdots \circ \phi^{X_1}_{-\tau_1}(z) \quad \text{for any } z \in U \tag{8}$$

where $\xi = \{X_k,\ k \in A\} \subset \hat{\mathcal{X}}$ and $\tau = (t_k)_{k\in A}$.
From (7) to each finite sequence $\phi^{X_m}_{-\tau_m} \circ \cdots \circ \phi^{X_1}_{-\tau_1}(z)$, we can associate an $\mathcal{X}$-piecewise smooth curve γ'_m which joins z to $z_m = \phi^{X_m}_{-\tau_m} \circ \cdots \circ \phi^{X_1}_{-\tau_1}(z)$. So given ϕ^{ξ}_{τ}, for any $z \in V$ we have a family of $\mathcal{X}$-piecewise smooth curves γ'_m whose origin is z and whose sequence of ends converges to $\phi^{\xi}_{\tau}(z)$.
Now, consider the case $y = \hat{\phi}^{\xi}_{\tau}(x) = [\phi^{\xi}_{\tau}]^{-1}(x)$. Again, from Theorem 2, there exists some open neighbourhood U of x such that $\hat{\phi}^{\xi}_{\tau}$ is a local diffeomorphism on U of x and we have

$$\hat{\phi}^{\xi}_{\tau}(z) = \lim_{m\to\infty} \phi^{X_1}_{-\tau_1} \circ \cdots \circ \phi^{X_m}_{-\tau_m}(z) \quad \text{for any } z \in U \tag{9}$$

where $\xi = \{X_k,\ k \in A\} \subset \hat{\mathcal{X}}$ and $\tau = (t_k)_{k\in A}$.
Again from (7) to each finite sequence $\phi^{X_1}_{-\tau_1} \circ \cdots \circ \phi^{X_m}_{-\tau_m}(z)$, we can associate an $\mathcal{X}$-piecewise smooth curve γ'_m which joins z to $z_m = \phi^{X_1}_{-\tau_1} \circ \cdots \circ \phi^{X_m}_{-\tau_m}(z)$. So given $\hat{\phi}^{\xi}_{\tau}$, for any $z \in V$ we have a family of $\mathcal{X}$-piecewise smooth curves γ'_m whose origin is z and whose sequence of ends converges to $\hat{\phi}^{\xi}_{\tau}(z)$. This is in particular true for the previous fixed pair (x, y).
In the general case when $y = \Phi(x)$ for some $\Phi \in \mathcal{G}_{\mathcal{X}}$, we have $\Phi = \phi_n \circ \cdots \circ \phi_1 \in \mathcal{G}_{\mathcal{X}}$. Set $x_1 = \phi_1(x)$. From the previous partial results, either we have an $\mathcal{X}$-piecewise smooth curve which joins x to x_1 or there exists a sequence γ_k of $\mathcal{X}$-piecewise smooth curves whose origin is x (for all curves) and whose sequence of ends converges to x_1. At first, assume that we are in the first case. Now, if we have an $\mathcal{X}$-piecewise smooth curve which joins x to x_1, applying the previous argument in x_1 by concatenation, we get either an $\mathcal{X}$-piecewise smooth curve which joint x to $x_2 = \phi_2(x_1)$ or we get a sequence of a sequence of $\mathcal{X}$-piecewise smooth curves whose origin is x (for all curves) and whose sequence of ends converges to x_2. If we are in the second case, where V is a neighborhood of x_1 on which (8) or (9) is true. For k large enough, $\gamma_k(x_1)$ belongs to V. So for each k, we have a family of $\mathcal{X}$-piecewise smooth curves $\gamma'_{(k,n)}$ whose origin is $\gamma_k(x)$ (for all curves) and whose sequence of ends converges to $\phi_2(\gamma_k(x_k))$. As $\lim \phi_2(\gamma_k(x_k)) = x_2$, there exists an increasing sequence n_k such that the sequence $\hat{\gamma}_k$ of the concatenations γ_k with $\gamma'_{(k,n_k)}$ is a sequence of $\mathcal{X}$-piecewise smooth curves whose origin is x (for all curves) and whose sequence of the ends converges to x_2. By finite induction on k, we get the same result for the pair (x, y).

To $\mathcal{G}_{\mathcal{X}}$ is naturally associated the following equivalence relation on M:

$$x \equiv y \text{ if and only if there exists } \Phi \in \mathcal{G}_{\mathcal{X}} \text{ such that } \Phi(x) = y.$$

An equivalence class is called an l^1-orbit of $\mathcal{X}$ or an $\mathcal{X}$-orbit.

The term “l^1-orbit” will be justified by the following result which sums up the previous commentaries and Lemma 3.1 part (ii):

Proposition 3.4.

1. *Each point of the $\mathcal{X}$-orbit of x can be joined from x by an l^1-curve whose each connected smooth part is tangent to Y or $-Y$ for some $Y \in \hat{\mathcal{X}}$.*
2. *For each pair (x, y) in the same $\mathcal{X}$-orbit, either we have an $\mathcal{X}$-piecewise smooth curve which joins x to y or there exists a sequence γ_k of $\mathcal{X}$-piecewise smooth curves whose origin is x (for all curves) and whose sequence of the ends converges to y.*
3. *Let $\mathcal{G}_{\hat{\mathcal{X}}}$ be the pseudogroup naturally associated to $\hat{\mathcal{X}}$. Then we have $\mathcal{G}_{\hat{\mathcal{X}}} = \mathcal{G}_{\mathcal{X}}$. In particular each $\hat{\mathcal{X}}$-orbit is a $\mathcal{X}$-orbit.*

3.2. Preliminaries on weak distributions

Recall that, according to the proof of Sussmann's theorem on reachable sets in [16], we want to associate to $\mathcal{X}$ and $\hat{\mathcal{X}}$ weak distributions $\mathcal{D}$ and $\hat{\mathcal{D}}$ respectively, such that $\mathcal{D}_x \subset \hat{\mathcal{D}}_x$ for any $x \in M$, $\hat{\mathcal{D}}$ is invariant by any flow of vector fields in $\mathcal{X}$ and which is minimal (in some sense) for these properties.

Before beginning this construction, we need to recall some definitions on weak distributions which will be used in the next subsection.

• Given a finite or countable or eventually uncountable ordered set A of indexes, a family $\{\epsilon_\alpha,\ \alpha \in A\}$ is said to be an unconditional basis of $\mathbb{R}^A$ if, for every $\tau \in \mathbb{R}^A$ there is a unique family of scalars $\{\tau_\alpha;\ \alpha \in A\}$ such that $\tau = \sum_{\alpha \in A} \tau_\alpha \epsilon_\alpha$ (unconditional convergence); such a basis is symmetric if for any sequence $(\alpha_k) \in A$ with $k \in K \subset \mathbb{N}$, the basic sequence $\{\tau_{\alpha_k},\ k \in K\}$ is equivalent to the canonical basis of $\mathbb{R}^K$ (see for instance [12]). It is well known that all unconditional symmetric basis of $\mathbb{R}^A$ are equivalent to the canonical basis of $\mathbb{R}^A$.

• A **weak submanifold** of M of class C^p (resp. smooth) is a pair (N, f) of a connected Banach manifold N of class C^p (resp. smooth) (modeled on a Banach space F) and a map $f : N \to M$ of class C^p (resp. smooth) such that: [6,13]:

– there exists a continuous injective linear map $i : F \to E$ between these two Banach spaces;
– f is injective and the tangent map $T_x f : T_x N \to T_{f(x)} M$ is injective for all $x \in N$.

Note that for a weak submanifold $f : N \to M$, on the subset $f(N)$ in M we have two topologies:

– the induced topology from M;
– the topology for which f is a homeomorphism from N to $f(N)$.

With this last topology, via f, we get a structure of Banach manifold modeled on F. Moreover, the inclusion from $f(N)$ into M is continuous as a map from the Banach manifold $f(N)$ to M. In particular, if U is an open set of M, then, $f(N) \cap U$ is an open set for the topology of the Banach manifold on $f(N)$.

• According to [13], a **weak distribution** on an M is an assignment $\mathcal{D} : x \mapsto \mathcal{D}_x$ which, to every $x \in M$, associates a vector subspace $\mathcal{D}_x$ in $T_x M$ (not necessarily closed) endowed with a norm $\| \ \|_x$ such that $(\mathcal{D}_x, \| \ \|_x)$ is a Banach space (denoted by $\tilde{\mathcal{D}}_x$) and such that the natural inclusion $i_x : \tilde{\mathcal{D}}_x \to T_x M$ is continuous.

When $\mathcal{D}_x$ is closed, we have a natural Banach structure on $\tilde{\mathcal{D}}_x$, induced by the Banach structure on T_xM, and so we get the classical definition of a distribution; in this case we will say that $\mathcal{D}$ is **closed**.

A vector field $Z \in \mathcal{X}(M)$ is **tangent** to $\mathcal{D}$, if for all $x \in \mathrm{Dom}(Z)$, $Z(x)$ belongs to $\mathcal{D}_x$. The set of local vector fields tangent to $\mathcal{D}$ will be denoted by $\mathcal{X}_{\mathcal{D}}$.

• We say that $\mathcal{D}$ **is generated by a subset** $\mathcal{X} \subset \mathcal{X}(M)$ if, for every $x \in M$, the vector space $\mathcal{D}_x$ is the linear hull of the set $\{Y(x),\ Y \in \mathcal{X},\ x \in \mathrm{Dom}(Y)\}$.

For a weak distribution $\mathcal{D}$, on M we have the following definitions:

• $\mathcal{D}$ is **lower (locally) trivial** at x if there exist an open neighborhood V of x in M, a smooth map $\Phi : \tilde{\mathcal{D}}_x \times V \to TM$ (called **lower trivialization**) such that:

(i) $\Phi(\tilde{\mathcal{D}}_x \times \{y\}) \subset \mathcal{D}_y$ for each $y \in V$;
(ii) for each $y \in V$, $\Phi_y \equiv \Phi(\,,y) : \tilde{\mathcal{D}}_x \to T_yM$ is a continuous operator and $\Phi_x : \tilde{\mathcal{D}}_x \to T_xM$ is the natural inclusion i_x;
(iii) there exists a continuous operator $\tilde{\Phi}_y : \tilde{\mathcal{D}}_x \to \tilde{\mathcal{D}}_y$ such that $i_y \circ \tilde{\Phi}_y = \Phi_y$, $\tilde{\Phi}_y$ is an isomorphism from $\tilde{\mathcal{D}}_x$ onto $\Phi_y(\tilde{\mathcal{D}}_x)$ and $\tilde{\Phi}_x$ is the identity of $\tilde{\mathcal{D}}_x$.

We say that $\mathcal{D}$ is **lower (locally) trivial** if it is lower trivial at any $x \in M$.

• $\mathcal{D}$ is called a l^1**-distribution** if each Banach space $\tilde{\mathcal{D}}_x$ is isomorphic to $\mathbb{R}^A$, for some appropriate finite, countable or eventually uncountable ordered set A of indexes (which depends of x).

• An **integral manifold** of class C^p, with $p \geqslant 1$ (resp. smooth) of $\mathcal{D}$ through x is a weak submanifold $f : N \to M$ of class C^p (resp. smooth) such that there exists $x_0 \in N$ with $f(x_0) = x$ and $T_zf(T_zN) = \mathcal{D}_{f(z)}$ for all $z \in N$. An integral manifold through $x \in M$ is called **maximal** if, for any integral manifold $g : L \to M$ through x, the set $g(L)$ is an open submanifold of $f(N)$, according to the structure of Banach manifold on $f(N)$ induced by N via f.

• $\mathcal{D}$ is called **integrable** of class C^p (resp. smooth) if for any $x \in M$ there exists an integral manifold N of class C^p (resp. smooth) of $\mathcal{D}$ through x.

• If $\mathcal{D}$ is generated by $\mathcal{X} \subset \mathcal{X}(M)$, then $\mathcal{D}$ is called $\mathcal{X}$**-invariant** if for any $X \in \mathcal{X}$, the tangent map $T_x\phi_t^X$ send $\mathcal{D}_x$ onto $\mathcal{D}_{\phi_t^X(x)}$ for all $(t,x) \in \Omega_X$. $\mathcal{D}$ is **invariant** if $\mathcal{D}$ is $\mathcal{X}_{\mathcal{D}}$-invariant.

3.3. Characteristic distribution associated to $\mathcal{X}$

Consider any set $\mathcal{Y}$ of local vector fields such that, conditions (Hi) and (Hii) are satisfied. We denote by $\mathcal{Y}_x$ the set of vector fields $Y \in \mathcal{Y}$ such that x belongs to $\mathrm{Dom}(Y)$. The distribution $l^1(\mathcal{Y})$ defined by

$$l^1(\mathcal{Y})_x = \left\{ X \in T_xM \text{ such that } X = \sum_{Y \in \mathcal{Y}_x} \tau_Y Y(x) \text{ with } \sum_{Y \in \mathcal{Y}} |\tau_Y| \text{ summable} \right\}$$

is called **the** l^1**-characteristic distribution** generated by $\mathcal{Y}$.

For $x \in M$ fixed, let Λ be any (ordered) set of indexes of same cardinal as $\mathcal{Y}_x$ so that each element of $\mathcal{Y}_x$ can be indexed as Y_λ, $\lambda \in \Lambda$. We then have a surjective linear map: $T : l^1(\Lambda) \to T_xM$ defined by $T((\tau_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} \tau_\lambda Y_\lambda(x)$ and whose range is $l^1(\mathcal{Y})_x$. So we get a bijective continuous map $\bar{T}$ from the quotient $l^1(A)/\ker T$ onto $l^1(\mathcal{Y})_x$. So we can put on $l^1(\mathcal{Y})_x$ a structure of Banach space such that $\bar{T}$ is an isometry. Finally, $l^1(\mathcal{Y})$ is a weak distribution. $l^1(\mathcal{Y})_x$ will always be equipped with this Banach structure.

Remark 3.5.

1. For the existence of $l^1(\mathcal{Y})_x$ we only need that for all $X \in \mathcal{Y}_x$

   $$\sup\{\|X(x)\|_\phi,\ X \in \mathcal{X}\} < \infty.$$

   So the condition (Hii) is much too strong in this way. However, independently of the existence of $l^1(\mathcal{Y})_x$, in this paper, we need to consider the set $\mathcal{Y}$ of local vector fields which satisfies condition (Hii).
2. The Banach space $l^1(\mathcal{Y})_x$ is isomorphic to $l^1(A)$ for some ordered set A if and only if, with the previous notations, $\ker T$ is complemented. In this case, A has the same cardinal as Λ (see [9]). In particular, if the distribution $l^1(\mathcal{Y})$ is upper trivial (see Section 4.3), then $l^1(\mathcal{Y})_x$ is isomorphic to some $l^1(A)$ for any $x \in M$.

The **characteristic distribution** $\mathcal{D}$ **associated to** $\mathcal{X}$ is defined by

$$\mathcal{D}_x = l^1(\mathcal{X}_x).$$

Note that, from assumptions (Hi) and (Hii), $\mathcal{D}_x$ is well defined for any $x \in M$. Moreover, the natural inclusion of $\mathcal{D}_x$ into T_xM is continuous.

In the same way, the **characteristic distribution** $\hat{\mathcal{D}}$ **associated to** $\hat{\mathcal{X}}$ is defined by

$$\hat{\mathcal{D}}_x = l^1(\hat{\mathcal{X}}_x).$$

From Lemma 3.1 part (i) it follows that $\hat{\mathcal{D}}$ is well defined and, again, the natural inclusion of $\hat{\mathcal{D}}_x$ in T_xM is continuous. Moreover, as $\mathcal{X}_x \subset \hat{\mathcal{X}}_x$, we have $\mathcal{D}_x \subset \hat{\mathcal{D}}_x$ for any $x \in M$. The other relative properties of $\mathcal{D}$ and $\hat{\mathcal{D}}$ are given in the following proposition.

Proposition 3.6.

1. *$\hat{\mathcal{D}}$ is $\mathcal{X}$-invariant and also $\hat{\mathcal{X}}$-invariant.*
2. *Let $\mathcal{Y}$ be any family of local vector fields which satisfies* (Hi) *and* (Hii) *and which contains $\mathcal{X}$. If the associated distribution $l^1(\mathcal{Y})$ is $\mathcal{X}$-invariant then $l^1(\mathcal{Y})_x$ contains $\hat{\mathcal{D}}_x$ for any $x \in M$. In particular, if $\mathcal{D}$ is $\mathcal{X}$-invariant, then $\mathcal{D} = \hat{\mathcal{D}}$.*
3. *Given $x \in M$ and assume that we have the following properties*:
   (i) *there exists a finite countable or eventually uncountable set A of indexes such that $\hat{\mathcal{D}}_x$ is isomorphic to $\mathbb{R}^A$;*
   (ii) *there exists a chart domain V_x centered at x and a family $\{X_\alpha,\ \alpha \in A\} \subset \hat{\mathcal{X}}_x$ such that $\{X_\alpha,\ \alpha \in A\}$ satisfies the condition* (LB($s+2$)) *on V_x, for some $s > 0$, and, $\{X_\alpha(x),\ \alpha \in A\}$ is a symmetric unconditional basis of $\hat{\mathcal{D}}_x \equiv \mathbb{R}^A$.*

   Then, there exists a weak Banach manifold $\Theta : B(0,\rho) \to M$ of class C^s, which is an integral manifold of $\hat{\mathcal{D}}$ through x, where $B(0,\rho)$ is the open ball in the Banach space $\mathbb{R}^A$. Such a manifold will be called a **slice centered at** *x.*
4. *Let $f : N \to M$ be a smooth connected integral manifold such that $x \in f(N)$. Assume that the hypothesis of part* 3 *are satisfied at x. Then, for ρ small enough , $\Theta(B(0,\rho))$ is contained in $f(N)$ and $f^{-1}(\Theta(B(0,\rho)))$ is an open set in N.*

Remark 3.7. Classically, a distribution on M is an assignment $\triangle : x \mapsto \triangle_x$ where $\triangle_x$ is a vector subspace of T_xM. As in [16], on the set of distributions, we can consider the partial order:

$$\triangle \subset \triangle' \quad \text{if and only if} \quad \triangle_x \subset \triangle'_x \quad \text{for any } x \in M.$$

So the result of part 2 of Proposition 3.6 can be interpreted in the following way:

The distribution $\hat{\mathcal{D}} = l^1(\hat{\mathcal{X}})$ is minimal among all the l^1-characteristic distribution $l^1(\mathcal{Y})$, generated by the family of vector fields $\mathcal{Y}$ which satisfies (Hi) and (Hii), contains $\mathcal{X}$ and which are $\mathcal{X}$-invariant.

Proof.

•*Proof of part* 1.

We want to prove that $T_z\Phi[\hat{\mathcal{D}}_z] = \hat{\mathcal{D}}_{\Phi(z)}$ for any $z \in \mathrm{Dom}(\Phi)$ and for any flow Φ of vector field of $\mathcal{X}$ and $\hat{\mathcal{X}}$.

We first show that this is true for any flow ϕ_t^X where $X \in \mathcal{X}$. Take any $Z \in \hat{\mathcal{X}}$ such that z belongs to $\mathrm{Dom}(Z)$ and set $x = \phi_t^X(z)$. There exists $Y \in \mathcal{X}$ and a finite composition Φ of flows of vector fields of $\mathcal{X}$ such that $Z = \Phi_*(\nu Y)$ for some $\nu > 0$. So we have $Z' = (\phi_t^X)_*(Z) = \Phi'_*(\nu X)$ where $\Phi' = (\phi_t^X \circ \Phi)$. But, there exists $\nu' > 0$ such that $\nu' Z$ belongs to $\hat{\mathcal{X}}_x$, in particular $Z'(x)$ belongs to $\hat{\mathcal{D}}_x$. As $\hat{\mathcal{D}}_x$ is generated by $\{Y(x),\ Y \in \hat{\mathcal{X}}_x\}$ we then have

$$T_z\phi_t^X[\hat{\mathcal{D}}_z] \subset \hat{\mathcal{D}}_x. \tag{10}$$

As $(\phi_t^X)^{-1} = \phi_{-t}^X$, by the same argument we get $T_x(\phi_t^X)^{-1}[\hat{\mathcal{D}}_x] \subset \hat{\mathcal{D}}_z$ and from (10) we get

$$T_z\phi_t^X\big[T_x\big(\phi_t^X\big)^{-1}[\hat{\mathcal{D}}_x]\big] = \hat{\mathcal{D}}_{\phi_t^X(z)} = \hat{\mathcal{D}}_x \subset T_z\phi_t^X[\hat{\mathcal{D}}_z].$$

Now, from (7) and the previous argument, we also have $T_z\phi_t^Y[\hat{\mathcal{D}}_z] = \hat{\mathcal{D}}_{\phi_t^Y(z)}$ for any $z \in \mathrm{Dom}(\phi_t^Y)$ and for any flow ϕ_t^Y with $Y \in \hat{\mathcal{X}}$.

•*Proof of part* 2.

Let be $x \in M$ and $Z \in \hat{\mathcal{X}}$ such that $x \in \mathrm{Dom}(Z)$. As before, we have $Z = \Phi_*(\nu Y)$ for some finite composition of flows of vector fields of $\mathcal{X}$ and Y is a vector field of $\mathcal{X}$ and $\nu > 0$.

$$\begin{aligned} Z(x) &= Z\big(\Phi\big(\Phi^{-1}(x)\big)\big) \\ &= T_{\Phi^{-1}(x)}\Phi\big(\nu Y\big(\Phi^{-1}(x)\big)\big). \end{aligned}$$

As $\triangle$ is $\mathcal{X}$-invariant we obtain that $Z(x)$ belongs to $\triangle_x$ and we get $\hat{\mathcal{D}}_x \subset \triangle_x$. In particular, if $\triangle = \mathcal{D}$, it is obvious that $\hat{\mathcal{D}}_x$ contains $\mathcal{D}_x$, so we get an equality.

This ends the proof of part 2.

•*Proof of part* 3.

In this proof we will use some notations and results proved in Section 6. In each case, we will mention the precise references of these notations and results.

Let be $x \in M$ for which all assumptions in part 3 are satisfied. Denote by (V_x, ϕ) the chart centered at x such that $\{X_\alpha,\ \alpha \in A\} \subset \hat{\mathcal{X}}_{V_x}$. Then, $V_x \subset \mathrm{Dom}(X_\alpha)$ for each $\alpha \in A$ and we set

$$k = \sup\big\{\big\| J^{s+2}(X_\alpha)(y)\big\|_\phi,\ \alpha \in A,\ y \in V_x\big\}.$$

Without loss of generality, we can assume that V_x is an open subset V of the Banach space $E \equiv T_xM$ and also that $TM \equiv V \times E$ on V_x. We choose $r > 0$ such that $B(x, 2r)$ is contained in V. For the sake of simplicity, we denote by

$$\big\{\epsilon_\alpha = X_\alpha(x)\big\}_{\alpha \in A}$$

the symmetric unconditional basis of $\hat{\mathcal{D}}_x$.

There exists an isomorphism T from $\hat{\mathcal{D}}_x$ to $\mathbb{R}^A$ such that: $T(\epsilon_\alpha) = e_\alpha$ where $\{e_\alpha\}_{\alpha \in A}$ is the canonical basis of $\mathbb{R}^A$. So we can choose $\rho > 0$ such that the image by T of the open ball $B(0, \rho) \subset \hat{\mathcal{D}}_x$ is contained in $B(0, \frac{r}{k}) \subset \mathbb{R}^A$.

Given any fixed $w = \sum_{\alpha \in A} t_\alpha \epsilon_\alpha \in B(0, \rho)$, we set $T(w) = \tau = (\tau_\alpha)_{\alpha \in \alpha}$. Of course, $T(w) \in B(0, \frac{r}{k})$. By application of Theorem 1 on V in the particular case where:

$\xi = \{X_\alpha\}_{\alpha \in A}$, $I = \mathbb{R}$, $u = \Gamma^\tau$ (see Section 6, Section 6.1), $t_0 = 0$, T' is any real number, large enough, and $T_0 = \|\tau\|_1$.

We have already proved that

$$\Gamma^\tau \in L^1_b(\mathbb{R}) \quad \text{with } \left\| \Gamma^\tau \right\|_\infty = 1.$$

Let be $I_0 = [-T_0, T_0]$ and $B_0 = B(x, r - kT_0)$. As $T_0 < \frac{r}{k}$, there exists a flow Φ_{Γ^τ} defined on $J_0 \times B_0$. From Theorem 2, $\Theta = \Psi^x \circ T$, is a map of class C^s from $B(0, \rho) \subset \hat{\mathcal{D}}_x$ with values in an open set of E contained in V. We then have

$$\Theta(w) = \Psi^x(\tau) = \phi^\xi_\tau(x) = \Phi_{\Gamma^\tau}\big(\|\tau\|_1, x\big). \tag{11}$$

The exact expression of ψ^x is given in Section 6.

It follows from Theorem 2 that Θ is a map of class C^s with $s > 0$ from $B(0, \rho)$ into V. We can consider $D\Theta_w$ as a field on $B(0, \rho)$ of operators from $\hat{\mathcal{D}}_x \equiv \mathbb{R}^A$ into $T_x M \equiv E$. On the other hand, we have

$$\begin{aligned} D\Theta_0(\epsilon_\alpha) &= D\Psi^x_{(0)}\big(T(\epsilon_\alpha)\big) \\ &= D\Psi^x_{(0)}(e_\alpha) \\ &= \epsilon_\alpha. \end{aligned}$$

So $D\Theta_0$ is an injective operator from $\hat{\mathcal{D}}_x$ into $T_x M$.

Now from [13] we have:

Lemma 3.8.

1. *Consider two Banach spaces E_1 and E_2 and $i : E_1 \to E_2$ an injective continuous operator. Let Θ_y be a continuous field of continuous operators of $L(E_1, E_2)$ on an open neighbourhood V of $x \in E_1$ such that $\Theta_x = i$. Then there exists a neighbourhood W in V such that Θ_y is an injective operator on W.*
2. *Let $f : U \to V$ be a map of class C^1 from two open sets U and V in Banach spaces E_1 and E_2 respectively such that $T_u f$ is injective at $u \in U$. Then there exists an open neighbourhood W of u in U such that the restriction of f to W is injective.*

By applying this lemma, we conclude that, for ρ small enough, $\Theta : B(0, \rho) \to V$ is a weak submanifold of class C^s.

It remains to show that $D\Theta_w(\hat{\mathcal{D}}_x) = \hat{\mathcal{D}}_{\Theta(w)}$.

Given $v = \sum_{\alpha \in A} v_\alpha \epsilon_\alpha$, we set $\sigma = T(v)$. From (29) (Section 6), we have

$$D\Theta_w(v) = D\Psi^x_{T(w)} = \triangle\Psi^x(\tau) \circ \mathcal{R}(\tau)\left(\sum_{\alpha \in A} \sigma_\alpha \epsilon_\alpha\right).$$

On one hand, the map $\mathcal{R}(\tau)(\sum_{\alpha \in A} \sigma_\alpha \epsilon_\alpha) = \sum_{\alpha \in A} \sigma_\alpha \triangle \hat{\Psi}^x_\alpha((-\tau)^\alpha)[X_i(\Psi^x_\alpha(\tau^\alpha)]$ is a continuous field $\tau \mapsto \mathcal{R}(\tau)$ of endomorphisms of $\hat{\mathcal{D}}_x$ (see Lemma 6.8). As at $\tau = 0$, the operator $\mathcal{R}(0)$ is the identity of $\hat{\mathcal{D}}_x$, for ρ small enough, $w \mapsto \mathcal{R} \circ T(w)$ is a field of isomorphisms of $\hat{\mathcal{D}}_x$.

On the other hand we have $\Theta(w) = \phi^\xi_{T(w)}(x)$. As $\phi^\xi_{T(w)}$ belongs to $\mathcal{G}_\mathcal{X}$, from part 1 of this proposition, we have: $D\phi^\xi_{T(w)}(\hat{\mathcal{D}}_x) = \hat{\mathcal{D}}_{\phi^\xi_{T(w)}(x)} = \hat{\mathcal{D}}_{\Theta(w)}$.

So we obtain the result required for ρ small enough. This ends the proof of part 3.

•*Proof of part* 4.

The point $x \in M$ for which the assumptions of part 3 of the proposition is true will be fixed, and we suppose that TM is trivializable on the chart domain V (around x). We then have:

Lemma 3.9. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of vector fields on $U \subset V$ which satisfies the condition* (LB$(s+2)$) *on U and which is an unconditional symmetric basis of $\hat{\mathcal{D}}_x$.*

1. *There exists a morphism $\Psi : U \times \hat{\mathcal{D}}_x \to TM$ which is a lower trivialization at x such that $\Psi_y(X_\alpha(x)) = X_\alpha(y)$ for any $\alpha \in A$.*
2. *For any integral manifold $f : N \to U$ of $\hat{\mathcal{D}}$ of class C^s through x, there exists a family $\{Y_\alpha\}_{\alpha\in A}$ of vector fields on N defined on a neighbourhood of $f^{-1}(x)$ such $f_*Y_\alpha = X_\alpha$ and $\eta = \{Y_\alpha\}_{\alpha\in A}$ satisfies the condition* (LB$(s+2)$) *at $f^{-1}(x)$.*

Proof. Consider $\tilde{\Psi} : \hat{\mathcal{D}}_x \times U \to \hat{\mathcal{D}}$ defined in the following way:

$$\text{if } w = \sum_{\alpha\in A} w_\alpha \epsilon_\alpha \text{ we set } \Psi(w, y) = \sum_{\alpha\in A} w_\alpha X_\alpha(y).$$

As usual, we set $\tilde{\Psi}_y = \tilde{\Psi}(, y)$. Denote by $\bar{\mathcal{D}}_y$ the normed subspace defined by $\hat{\mathcal{D}}_y$ from the structure of Banach space on T_yM, and $i_y : \hat{\mathcal{D}}_y \to T_yM$ the natural inclusion.

At first, as by definition, $\sum_{\alpha\in A} w_\alpha$ is absolutely summable, from the property LB$(s+2)$, it follows that $\Psi(w, y) \in T_yM$ is well defined and Ψ_y is a continuous operator from $\hat{\mathcal{D}}_x$ to $\hat{\mathcal{D}}_y$ such that $\|\Psi_y\| \leqslant K$. We set $\Psi_y = i_y \circ \tilde{\Psi}_y$. It is clear that the field $y \to i_y \circ \tilde{\Psi}(y)$ is smooth. From this construction, it is easy to see that $\Psi(w, y) = i_y \circ \tilde{\Psi}_y(w)$ is a lower trivialization at x such that $\Psi_y(X_\alpha(x)) = X_\alpha(y)$ for any $\alpha \in A$.

Let $f : N \to U$ be an integral manifold of $\hat{\mathcal{D}}$ through x of class C^s. Then, N is a Banach manifold modeled on the Banach space $\hat{\mathcal{D}}_x$. For any open neighborhood W of x the set $\tilde{W} = f^{-1}(W)$ is an open neighborhood of $\tilde{x} = f^{-1}(x)$. Without loss of generality, we may assume that N is an open set in $\hat{\mathcal{D}}_x$, with $\tilde{x} = 0$, and M is an open set in $E \equiv T_xM$. Modulo these identifications, f is the natural inclusion of N in M, that is the restriction to N of the natural inclusion $i_x : \hat{\mathcal{D}}_x \to T_xM$. In this context, on $i(N) \subset M$, $y \to \Psi_y$ is a C^s field of continuous linear operators from $\hat{\mathcal{D}}_x$ into $i_y(\mathcal{D}_y) \equiv i_x(\hat{\mathcal{D}}_x) \times \{y\} \subset E \times \{y\} \equiv T_yM$. From Lemma 2.10 in [13] $y \mapsto \tilde{\Psi}_y$ is also a C^s field of linear operators from $\hat{\mathcal{D}}_x$ into $\hat{D}_y \times \{y\} \equiv i_x(\hat{\mathcal{D}}_x) \times \{y\} \equiv T_yN$. It follows that, for any $\alpha \in A$, $Y_\alpha(y) = \tilde{\Psi}(\epsilon_\alpha)$ is a C^s vector field on N such that $(i_x)_*Y_\alpha \equiv f_*Y_\alpha = X_\alpha$. From the previous definition of Y_α, it follows that $\eta = \{Y_\alpha\}_{\alpha\in A}$ satisfies the condition (LB$(s+2)$). □

Now we come back to the proof of part 3. Consider an integral manifold $f : N \to M$ of $\hat{\mathcal{D}}$ of class C^s through x and suppose that the assumption of part 2 is satisfied at x. On N, the family $\eta = \{Y_\alpha\}_{\alpha\in A}$ satisfies the condition (LB$(s+2)$) at $\tilde{x} = f^{-1}(x)$. So for ρ small enough, given any $\tau \in B(0, \rho) \subset \mathbb{R}^A$, we can apply Theorem 1 to the family η and $u = \Gamma^\tau$ and Theorem 2 on N. We then get:

- A C^s flow $\tilde{\Phi}_{\Gamma^\tau}(t,)$ of $\tilde{Z} = \sum_{\alpha\in A} \Gamma^\tau_\alpha Y_\alpha$ (see Section 6) such that for any z in a small neighborhood W of $\tilde{x}$ we have

$$\Phi_{\Gamma^\tau}\big(t, f(z)\big) = f \circ \tilde{\Phi}_{\Gamma^\tau}(t, z)$$

where $\Phi_{\Gamma^\tau}(t, \,)$ is the flow of $Z = \sum_{\alpha\in A} \Gamma^\tau_\alpha X_\alpha$.

- On N, the associated flow $\tilde{\phi}^\eta_\tau(x) = \tilde{\Phi}_{\Gamma^\tau}(T, \tilde{x})$ and as in (11), the associated map $\tilde{\Theta}(w) = \tilde{\phi}^\eta_{T(w)}(\tilde{x})$.

Moreover, as $T_0\tilde{\Theta}$ is an isomorphism, so for ρ small enough, $\tilde{\Theta}$ is a diffeomorphism from $B(0, \rho)$ on an open neighborhood W of $\tilde{x}$ in N. On the other hand, from the previous construction, for ρ small enough, we have $\Theta = f \circ \tilde{\Theta}$. It follows that $f^{-1}(\Theta(B(0, \rho))) = \tilde{\Theta}(B(0, \rho)) = W$. This ends the proof of part 4. □

4. Structure of weak submanifold on $\mathcal{X}$-orbits

In this section, we will give sufficient conditions under which each $\mathcal{X}$-orbits has a structure of weak submanifold of M. The first one imposes some local conditions on the set $\hat{\mathcal{X}}$ which leads to integrability of $\hat{\mathcal{D}}$ (Theorem 3) and can be seen as a generalization of Sussmann's arguments used in [16]. The second one imposes that $\hat{\mathcal{D}}$ is upper trivial and also some local involutivity conditions on $\hat{\mathcal{X}}$.

4.1. Structure of manifold and $\mathcal{X}$-orbits

Now we will prove some results about integrable distributions which contain $\mathcal{D}$ and $\mathcal{X}$-orbits. This result will be used in each two following subsections.

Consider any set $\mathcal{Y}$ of local vector fields which contains $\hat{\mathcal{X}}$ and satisfies conditions (Hi). Assume that there exists a weak distribution generated by $\mathcal{Y}$: for instance if $\mathcal{Y}$ satisfies (Hii) then we can choose $\triangle = l^1(\mathcal{Y})$ (see Section 3.3). Assume that $\triangle$ is integrable on M and for each $x \in M$ there exists a lower trivialization $\Theta : F \times V \to TM$ for some Banach space F (which depends of x) and some neighborhood V of x in M. Let N be the union of all integral manifolds $i_L : L \to M$ through x_0. Then $i_N : N \to M$ is the maximal integral manifold of $\triangle$ through x_0 (see Lemma 2.14 [13]).

For the clarity of the proof of results in this subsection, for any point $z \in N$, when N is equipped with the induced topology of M, we denote by $\tilde{z}$ the same point of N but when N is equipped of its Banach manifold structure.

Proposition 4.1. *As previously, let $f \equiv i_N : N \to M$ be the maximal integral manifold of $\triangle$ through x.*

1. *Let $Z \in \mathcal{X}(M)$ be such that* $\mathrm{Dom}(Z) \cap f(N) \neq \emptyset$ *and Z is tangent to $\triangle$. Set $\tilde{V}_Z = f^{-1}(\mathrm{Dom}(Z) \cap f(N))$. Then $\tilde{V}_Z$ is an open set in N and there exists a vector field $\tilde{Z}$ on N such that* $\mathrm{Dom}(\tilde{Z}) = \tilde{V}_Z$ *and $f_*\tilde{Z} = Z \circ f$.*
   Moreover, if $]a_x, b_x[$ is the maximal interval on which the integral curve $\gamma : t \to \phi^Z(t, x)$ is defined in M, then the integral curve $\tilde{\gamma} : t \to \phi^{\tilde{Z}}(t, \tilde{x})$ is also defined on $]a_x, b_x[$ and we have
   $$\gamma = f \circ \tilde{\gamma}. \tag{12}$$
2. *Let $\xi = \{X_\beta,\ \beta \in B\} \subset \hat{\mathcal{X}} \subset \mathcal{Y}$ be which satisfies the conditions* (LB($s + 2$)) *on a chart domain V centered at $x \in f(N)$ and consider ϕ^ξ_τ for some $\tau \in \mathbb{R}^B$ as defined in Theorem 2*

and let γ be the l^1-curve on $[0, \|\tau_1\|_1]$ associated to ϕ_τ^ξ as in Remark 2.7. Then there exists an l^1-curve $\tilde{\gamma} : [0, \|\tau\|_1[\to N$ such that

$$f \circ \tilde{\gamma} = \gamma \quad \text{on } \big[0, \|\tau\|_1\big[. \tag{13}$$

When $\triangle$ is a closed distribution, we extend $\tilde{\gamma}$ to $[0, \|\tau\|_1]$ so that (13) is true on $[0, \|\tau\|_1]$. Moreover, under this last assumption, to the local diffeomorphism $[\phi_\tau^\xi]^{-1}$, consider the associated l^1 curve $\hat{\gamma}$. Then the curve $\tilde{c}(s) = \tilde{\gamma}(T - s)$ is an l^1-curve which satisfies (13) relatively to $\hat{\gamma}$.

Proof.

• *Proof of part* 1.

Fix some $Z \in \mathcal{X}(M)$ as in lemma. As f (resp. $T_{\tilde{y}} f$ for any $\tilde{y} \in N$) is injective, there exists a field $\tilde{Z} : \tilde{y} \to \tilde{Z}(\tilde{y}) \in T_{\tilde{y}} N$ such that

$$T_{\tilde{y}} f\big[\tilde{Z}(\tilde{y})\big] = Z\big(f(\tilde{y})\big) \quad \text{for any } \tilde{y} \in \tilde{V}_Z = f^{-1}\big[\mathrm{Dom}(Z) \cap f(N)\big]. \tag{14}$$

It remains to show that the vector field $\tilde{Z}$ is smooth on $\tilde{V}_Z$.

In fact, it is sufficient to prove this property on some neighborhood $\tilde{V}$ of any point $\tilde{x} \in \tilde{V}_Z$.

Note at first that from our assumption about the lower trivialization, we have $\tilde{\triangle}_x = T_{\tilde{x}} N \equiv F$. So F is independent of $x \in f(N)$. For any $x \in f(N)$ and an associated lower trivialization $\Theta : \mathbb{R}^A \times V \to TM$ we will always choose V such that $TM_{|V} \equiv E \times V$. Of course, $f^{-1}(V)$ is an open neighborhood of $\tilde{x}$ in N. We also always choose an open neighborhood $\tilde{V}$ of $\tilde{x}$ in $f^{-1}(V)$ such that $TN_{|\tilde{V}} \equiv F \times \tilde{V}$.

We assert that the vector field $\tilde{Z}$ is smooth on $\tilde{V}$.

Indeed, from convenient analysis (see [10]), recall that for a map g from an open set U in a Banach space E_1 to a Banach space E_2 we have the equivalent following properties:

(i) g is smooth;
(ii) for any smooth curve $c : \mathbb{R} \to U$ the map $t \mapsto g \circ c(t)$ is smooth;
(iii) the map $t \mapsto \langle \alpha, g \circ c(t) \rangle$ is smooth for any $\alpha \in E_2^*$.

Fix some $\tilde{y} \in \tilde{V}_Z$. As we have already seen, we can choose a neighborhood $\tilde{V}$ of $\tilde{y} \in \tilde{V}_Z$ such that $TN_{|\tilde{V}} \equiv F \times \tilde{V}$. So, without loss of generality, we can suppose that $\tilde{V}$ is an open set in F and V an open set in E and $f \equiv T_{\tilde{y}} f$ on $\tilde{V}$. For simplicity, let be $\theta = T_{\tilde{y}} f : T_{\tilde{y}} N \equiv F \to T_y M \equiv E$ where $y = f(\tilde{y})$ with our conventions. In these conditions, $\tilde{Z}$ is a map from $\tilde{V}$ to F and Z is a smooth map from V to E. Note that, according to (14), we have

$$\theta \circ \tilde{Z}(\tilde{y}) = Z \circ \theta(\tilde{y})$$

for any $\tilde{y} \in \tilde{V}$. Choose any $\omega \in E^*$. For any smooth curve $c : \mathbb{R} \to \tilde{V}$, we then have

$$\langle \omega, Z \circ \theta \circ c \rangle = \langle \omega, \theta \circ \tilde{Z} \circ c \rangle = \big\langle \theta^t(\omega), \tilde{Z} \circ c \big\rangle.$$

As the adjoint θ^t of θ is surjective, according to the previous argument of convenient analysis we conclude that $\tilde{Z}$ is smooth on $\tilde{V}$.

Now, if $x = f(\tilde{x})$, from the relation $f_* \tilde{Z} = Z \circ f$ we get

$$\phi^Z(t, x) = f \circ \phi^{\tilde{Z}}(t, \tilde{x})$$

for any t for which $\phi^{\tilde{Z}}(t,\tilde{x})$ is defined. In particular, this relation exists for some interval $]-\varepsilon,\varepsilon[$.

Given the maximal interval $]a_x,b_x[$ as in the lemma, choose any $\tau\in[0,b_x[$. For each $t\in[0,\tau]$ we have an integral manifold $f_t:L_t\to M$ which is an integral manifold of $\triangle$ through $\phi^Z(t,x)$. As $\phi^{\tilde{Z}}(t+s,x)=\phi^{\tilde{Z}}(t,\phi^{\tilde{Z}}(s,x))$, by the previous argument, there exists some subinterval on which the curve $s\to\phi^{\tilde{Z}}(s,x)$ belongs to L_t. If we set $L_\tau=\bigcup_{t\in[0,\tau]}L_t$, by connexity argument, using Lemma 2.14 [13], it follows that $i_{L_\tau}:L_\tau\to M$ is an integral manifold of $\triangle$ through x. But by construction L_τ is an open submanifold of N. It follows that (12) is true on $[0,b_x[$; the same arguments works for any $\tau\in]a_x,0]$. This ends the proof of part 1.

• *Proof of part* 2.

Now, let be some $\xi=\{X_\beta,\ \beta\in B\}\subset\hat{\mathcal{X}}\subset\mathcal{Y}$ satisfying the required conditions. According to Theorem 2, we have a map Ψ^x from some neighborhood U of $0\in\mathbb{R}^B$ into V of class C^s. From part 1, on N, we have a family of smooth vector fields $\tilde{\xi}=\{\tilde{X}_\beta,\ \beta\in B\}$ such that $\mathrm{Dom}(\tilde{X}_\beta)=\tilde{V}=f^{-1}(V)$ for any $\beta\in B$. Fix some $\tau\in U$. According to Remark 2.7, and (14), by induction, we can construct a curve $\tilde{\gamma}_\tau:[0,\|\tau\|_1[$ such that

$$f\circ\tilde{\gamma}_\tau=\Phi^{\xi}_{\tau}(t,x)\quad\text{for any }t\in\big[0,\|\tau\|_1\big[.\tag{15}$$

Suppose that $\triangle$ is closed. So $\triangle_z$ is closed in T_zM for any $z\in N$ and it follows that the topology of N as weak manifold is nothing but the induced topology of M on N. The endpoint $y=\gamma(\|\tau\|_1)$ belongs to V. So $\hat{\gamma}:[0,\|\tau\|_1]\to M$ defined by $\hat{\gamma}(s)=\gamma(\|\tau\|_1-s)$ is an integral curve of the vector field

$$Z=\sum_{\beta\in B}u_\beta X_\beta$$

where $(u_\beta)=\hat{\Gamma}^\tau$ is associated to $\hat{\phi}^{\xi}_{\tau}$.

On the other hand, we have an integral manifold $i_L:L\to M$ through y. We choose a neighborhood $U\subset V$ of y such that we have $TM_{|U}\equiv U\times T_yM$. From our assumption, again, the topology of L as weak manifold is nothing but the induced topology of M on L.

From part 1, $\tilde{U}=U\cap L=(i_L)^{-1}(U\cap L)$ is an open neighborhood of $\tilde{y}=(i_L)^{-1}(y)$ in L and we have a family $\xi=\{\tilde{Y}_\beta,\ \beta\in B\}$ such that $(i_L)_*(\tilde{Y}_\beta)=[X_\beta]_{|\tilde{U}}$.

From our notations we have $TM_{|U}\equiv U\times T_yM$ and T_yL is a Banach subspace of T_yM. So for each $z\in\tilde{U}$ we have an induced norm on the finite order jets of vector fields induced from $\|.\|_\phi$ on the finite jets of vector fields on $\tilde{U}$. As $\{X_\beta,\ \beta\in B\}$ satisfies the conditions (LB$(s+2)$) on V, and $U\subset V$, the family $\{\tilde{Y}_\beta;\ \beta\in B\}$ will also satisfies the condition (LB$(s+2)$) on $\tilde{U}$. So by application of Theorem 1 on $\tilde{U}$ to Z and the unicity of the integral curve through y we have obtained that $\gamma(\|\tau\|_1-s)=\hat{\gamma}(s)$ belongs to L for $0\leqslant s<\varepsilon$ with $\varepsilon>0$ small enough. We then have $N\cap L\neq\emptyset$. It follows that $\tilde{U}$ is an open set of N and in particular y belongs to N and we can extend $\tilde{\gamma}$ to $[0,\|\tau\|_1]$.

For the last part, the l^1 curve associated to $[\phi^{\xi}_{\tau}]^{-1}$ is $\hat{\gamma}(s)=\gamma(\|\tau\|_1-s)$ on $[0,\|\tau\|_1]$ and we trivially obtain the result from the previous proof. □

4.2. *Structure of weak submanifold on $\mathcal{X}$-orbits under local regularity conditions*

Now we suppose that $\mathcal{X}$ is a set of vector fields on M which satisfies the assumptions (H) = (Hi,Hii,Hiii) that is to say previous conditions (Hi) and (Hii) and also the assumption of Proposition 3.6, part 3:

(Hiii) there exists a finite, countable or eventually uncountable set A of indexes such that $\hat{\mathcal{D}}_x$ is isomorphic to $\mathbb{R}^A$ and a family $\{X_\alpha,\ \alpha \in A\} \subset \hat{\mathcal{X}}_x$ such that $\{X_\alpha,\ \alpha \in A\}$ satisfies the condition (LB($s+2$)), for some $s > 0$, and $\{X_\alpha(x),\ \alpha \in A\}$ is a symmetric unconditional basis of $\hat{\mathcal{D}}_x \equiv \mathbb{R}^A$.

Proposition 4.2.

1. *For all x in M, the $\hat{\mathcal{X}}$-orbit and the $\mathcal{X}$-orbit passing through x are equal.*
2. *The distribution $\hat{\mathcal{D}}$ is lower trivial on M.*
3. *The distribution $\hat{\mathcal{D}}$ is integrable. Each maximal integral manifold of $\hat{\mathcal{D}}$ has a natural smooth structure of weak connected Banach submanifold, modeled on some $\mathbb{R}^A$ where A is a finite, countable or eventually uncountable set of indexes. Moreover, any maximal integral manifold of $\hat{\mathcal{D}}$ is contained in an $\mathcal{X}$-orbit.*

Theorem 3. *If $\mathcal{X}$ satisfies the assumptions* (H) *at each point of M, then $\hat{\mathcal{D}}$ is integrable. Moreover, we have the following properties*:

(i) *Each $\mathcal{X}$-orbit $\mathcal{O}$ is the union of the maximal integral manifolds which meet $\mathcal{O}$ and such an integral manifold is dense in $\mathcal{O}$.*
(ii) *Let $\bar{\mathcal{D}}$ be the closed distribution generated by $\hat{\mathcal{X}}$. If $\bar{\mathcal{D}}$ is lower trivial and integrable, then, the $\mathcal{X}$-orbit of x is a dense subset in the maximal integral manifold through x.*
(iii) *If $\hat{\mathcal{D}}$ is a closed distribution then each $\mathcal{X}$-orbit is a maximal integral manifold of $\hat{\mathcal{D}}$ modeled on some $\mathbb{R}^A$.*

Remark 4.3. At any point $x \in M$ where $\hat{\mathcal{D}}$ is a finite dimensional vector space, $\hat{\mathcal{D}}_x$ is isomorphic to some $\mathbb{R}^n$ and we can always choose a finite set $\{X_1, \ldots, X_n\} \subset \hat{\mathcal{X}}$ such that $\{X_1(x), \ldots, X_n(x)\}$ is a basis of $\hat{\mathcal{D}}_x$. Moreover for finite set $\{X_1, \ldots, X_n\}$ we can always find an open neighborhood of x so that the condition (LB($s+2$)) is satisfied on V by this set. So, in this case, the assumption (H) is satisfied at x. So, if $\hat{\mathcal{D}}$ is finite dimensional, from Theorem 3 any $\mathcal{X}$-orbit is a finite dimensional submanifold of M.

Proof of Theorem 3. The integrability of $\hat{\mathcal{D}}$ is a direct consequence of part 3 of Proposition 4.2.

Moreover, again from part 3 of Proposition 4.2 we know that each maximal integral manifold N is contained in an $\mathcal{X}$-orbit $\mathcal{O}$. It remains to show that such an integral manifold is dense in $\mathcal{O}$. As the binary relation associated to the $\mathcal{X}$-orbit is symmetric, $\mathcal{O}$ is the $\mathcal{X}$-orbit of any point of $\mathcal{O}$. So, if L contains x, then, from Proposition 3.4 part 2 and Proposition 4.1, any $y \in \mathcal{O}$ must belong to the closure of L (in M).

Now, assume that the closed distribution $\bar{\mathcal{D}}$ generated by $\hat{\mathcal{X}}$ is lower trivial and integrable. Denote by $\mathcal{O}$ the $\mathcal{X}$-orbit of x. Choose some $y \in \mathcal{O}$ and let $\Psi \in \mathcal{G}_{\mathcal{X}}$ be such that $\Psi(x) = y$. According to Comments 3.3, we can associate to Ψ a finite sequence of points $(x_k)_{k=0,\ldots,n}$ and a finite family $\{\gamma_k\}_{k=1,\ldots,n}$ of l^1-curves associated to some $\phi^{\xi_k}_{\underline{\tau_k}}$ which joins x_{k-1} to x_k and with $x_0 = x$ and $x_n = y$. Let N be the maximal integral manifold of $\bar{\mathcal{D}}$ through x. As $x_0 = x$, according to Proposition 4.1, there exists an l^1 curve $\tilde{\gamma}_1$ in N such that $i_N \circ \tilde{\gamma}_1 = \gamma_1$ so x_1 belongs to N. By induction we can construct an l^1-curve $\tilde{\gamma}_k$ in N such that $i_N \circ \tilde{\gamma}_k = \gamma_k$ and then x_k belongs to N. So, for $k = n$ we obtain that $x_n = y$ belongs to N. In particular, by part 1, each maximal integral manifold L of $\hat{\mathcal{D}}$ which meets $\mathcal{O}$ is contained in N. So, as $\bar{\mathcal{D}}$ is closed, the topology of Banach manifold on N is the induced topology as subset of M. So, any maximal integral manifold of $\hat{\mathcal{D}}$ contained in $\mathcal{O}$ is dense in $\mathcal{O}$, as subset of N.

Denote by $\bar{\mathcal{O}}$ the closure of $\mathcal{O}$ in N. So $\bar{\mathcal{O}}$ is a connected closed subset of N. Consider any $y \in \bar{\mathcal{O}}$. Let L be the maximal integral manifold of $\hat{\mathcal{D}}$ through y. As L is arc-connected and the inclusion of L (with the topology of Banach manifold) in N is continuous, it follows that L is contained in N. Let $\mathcal{O}'$ be the $\mathcal{X}$-orbit of y. From previous arguments, $\mathcal{O}'$ is also contained in N. Let (y_k) be a sequence in $\mathcal{O}$ which converges to y. Given any $z \in \mathcal{O}'$ let be $\Phi \in \mathcal{G}_{\mathcal{X}}$ such that $\Phi(y) = z$. Now as each y_k belongs to $\mathcal{O}$ and $\mathcal{O}$ is invariant by any local diffeomorphism of $\mathcal{G}_{\mathcal{X}}$, it follows that $z_k = \Phi(y_k)$, for k large enough. So $z = \lim_{k\to\infty} z_k$, and then z belongs to $\bar{\mathcal{O}}$. Finally we get $\mathcal{O}' \subset \bar{\mathcal{O}}$ and, in particular, the maximal integral manifold L' of $\hat{\mathcal{D}}$ through y is contained in $\bar{\mathcal{O}}$.

On the other hand, as T_yN is the closure in T_yM of the normed subspace T_yL', there exists a neighborhood U of y in L' (for the two topologies on L) such that, the closure of U in N is a closed set of N with non-empty interior. But $U \subset L' \subset \bar{\mathcal{O}}$, so it follows that $\bar{\mathcal{O}}$ is open. By connexity argument, we get $\bar{\mathcal{O}} = N$.

Now if $\hat{\mathcal{D}}$ is a closed distribution, obviously we have $\bar{\mathcal{D}} = \hat{\mathcal{D}}$ so, the assumptions of property are satisfied. So part (iii) is a direct consequence of properties (i) and (ii). □

Proof of Proposition 4.2.

•*Proof of part* 1.

This result comes from $\mathcal{G}_{\mathcal{X}} = \mathcal{G}_{\hat{\mathcal{X}}}$ (Proposition 3.4 part 3).

•*Proof of part* 2.

This result is a consequence of Lemma 3.9 part 1.

•*Proof of part* 3.

From Proposition 3.6 part 3, for any $x \in M$ we have a C^s integral manifold through x, with $s \geqslant 1$. As $\hat{\mathcal{D}}$ is a lower trivial weak distribution, consider the set

$$\mathcal{X}_{\mathcal{D}}^- = \big\{X(u) = \Psi_x(u, y) \text{ for any lower trivialization} \\ \Psi_x : \hat{\mathcal{D}}_x \times V \to TM \text{ and any } x \in M\big\}.$$

As through x, we have an integral manifold of class C^s, $s \geqslant 1$, from the proof of Proposition 2.8 in [13], it follows that $\mathcal{D}$ is $\mathcal{X}_{\mathcal{D}}^-$-invariant. So from Theorem 1 of [13] we have a smooth integral manifold through x. Moreover, if we consider the following equivalence relation on M:

$$x\mathcal{R}y \text{ iff there exists an integral manifold of } \hat{\mathcal{D}} \text{ passing through } x \text{ and } y$$

then each equivalence class L has a natural structure of weak Banach submanifold modeled on $\hat{\mathcal{D}}_x$ for any $x \in L$ and L is an integral manifold of $\hat{\mathcal{D}}$. Take such an equivalence class L and denote by i_L the natural inclusion of i_L of L (endowed with its Banach structure) into M. From Proposition 3.6 part 3, for any $x \in L$, there exist an open ball $B(0, \rho_x) \subset \mathbb{R}^A \equiv \hat{\mathcal{D}}_x$ and a C^s map $\Theta_x : B(0, \rho_x) \to M$ which is a C^s integral manifold of $\hat{\mathcal{D}}$ through x and such that $\Theta_x(B(0, \rho_x))$ is an open set of L. So, $P_x = \Theta_x(B(0, \rho_x))$ has an induced structure of smooth Banach manifold modeled on $\hat{\mathcal{D}}_x$ (isomorphic to $\mathbb{R}^A$ for some appropriate set of indexes A). In particular, the natural inclusion $i_x : P_x \to M$ is a smooth integral manifold of $\hat{\mathcal{D}}$ through x. Now take some $x \in L$. For any $y \in L$ we have a continuous curve $\gamma : [a, b] \subset \mathbb{R} \to L$ such that $\gamma(a) = x$ and $\gamma(b) = y$. By compactness of $\gamma([a, b])$ we have a finite covering of $\gamma([a, b])$ by a family of open sets $\{\Theta_{x_i}(B(0, \rho_{x_i}))\}_{i=1,\dots,n}$ such that $x_i \in \gamma([a, b])$, $x_1 = x$ and $x_n = y$. Now choose any $y_i \in \Theta_{x_i}(B(0, \rho_{x_i})) \cap \Theta_{x_{i+1}}(B(0, \rho_{x_{i+1}})) \cap \gamma([a, b])$ for $i = 1, \dots, n - 1$. From the construction of each Θ_x, there exists $\Phi_i \in \mathcal{G}_\mathcal{X}$ (resp. $\Phi'_i \in \mathcal{G}_\mathcal{X}$) such that $\Phi_i(x_i) = y_i$ (resp. $\Phi'_i(x_{i+1}) = y_i$). So the composition:

$$\Phi = \Phi_1 \circ \left[\Phi'_1\right]^{-1} \circ \cdots \circ \Phi_{n-1} \circ \left[\Phi'_{n-1}\right]^{-1} \tag{16}$$

is an element of $\mathcal{G}_\mathcal{X}$ such that $\Phi(x) = y$. It follows that L is contained in the $\mathcal{X}$-orbit of x. □

4.3. Structure of weak submanifold on $\mathcal{X}$-orbits under involutivity conditions

A weak distribution $\triangle$ is called **(locally) upper trivial** (upper trivial for short) if, for each $x \in M$, there exist an open neighborhood V of x, a Banach space F and a smooth map $\Phi : F \times V \to TM$ (called **upper trivialization**) such that:

(i) for each $y \in V$, $\Phi_y \equiv \Phi(\,,y) : F \to T_yM$ is a continuous operator with $\Phi_y(F) = \triangle_y$;
(ii) $\ker \Phi_x$ complemented in F;
(iii) if $F = \ker \Phi_x \oplus S$, the restriction θ_y of Φ_y to S is injective for any $y \in V$;
(iv) $\Theta(u, y) = (\theta_y \circ [\theta_x]^{-1}(u), y)$ is a lower trivialization of $\mathcal{D}$.

In this case the map Θ is called the **associated lower trivialization**.

In this case, each lower section $X_v = \Theta(v,\,)$ with $v \in \triangle_x$ can be written as $X_v = \Theta(\Phi(v', x),\,)$ for any $v' \in F$ such that $\Phi(v', x) = v \in \triangle_x$.

An upper trivial weak distribution $\triangle$ is called **Lie bracket invariant** if, for any $x \in M$, there exists an upper trivialization $\Phi : F \times V \to TM$ such that, for any $u \in F$, there exists $\varepsilon > 0$, and, for all $0 < \tau < \varepsilon$ there exists a smooth field of operators $C : [-\tau, \tau] \to L(F, F)$ with the following property

$$[X_u, Z_v]\big(\gamma(t)\big) = \Phi\big(C(t)[v], \gamma(t)\big) \quad \text{for any } Z_v = \Phi(v,\,) \text{ and any } v \in F \tag{17}$$

along the integral curve $t \mapsto \phi_t^{X_u}(x)$ on $[-\tau, \tau]$ of the lower section $X_u = \Theta(\Phi(u, x),\,)$.

With these definitions we have:

Theorem 4.4. *Let $\triangle$ be an upper trivial weak distribution. Then $\triangle$ is integrable if and only if $\triangle$ is Lie bracket invariant.*

We now come back to our original context. Consider any set $\mathcal{Y}$ of local vector fields which contains $\mathcal{X}$ and which satisfies properties (Hi) and (Hii). We have seen that if Λ is any ordered set of indexes of same cardinal as the set

$$\mathcal{Y}_x = \big\{Y \in \mathcal{Y} \text{ such that } x \in \mathrm{Dom}(Y)\big\}$$

then we have a surjective linear map: $T : l^1(\Lambda) \to l^1(\mathcal{Y})_x$.

Let $\triangle$ be the weak distribution $l^1(\mathcal{Y})$ and index the set $\mathcal{Y}_x$ as set $\{Y_\lambda,\ \Lambda \in \Lambda\}$. Assume that $\triangle$ has the following properties labelled (H′):

(H′1) for any $x \in M$ there exists an upper trivialization $\Phi : l^1(\Lambda) \times V \to TM$ such that $\Phi(e_\lambda) = Y_\lambda$ for each $\Lambda \in \Lambda$ where $\{e_\lambda\}_{\lambda \in \Lambda}$ is the canonical basis of $l^1(\Lambda)$;
(H′2) for any $x \in M$ there exists a neighborhood V of x such that $V \subset \bigcap_{\lambda \in \Lambda} \mathrm{Dom}(Y_\lambda)$, and a constant $C > 0$ such that we have

$$[Y_\lambda, Y_\mu](y) = \sum_{\nu\in\Lambda} C^\nu_{\lambda\mu}(y) Y_\nu(y) \quad \text{for any } \lambda, \mu \in \Lambda \tag{18}$$

where each $C^\nu_{\lambda\mu}$ is a smooth function on V, for any $\lambda, \mu, \nu \in \Lambda$ and we have

$$\sum_{\nu\in A} \left|C^\nu_{\lambda\mu}(y)\right| \leqslant C$$

for any $y \in V$.

Theorem 4.

1. *Under the previous assumptions* (H′)*, the distribution* $\triangle$ *is integrable.*
2. *If* $\triangle$ *is an integrable distribution which satisfies assumption* (H′1)*, then* $\hat{\mathcal{D}}_x$ *is contained in* $\triangle_x$ *for any* $x \in M$*. Moreover if* $\triangle$ *is closed then each* $\mathcal{X}$*-orbit is contained in a maximal integral manifold of* $\triangle$*.*

To the set $\mathcal{X}$ we can associate the sequence of families

$$\begin{aligned}\mathcal{X} &= \mathcal{X}^1 \subset \mathcal{X}^2 = \mathcal{X} \cup \left\{[X, Y],\ X, Y \in \mathcal{X}\right\} \subset \cdots \subset \mathcal{X}^k \\ &= \mathcal{X}^{k-1} \cup \left\{[X, Y],\ X \in \mathcal{X},\ Y \in \mathcal{X}^{k-1}\right\} \subset \cdots.\end{aligned}$$

The set $\mathcal{X}^k$ always satisfies the condition (Hi). Moreover, if it satisfies condition (Hii), the distribution $\mathcal{D}^k = l^1(\mathcal{X}^k)$ is well defined.

By application of the previous result to $\triangle = \hat{\mathcal{D}}$ or $\triangle = \mathcal{D}^k$ we get:

Theorem 5.

1. *If the distribution* $\hat{\mathcal{D}}$ *satisfies the assumptions* (H′)*, then* $\hat{\mathcal{D}}$ *is integrable and we have the following properties*:
   - (i) *Each* $\mathcal{X}$*-orbit* $\mathcal{O}$ *is the union of the maximal integral manifolds which meet* $\mathcal{O}$ *and such an integral manifold is dense in* $\mathcal{O}$*.*
   - (ii) *Assume that the closed distribution* $\bar{\mathcal{D}}$ *generated by* $\hat{\mathcal{X}}$ *is lower trivial and integrable. Then the* $\mathcal{X}$*-orbit of* x *is a dense subset in the maximal integral manifold through* x*.*
   - (iii) *If* $\hat{\mathcal{D}}$ *is a closed distribution then each* $\mathcal{X}$*-orbit is a maximal integral manifold of* $\hat{\mathcal{D}}$ *modeled on some* $\mathbb{R}^A$*.*
2. *If* $\mathcal{X}$ *satisfies* (LB s)*, and if* $\mathcal{D}^k$ *satisfies assumptions* (H′) *for some* $k \leqslant s$*, then we have* $\mathcal{D}^k = \hat{\mathcal{D}}$ *and* $\mathcal{D}^k$ *is integrable. Moreover,* $\mathcal{D}^k$ *satisfies all the previous properties* (i), (ii) *and* (iii).

Example 4.5. As in Example 2.5(3), consider a finite family $\mathcal{X} = \{X_1, \ldots, X_n\}$ of global vector fields on M. We have seen that the condition (LB s) is satisfied for any $s > 0$. Then each set $\mathcal{X}^k$ is finite and then, it is clear that each distribution $\mathcal{D}^k$ is upper trivial:

If n_k is the cardinal of $\mathcal{X}^k$, we can order $\mathcal{X}^k$ in a sequence $\{Z_1, \ldots, Z_{n_k}\}$ and on each open set V according to the identification $TM \equiv V \times T_x M$ we can consider the upper trivialization $\Phi : V \times \mathbb{R}^{n_k} \to TM$ defined by

$$\Phi\big(y, (t_1, \ldots, t_{n_k})\big) = \sum_{i=1}^{n_k} t_i Z_i(y); \ \text{in fact it is an upper trivialization.}$$

Suppose that the condition (H′2) for $\mathcal{X}^k$ is satisfied, then from Theorem 5, the closed distribution $\mathcal{D}^k$ is integrable, and each maximal integral is a Banach submanifold of M which is also an $\mathcal{X}$-orbit.

The reader can find such a context in [15] where M is the set (denoted "Conf") of "configurations" of the snake (which is a Banach manifold), $\mathcal{X}$ is the set of global vector fields $\{\xi_1, \dots, \xi_d\}$ on Conf (in notations [15]). Then $\mathcal{X}^1$ satisfies the condition (H′2). Each $\mathcal{X}$-orbit is nothing but an orbit of the action Möb on Conf (see [15]). From Theorem 5 we directly obtain that each orbit is a closed (finite dimensional) submanifold of Conf.

In [14], the reader can find a generalization of the results of [15] in the context of Hilbert space and get an application of the previous result for a countable set $\mathcal{X}$ of global vector fields on a Banach manifold.

Proof Theorem 4.

• *Proof of part* 1.

According to Theorem 4.4, it is sufficient to show that $\triangle$ is Lie bracket invariant. So fix some $x \in M$ and consider an upper trivialization $\Phi : l^1(\Lambda) \times V \to TM$ as in the previous assumption. As $\ker \Phi_x$ is complemented, we have $l^1(\Lambda) = \ker \Phi_x \oplus S$. So there exists a family $\{\epsilon_\alpha\}_{\alpha \in A}$ (resp. $\{\epsilon'_\beta\}_{\beta \in B}$) of $l^1(\Lambda)$ which is a normalized symmetric unconditional basis of S (resp. $\ker \Phi_x$) (see Remark 3.5). Now, the canonical unconditional basis $\{e_\lambda\}_{\lambda \in \Lambda}$ has a (unique) decomposition:

$$e_\lambda = \sum_{\alpha \in A} f_\lambda^\alpha \epsilon_\alpha + \sum_{\beta \in B} f_\lambda^{\prime \beta} \epsilon'_\beta \tag{19}$$

such that $\sum_{\alpha \in A} |f_\lambda^\alpha| \leqslant 1$ and $\sum_{\beta \in B} |f_\lambda^{\prime \beta}| \leqslant 1$ for any $\lambda \in \Lambda$.

Any lower section can be written as $X_u = \Phi(u, .)$ for some $u = (u_\lambda) \in l^1(\Lambda)$. Such a section can be written

$$X_u = \sum_{\lambda \in \Lambda} u_\lambda Y_\lambda.$$

On the other hand consider a neighborhood V' of x in which (H′2) is true and the neighborhood $V \cap V'$ (again denoted by V). As previously fix some lower section $X_u = \Phi(u, .)$ and consider $\varepsilon > 0$ such that the integral curve $\gamma(t) = \phi_t^{X_u}(x)$ is defined on $]-\varepsilon, \varepsilon[$ in V. According to (H′2), for any $0 < |\tau| < \varepsilon$, we define $C : [-\tau, \tau] \to L(l^1(\Lambda), l^1(\Lambda))$ in the following way:

$$C(t)[v] = \sum_{\lambda, \mu, \nu \in \Lambda} C_{\lambda\mu}^\nu\big(\gamma(t)\big) u_\lambda v_\mu e_\nu$$

where $v = (v_\mu) \in l^1(\Lambda)$. So from assumption (H′2), we have

$$\big\|C(t)[v]\big\| \leqslant C \sum_{\lambda, \mu \in \Lambda} |u_\lambda||v_\mu| \leqslant C\bigg[\sum_{\lambda \in \Lambda} |u_\lambda|\bigg]\bigg[\sum_{\mu \in \Lambda} |v_\mu|\bigg] = C\|u\|_1\|v\|_1$$

then $C(t)$ is a field of continuous endomorphisms of $l^1(\Lambda)$.

On the other hand, for any $v = (v_\mu) \in l^1(\Lambda)$, we have

$$Z_v = \Phi(v, .) = \sum_{\mu \in \Lambda} v_\mu Y_\mu.$$

So we get

$$[X_u, Z_v]\big(\gamma(t)\big) = \Phi\big(C(t)[v], \gamma(t)\big).$$

From Theorem 4.4 it follows that $\triangle$ is integrable.

• *Proof of part* 2.

Now suppose that $\triangle$ is integrable. Fix some $x \in M$ and let $f \equiv i_N : N \to M$ be a maximal integral manifold through x. We want to show that $\hat{\mathcal{D}}_x$ is contained in $\triangle_x$. It is sufficient to prove that for any $Y \in \hat{\mathcal{X}}_x$, $Y(x)$ belongs to $\triangle_x$. For such a vector field there exist vector fields $X_1, \dots, X_p, X \in \mathcal{X}$ and $\nu > 0$ such that

$$Y = \left(\phi_{t_p}^{X_p} \circ \cdots \circ \phi_{t_1}^{X_1}\right)_*(\nu X).$$

Let $z = (\phi_{t_p}^{X_p} \circ \cdots \circ \phi_{t_1}^{X_1})^{-1}(x)$ be. Consider the integral curve γ_1 of X_1 through z: $\gamma_1(t) = \phi_t^{X_1}(z)$ for $t \in [0, t_1]$. As $\mathcal{X} \subset \mathcal{Y}$, from Proposition 4.1, we have a curve $\tilde{\gamma}_1 : [0, t_1] \to N$ such that $f \circ \tilde{\gamma}_1 = \gamma_1$, and for any $s \in [0, t_1]$, a neighborhood $\tilde{V}_s$ of $\tilde{\gamma}_1(s)$ in N, and a vector field $\tilde{Y}_s$ on $\tilde{V}_s$ such that $f_* \tilde{Y}_s = X_1$. In particular we also have

$$f \circ \phi_r^{\tilde{Y}_s}\left(\tilde{\gamma}_1(s)\right) = \phi_r^{X_1}\left(\gamma_1(s)\right) \quad \text{for any } r \text{ small enough.} \tag{20}$$

Moreover, we can find $\tilde{X}$ on the neighborhood $\tilde{V}_0$ of $\tilde{z}$ such that $f_*(\tilde{X}) = \nu X$, after having restricted $\tilde{V}_0$ if necessary. By compactness, we can cover $\tilde{\gamma}_1([0, t_1])$ by a finite number $\tilde{V}_{s_0}, \dots, \tilde{V}_{s_m}$. On $\tilde{V}_{s_0}$ we have $T_{\tilde{z}}[\phi_t^{\tilde{Y}_{s_0}}](\tilde{X}((\tilde{z})))$ which belongs to $T_{\tilde{\gamma}_1(t)} N = \tilde{D}_{\tilde{\gamma}_1(t)}$ for any t so that $\tilde{\gamma}_1(t)$ belongs to $\tilde{V}_{s_0}$. From properties of $\tilde{Y}_{s_0}$ and $\tilde{\gamma}_1$, it follows that

$$\left[\left(\phi_t^{X_1}\right)_*(\nu X)\right]\left(\gamma(t)\right) \quad \text{belongs to } \triangle_{\gamma(t)}. \tag{21}$$

Choose σ_1 such that $\tilde{\gamma}_1(\sigma_1)$ belongs to $\tilde{V}_{s_1}$. So we have (21) for $t = \sigma_1$. By applying the same argument to $T_{\tilde{z}}[\phi_{\sigma_1}^{\tilde{Y}_{s_0}}](\tilde{X}((\tilde{z})))$ by choosing σ_2 such that $\tilde{\gamma}_1(\sigma_2)$ belongs to $\tilde{V}_{s_1} \cap \tilde{V}_{s_2}$, we obtain (21) for $t = \sigma_2$. Finally, by induction we get (21) for $t = t_1$. Then by same argument applied to $[(\phi_t^{X_1})_*(X)](\gamma(t_1))$ instead of $(\nu X)(x)$ and along the curve $\gamma_2(t_1 + t) = \phi_t^{X_2}(\gamma_1(t_1))$ we obtain that

$$\left(\phi_{t_2}^{X_2} \circ \phi_{t_1}^{X_1}\right)_*\left(\nu X(x)\right) \quad \text{belongs to } \triangle_{\gamma(t_2)}.$$

Again by induction, on $i = 2, \dots, p$, we finally obtain that $Y(x) = (\phi_{t_p}^{X_p} \circ \cdots \circ \phi_{t_1}^{X_1})_*(\nu X(z))$ belongs to $\triangle_x$.

Now we assume that $\triangle$ is a closed integrable manifold. Take $x \in M$ and again let be $f = i_N : N \to M$ the maximal integral manifold trough x. We want to show that for any $\Psi \in \mathcal{G}_{\mathcal{X}}$, the point $y = \Psi(x)$ belongs to $f(N)$. From the previous proof we also have obtained that if Ψ is a finite composition $(\phi_{t_p}^{X_p} \circ \cdots \circ \phi_{t_1}^{X_1})$, then y belongs to N. So, from (7), for $Y \in \hat{\mathcal{X}}$ and any $\tau \in \mathbb{R}$, $\phi_\tau^X(x)$ belongs to $f(N)$ (even when $\triangle$ is not closed).

Suppose that Ψ is reduced to some ϕ_τ^ξ, with $\xi = \{Y_\delta,\ \delta \in D\} \subset \hat{\mathcal{X}}$ and $\tau \in \mathbb{R}^D$.

Let $\gamma : [0, \|\tau\|_1] \to M$ be the curve $\gamma(t) = \Phi_\tau^\xi(t, x)$ where $\Phi_\tau^\xi(t, .)$ is the flow associated to ξ, τ and $u = \Gamma^\tau$ (see Remark 2.7.1) From Proposition 4.1, part 2 there exists an l^1-curve $\tilde{\gamma} : [0, \|\tau\|_1] \to N$ such that

$$f \circ \tilde{\gamma} = \gamma \quad \text{on } \left[0, \|\tau\|_1\right]. \tag{22}$$

As $\phi_\tau^\xi(x) = \Phi_\tau^\xi(\|\tau\|_1, x)$, we obtain that $y = \phi_\tau^\xi(x)$ belongs to $f(N)$.

For the case $\Psi = [\phi_\tau^\xi]^{-1}$, set again $y = [\phi_\tau^\xi]^{-1}(x)$ and let $\gamma : [0, T] \to M$ be the l^1-curve associated to ϕ_τ^ξ and we use the previous notations. Then the l^1 curve associated to Ψ is $\hat{\gamma}(s) =$

$\gamma(T-s)$ which satisfies $\hat{\gamma}(0)=x$ and $\gamma(\|\tau\|_1)$ and $\hat{\gamma}(\|\tau\|_1)=y=\gamma(0)$. From (22), we obtain that x belongs to the maximal integral manifold through y which, by maximality, is N.

In the general case we have $\Psi=\phi_n\circ\cdots\circ\phi_1$ where each ϕ_k is a local diffeomorphism of type $\phi^{Y_k}_{\tau_k}$ or $\phi^{\xi_k}_{\tau_k}$ or $[\phi^{\xi_k}_{\tau_k}]^{-1}$ for $k=1,\dots,n$ and all these vector fields belong to $\hat{\mathcal{X}}$. So, by finite induction on k, using the previous partial results, we get the proof of part 2 in the general case. □

Proof Theorem 5.

• *Proof of part* 1.

By application of Theorem 4, part 1 to $\hat{\mathcal{D}}$, it follows that $\hat{\mathcal{D}}$ is integrable. We must show that each maximal integral manifold which meets an $\mathcal{X}$-orbit $\mathcal{O}$ is contained in $\mathcal{O}$.

Fix some maximal integral manifold $i_N: N\to M$ of $\hat{\mathcal{D}}$. Fix some $x\in N$ and consider an upper trivialization $\Phi:\mathbb{R}^{\Lambda}\times V\to TM$ as in assumption (H′1). From this assumption, after restricting V if necessary, the set $\xi=\hat{\mathcal{X}}_x$ satisfies the condition (LB s) at any point of V and for any $s\in\mathbb{N}$ (see Example 2.5(2)). On the other hand, according to Lemma 2.10 in [13], we have a neighborhood $\tilde{V}$ of x, for the Banach structure of N, so that we have a smooth field of continuous operators $y\to\tilde{\Phi}_y$ from $\mathbb{R}^{\Lambda}$ to T_yN such that $\Phi_y(.)=Ti_N\circ\tilde{\Phi}_y$ on $\tilde{V}$. From Proposition 4.1, for each $\lambda\in\Lambda$ we have a smooth vector field on $\tilde{Y}_\lambda$ such that

$$Y_\lambda=(i_N)_*\tilde{Y}_\lambda\quad\text{on }\tilde{V}.\tag{23}$$

Note that, according to the notation used in the proof of Theorem 4 part 1, in fact we have $Y_\lambda(y)=\tilde{\Phi}_y(e_\lambda)$. So, as previously, after restriction of $\tilde{V}$ if necessary, the set $\tilde{\xi}=\{\tilde{Y}_\lambda,\ \lambda\in\Lambda\}$ satisfies the condition (LB$(s+2)$) for any $s\in\mathbb{N}$. Applying Theorem 2 to $\tilde{\xi}$ we get a map $\tilde{\Psi}^x: B(0,r)\subset\mathbb{R}^{\Lambda}\to L$ of class C^s. By the same argument applied to $\xi=\{Y_\lambda,\ \lambda\in\Lambda\}$ on M, we get a map $\Phi^x: B(0,r')\to M$ which is of class C^s. Using (23) we have $\Psi^x=i_N\circ\tilde{\Phi}^x$ on some $B(0,\rho)$ with ρ small enough. The linear map $T_0\tilde{\Phi}^x$ is surjective and its kernel is $\ker\Phi_x$. So, for ρ small enough, $\tilde{\Psi}^x$ is a submersion and in particular, $\tilde{P}(x,\rho)=\tilde{\Psi}^x(B(0,\rho))$ is an open set in N (with it Banach structure). If we set $P(x,\rho)=\Psi^x(B(o,\rho))$ by definition of an $\mathcal{X}$-orbit, the set $P(x,\rho)$ is contained in $\mathcal{O}$. But, by construction we have $P(x,\rho)=i_N(\tilde{P}(x,\rho))$ and then we have an open neighborhood $\tilde{P}(x,\rho)$ of x (for the Banach structure of N) such that $i_N(\tilde{P}(x,\rho))\subset\mathcal{O}$. As we can cover N by such open subsets and $\mathcal{O}$ is the $\mathcal{X}$-orbit of any $y\in\mathcal{O}$, we get $N\subset\mathcal{O}$. For the density of N in $\mathcal{O}$, we use the same arguments as in the proof of Theorem 3. The properties (ii) and (iii) have same proofs as in Theorem 3.

• *Proof of part* 2.

From Theorem 4 applied to $\triangle=\mathcal{D}^k$ we obtain that $\mathcal{D}^k$ is integrable and, for any $x\in M$ each $\mathcal{D}^k_x$ contains $\hat{\mathcal{D}}_x$. According to part 1 of Theorem 5, it remains to show that $\hat{\mathcal{D}}_x$ contains $\mathcal{D}^k_x$ for any $x\in M$.

Given $x\in M$, we can suppose that the upper trivialization $\Phi:\tilde{\mathcal{D}}^k_x\times V\to TM$ on a neighborhood V of x is such that $TM_{|V}\equiv E\times V$. Take any $X\in\mathcal{X}$ and $Y\in\hat{\mathcal{X}}$ so that x belongs to the domain of X and of Y. For $0<t<\varepsilon$ small enough so that the flow ϕ^X_t is defined on some neighborhood $U\subset V$ of x, we consider the curve $t\to\frac{1}{t}\{([\phi^X_t]_*Y)_x-Y_x\}$ in E. As $\hat{\mathcal{D}}$ is $\mathcal{X}$-invariant, the previous curve belongs to $\hat{\mathcal{D}}_x$, as Banach space. But we have

$$[X,Y]_x=\lim_{t\to 0}\frac{1}{t}\bigl\{\bigl(\bigl[\phi^X_t\bigr]_*Y\bigr)_x-Y_x\bigr\}.$$

As $\mathcal{D}^k$ satisfies the assumption (H′), the structure of Banach space for $\mathcal{D}^k_x$ is isomorphic to some $\mathbb{R}^{\Lambda}$. So $\mathcal{D}^k_x$ has the Schur property. By using an argument of weak convergency and Schur's property, $[X, Y]_x$ belongs to $\hat{\mathcal{D}}_x$. Now by induction, applying this result for $Y \in \mathcal{X}^{k-1}$, we obtain the inclusion $\mathcal{D}^k_x \subset \hat{\mathcal{D}}_x$ for any $x \in M$. □

5. Applications

5.1. Criteria of integrability for l^1-distribution

In this subsection we will give a criterion of integrability for l^1-distributions generated by sets $\mathcal{X}$ of vector fields on M which satisfies the assumption (H). We have the following result:

Theorem 6.

1. *Let $\mathcal{D}$ be an l^1-distribution generated by a set of* (*local*) *vector fields $\mathcal{X}$ on a Banach manifold M which satisfies the assumptions* (H). *Then $\mathcal{D}$ is lower trivial. Moreover, $\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is $\mathcal{X}$-invariant.*
2. *Let $\mathcal{D}$ be a lower trivial l^1-distribution on a Banach manifold M. Then there exists generating set $\mathcal{X}$ of $\mathcal{D}$ which satisfies assumptions* (Hi), (Hii) *and* (Hiii). *Given any such generating set $\mathcal{X}$ of $\mathcal{D}$, then $\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is $\mathcal{X}$-invariant.*

Remark 5.1. As any l^1-distribution $\mathcal{D}$ is a weak distribution, from Theorem 1 of [13], when $\mathcal{D}$ is lower trivial, it is integrable if and only if it is $\mathcal{X}^-_{\mathcal{D}}$-invariant ($\mathcal{X}^-_{\mathcal{D}}$ in the set of lower sections of $\mathcal{D}$ see [13]). So, for lower trivial l^1-distribution, Theorem 4 gives a necessary and sufficient condition of integrability for **any** generating set of $\mathcal{D}$ satisfying (Hi), (Hii) and (Hiii). Note that, if $\mathcal{D}$ is finitely generated at each point, these conditions are automatically satisfied. We then get a generalization of the famous criterion of integrability of Nagano–Sussmann in this context of Banach manifold for finite dimensional distribution. In this sense, Theorem 4 can be considered as a generalization of this Nagano–Sussmann's result in infinite dimension.

In Example 2.5(1), if the set $\{T(x_\alpha)\}_{\alpha \in A}$ is a family of linearly independent vectors, the conditions of Theorem 4 are satisfied. Of course, this result can be proved directly in an obvious way. Each leaf is the affine space in E associated to the l^1 normed space generated by $\mathcal{X}_0$. On the other hand in Example 2.5(2), even in analogue conditions, the characteristic distribution of $\mathcal{X}$ is not $\mathcal{X}$-invariant. Such a sufficient conditions will be carried by Ψ (see Theorem 4 in [13]).

Proof of Theorem 6.

Part 1.

From Proposition 3.6 we have $\mathcal{D} = \hat{\mathcal{D}}$ if and only if $\mathcal{D}$ is $\mathcal{X}$-invariant. On the other hand, $\mathcal{X}$ satisfies the assumption (H) (of Section 4). By application of Theorem 3, we obtain the first part.

Part 2.

Fix some $x \in M$. From the property of lower triviality, there exists an open neighborhood V of x in M, a smooth map $\Psi : \tilde{\mathcal{D}}_x \times V \to TM$ such that:

(i) $\Psi(\tilde{\mathcal{D}}_x \times \{y\}) \subset \mathcal{D}_y$ for each $y \in V$;

(ii) for each $y \in V$, $\Psi_y \equiv \Psi(, y) : \tilde{\mathcal{D}}_x \to T_yM$ is a continuous operator and $\Psi_x : \tilde{\mathcal{D}}_x \to T_xM$ is the natural inclusion i_x;

(iii) there exists a continuous operator $\tilde{\Psi}_y : \tilde{\mathcal{D}}_x \to \tilde{\mathcal{D}}_y$ such that $i_y \circ \tilde{\Psi}_y = \Psi_y$, $\tilde{\Psi}_y$ is an isomorphism from $\tilde{\mathcal{D}}_x$ onto $\Psi_y(\tilde{\mathcal{D}}_x)$ and $\tilde{\Psi}_x$ is the identity of $\tilde{\mathcal{D}}_x$.

As $\tilde{\mathcal{D}}_x$ is isomorphic to some $\mathbb{R}^A$ consider any unconditional symmetric basis $\{e_\alpha\}_{\alpha\in A}$ of $\mathbb{R}^A$ and set $X_\alpha(y)=\Psi(e_\alpha,y)$ for any $y\in V$. We set $\mathcal{X}_x=\{X_\alpha,\ \alpha\in A\}$ and after a choice of such a set $\mathcal{X}_x$ for any $x\in M$, let be $\mathcal{X}=\bigcup_{x\in M}\mathcal{X}_x$. By construction $\mathcal{X}$ satisfies (Hi) and (Hiii) but without (LB($s+2$)). Given $x\in M$, with the previous notations, we have $\|e_\alpha\|_1=1$ and as $y\mapsto\Psi_y$ is a smooth field of continuous operators from $\mathbb{R}^A$ to $T_xM\equiv E$, we get the property (Hii) at x after restriction of V if necessary and also (LB($s+2$)) at x for (Hiii).

Now given any generating set of $\mathcal{D}$ which satisfies assumption (H), by application of part 1, we get the result. □

5.2. *Attainable set in infinite dimensional control theory for a family of vector fields*

Let $\mathcal{X}$ be a family of local vector fields which satisfies conditions (Hi) and (Hii) on a Banach manifold M. In our context a **controlled trajectory** of the **controlled system** associated to $\mathcal{X}$ is a curve $\gamma: I\to M$ which is the integral curve of some vector field

$$Z(x,t,u)=\sum_{k=1}^{p}u_k(t)Z_k(x) \tag{24}$$

associated to a family $\zeta=\{Z_k\}_{k=1,\dots,p}\subset\mathcal{X}$ which satisfied the assumptions of Theorem 1 and where $u=(u_k)$ is a family of bounded curves of class L^1 on some interval of $\mathbb{R}$ (see Theorem 1). In these conditions, u is called **the control** associated to γ. An **admissible trajectory** is a curve $\gamma:[a,b]\to M$ such that there exists a finite partition $a=t_0\leqslant\cdots\leqslant t_n$ such that $\gamma:[t_i,t_{i+1}]\to M$ is a controlled trajectory of the controlled system associated to $\mathcal{X}$ for $i=0,\dots,n-1$.

This context can be found in many papers (see for example [4,8,18,2,3,1,15]). On the other hand, it is easy to see that any $\mathcal{X}$-smooth piecewise curve is an admissible trajectory (see Section 2.1).

According to the classic context in control theory for a family $\mathcal{X}$ of vector fields on M, the **exact attainable set** $\mathcal{A}(x)$ of a point $x\in M$ is the set of points y such there exists an admissible trajectory $\gamma:[0,T]\to M$ such that each $\gamma(0)=x$ and $\gamma(T)=y$. On the other end, the **approximate attainable set** of $x\in M$ is the closure $\bar{\mathcal{A}}(x)$ in M.

Remark 5.2. According to Proposition 3.4, if $\hat{\mathcal{D}}$ is integrable, for any ζ as in (24), on each maximal integral manifold N which meets $V=\bigcap_{k=1,\dots,p}\mathrm{Dom}(Z_k)$, there exist vector fields $\tilde{Z}_k$, such that $(i_N)_*\tilde{Z}_k=Z_k$. So if we set

$$\tilde{Z}(x,t,u)=\sum_{k=1}^{p}u_k(t)\tilde{Z}_k(x)$$

then we have $(i_N)_*\tilde{Z}(t,u,.)=Z(t,u,.)$ and then we obtain that each controlled trajectory with origin in L is contained in L. In this case, if $\mathcal{O}(x)$ is the $\mathcal{X}$-orbit of x, we have the inclusions:

$$\mathcal{A}(x)\subset\mathcal{O}(x)\subset\bar{\mathcal{A}}(x).$$

In finite dimension, we have $\mathcal{A}(x)=\mathcal{O}(x)$ and it is well known (from [16]) that $\mathcal{A}(x)$ is an **immersed submanifold** of M for any $x\in M$.

In our context, a corresponding result is given by the following theorem:

Theorem 7. *Assume that the set $\hat{\mathcal{X}}$ (resp. the characteristic distribution $\hat{\mathcal{D}} = l^1(\hat{\mathcal{X}})$) satisfies the conditions* (H) *(see Section 4.2) (resp.* (H′) *(see Section 4.3)) at any point $x \in M$. Then $\hat{\mathcal{D}}$ is integrable. The exact attainable set $\mathcal{A}(x)$ of any $x \in M$ is dense in the maximal integral manifold $L(x)$ of $\hat{\mathcal{D}}$ through x and the approximate attainable set $\bar{\mathcal{A}}(x)$ is the closure of $L(x)$ in M and also the closure of the $\mathcal{X}$-orbit of x. Moreover if the distribution $\hat{\mathcal{D}}$ is closed $\hat{\mathcal{A}}(x)$ is a weak submanifold of M for any $x \in M$.*

The reader will find an illustration of this theorem in [15] or in [14] (see also Example 4.5). Note that, if $\hat{\mathcal{D}}$ is finite dimensional, from Remark 4.3, the assumptions of Theorem 7 are always satisfied and the distribution $\hat{\mathcal{D}}$ is closed. In this case the attainable set is exactly an $\mathcal{X}$-orbit. So in particular, when M is finite dimensional we obtain Sussmann's result.

In finite dimension, to the distribution $\mathcal{D}$ we can associate a chain of distributions

$$\mathcal{D}^1 = \mathcal{D} \subset \cdots \subset \mathcal{D}^k \subset \cdots \tag{25}$$

where, for $k \geqslant 2$, $\mathcal{D}^k$ is generated by the set $\mathcal{X}^k$ of local vector fields of type $[X_1, [\cdots [X_{k-1}, X_k] \cdots]$ where $X_1, \ldots, X_k$ belongs to $\mathcal{X}$. The famous theorem of Chow–Rashevsky asserts that if, for any $x \in M$, there exists k such that $\mathcal{D}^k_x = T_x M$ then M is the attainable set of any point $x \in M$.

Classically, $\mathcal{X}$ is called **approximatively controllable** (resp. **exactly controllable**) if $\bar{\mathcal{A}}(x) = M$ (resp. $\mathcal{A}(x) = M$) for any $x \in M$. In order to give an analogue of theorem of Chow–Rashevsky we have already associated to $\mathcal{D}$, a chain of distributions as in (25) (see Section 4.3). As we have seen, if $\mathcal{X}$ satisfies condition (Hii) for some $s \in \mathbb{N}$, then the set $\mathcal{X}^k$ also satisfies (Hii) for $s' = s - k$ and then, the l^1-characteristic distribution $\mathcal{D}^k = l^1(\mathcal{X}^k)$ is well defined. So we have the following version of theorem of Chow–Rashevsky:

Theorem 8. *Assume that the set $\hat{\mathcal{X}}$ (resp. the characteristic distribution $\hat{\mathcal{D}} = l^1(\hat{\mathcal{X}})$) satisfies the conditions* (H) *(see Section 4.2) (resp.* (H′) *(see Section 4.3)) at any point $x \in M$. Moreover, we suppose that for any $x \in M$, there exists k such that $\mathcal{D}^k_x$ is defined, and is dense in $T_x M$ (resp. $\mathcal{D}^k_x = T_x M$). Then M is approximatively controllable (resp. exactly controllable).*

In the previous theorem, note that, according to the assumption we can have controllability only if the Banach manifold M is modeled on some $l^1(A)$ where A is a countable or uncountable set.

We say that a distribution $\mathcal{D}$ on M is finite co-dimensional if for each x, the normed space $\mathcal{D}_x$ is finite co-dimensional in $T_x M$. In this case $\mathcal{D}_x$ must be closed. In particular, finite co-dimensional l^1 distribution on M again imposes that M is modeled on $l^1(A)$ where A is a countable or uncountable set. In this case we have:

Corollary 5.3. *Let M be a Banach manifold modeled on some $l^1(B)$. Consider any set of vector fields $\mathcal{X}$ on M, which satisfies the conditions* (H). *If the characteristic distribution $\mathcal{D}$ is finite co-dimensional, then M is foliated by weak Banach submanifolds of M and each leaf is an $\mathcal{X}$-orbit. Moreover, each attainable set is dense in such a leaf.*

Proof of Theorem 7. By application of Theorem 3 or Theorem 5, we get the integrability of $\hat{\mathcal{D}}$. On one hand, for any $x \in M$, if y belongs to $\mathcal{A}(x)$ as in (24), the set ζ is finite, it follows from

Proposition 4.1 that each integral curve of such a Z is tangent to the leaf L through x. On the other hand, from Proposition 3.4, if y belongs to L, then y is adherent to $\mathcal{A}(x)$. According to Remark 5.2, we have

$$\mathcal{A}(x) \subset L(x) \subset \mathcal{O}(x) \subset \bar{\mathcal{A}}(x).$$

So, we get $\bar{L}(x) = \bar{\mathcal{O}}(x) = \bar{\mathcal{A}}(x)$.

The last part is also a consequence of Theorem 3 or Theorem 5. □

Proof of Theorem 8. From Theorem 7 we know that $\hat{\mathcal{D}}$ is integrable and, as Banach space is isomorphic to some $\mathbb{R}^A$. So by the same arguments as the ones used in the proof of Theorem 5 part 2, we have $\mathcal{D}^k_x \subset \hat{\mathcal{D}}_x$. It follows that $\hat{\mathcal{D}}_x$ is dense in T_xM or equal to T_xM. The result is then a consequence of properties (ii) or (iii) respectively of Theorem 4 or Theorem 5. □

Proof of Corollary 5.3. It is sufficient to prove that $\hat{\mathcal{X}}$ satisfies the condition (H) at each point $x \in M$. Given any $x \in M$, from our assumption we know that $\mathcal{X}$ satisfies the condition (H) at x. Take an unconditional symmetric basis $\{X_\alpha(x)\}_{\alpha\in A}$ such that $\{X_\alpha\}_{\alpha\in A} \subset \mathcal{X}_x$ and satisfies the condition (LB$(s+2)$) for $s > 0$. As $\hat{\mathcal{X}}_x$ contains $\mathcal{X}_x$ and as $\mathcal{D}_x$ is finite co-dimensional, we can choose in $\hat{\mathcal{D}}_x$ a finite number $Y_1, \dots, Y_p$ such that $\{X_\alpha(x)\}_{\alpha\in A} \cup \{Y_1(x), \dots, Y_p(x)\}$ is an unconditional symmetric basis and $\{X_\alpha\}_{\alpha\in A} \cup \{Y_1, \dots, Y_p\}$ satisfies the condition (LB$(s+2)$) for $s > 0$. We then apply Theorem 7. The last part can be shown as in the finite dimensional case (see [16]). □

6. Proof of Theorem 2

In this last section, we will use Theorem 1 to give a proof of Theorem 2.

Recall that $\xi = \{X_\alpha,\ \alpha \in A\}$ is a family of vector fields defined on an open neighborhood V of $x_0 \in E$ and satisfies the condition (LB$(s+2)$) at x_0 and with the relation (4) true for all $x \in V$.

6.1. Maps Γ^τ and $\hat{\Gamma}^\tau$

In this subsection we fix $\tau = (\tau_\alpha)_{\alpha\in A} \in \mathbb{R}^A$. Let B be any countable subset of A which contains all the indexes $\alpha \in A$ such that $\tau_\alpha \neq 0$. The set B can be written as a sequence $\{\beta_i,\ i \in \mathbb{N}\} \subset A$. For the sake of simplicity, we then denote by τ_i instead of τ_{β_i} the corresponding term of $(\tau_\alpha)_{\alpha\in A}$. With these notations we define the sequence $(\Gamma_i^\tau)_{i\in B}$ in the following way

- for $i = 1$,
  - if $\tau_1 = 0$ then $\Gamma_1^\tau(s) = 0$,
  - if $\tau_1 \neq 0$ then

$$\Gamma_1^\tau(s) = \begin{cases} \frac{\tau_1}{|\tau_1|} & \text{if } s \in [0, |\tau_1|[, \\ 0 & \text{othewise;} \end{cases}$$

- for $i > 1$,
  - if $\tau_i = 0$ then $\Gamma_i^\tau(s) = 0$,
  - if $\tau_i \neq 0$ then

$$\Gamma_i^\tau(s) = \begin{cases} \frac{\tau_i}{|\tau_i|} & \text{if } s \in [\sum_{j=1}^{i-1} |\tau_j|, \sum_{j=1}^{i} |\tau_j|[, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for all $\alpha \notin B$ we set $\Gamma^\tau_\alpha(s) = 0$ for all $s \in \mathbb{R}$.

Finally, we define the families $\Gamma^\tau(s)$ and $\hat{\Gamma}^\tau(s)$ in the following way by

$$\Gamma^\tau(s) = \big(\Gamma^\tau_\alpha(s)\big)_{\alpha\in A} \quad \text{and} \quad \hat{\Gamma}^\tau = \big(\hat{\Gamma}^\tau_\alpha(s)\big)_{\alpha\in A} = \big(\Gamma^\tau_\alpha\big(\|\tau\|_1 - s\big)\big)_{\alpha\in A}.$$

From this construction, it follows that

$$\forall s \in \mathbb{R}, \quad \big(\Gamma^\tau_\alpha(s)\big)_{\alpha\in A} \in \mathbb{R}^A \quad \text{and} \quad \big(\hat{\Gamma}^\tau_\alpha(s)\big)_{\alpha\in A} \in \mathbb{R}^A.$$

Now we consider the maps Γ^τ and $\hat{\Gamma}^\tau$ defined in the following way:

$$\begin{aligned}
&\Gamma^\tau : \mathbb{R} \to \mathbb{R}^A,\\
&s \mapsto \Gamma^\tau(s) = \big(\Gamma^\tau_\alpha(s)\big)_{\alpha\in A},\\
&\hat{\Gamma}^\tau : \mathbb{R} \to \mathbb{R}^A,\\
&s \mapsto \hat{\Gamma}^\tau(s) = \big(\hat{\Gamma}^\tau_\alpha(s)\big)_{\alpha\in A}.
\end{aligned}$$

Lemma 6.1. *Γ^τ and $\hat{\Gamma}^\tau$ belong to $L^1_b(\mathbb{R})$.*

6.2. *Proof of the first part of Theorem 2*

In this section $x \in V$ and $\tau = (\tau_\alpha)_{\alpha\in A} \in \mathbb{R}^A$ are fixed. We consider any element $\sigma = (\sigma_\alpha)_{\alpha\in A}$ of $\mathbb{R}^A$. We choose a countable subset B of A such that B contains all the indexes $\alpha \in A$ such that $t_\alpha \neq 0$ and also all indexes $\beta \in A$ such that $\sigma_\beta \neq 0$. Again the ordered set B can be written as $B = \{\beta_i,\ i \in \mathbb{N}\}$ and we then denote by $(\tau_i)_{\beta_i\in B}$ (resp. $(\sigma_i)_{\beta_i\in B}$) the corresponding subsequence or τ (resp. σ) and also we denote simply by X_i the vector field X_{β_i} of ξ for all $\beta_i \in B$.

With these notations, for any $n \in \mathbb{N}$ and any $\sigma \in \mathbb{R}^A$, we set $\sigma^n = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ and $\mathbb{R}^n$ is then considered as a subset of $\mathbb{R}^B \subset \mathbb{R}^A$

$$\begin{aligned}
&\psi^x_n\big(\tau^n\big) = \phi^{X_n}_{\tau_n} \circ \cdots \circ \phi^{X_2}_{\tau_2} \circ \phi^{X_1}_{\tau_1}(x),\\
&\hat{\psi}^x_n\big(\tau^n\big) = \phi^{X_1}_{\tau_1} \circ \cdots \circ \phi^{X_n}_{\tau_n}(x).
\end{aligned} \tag{26}$$

Lemma 6.2. *With the previous notations, for each $n \in \mathbb{N}$, the map ψ^x_n is differentiable on $\mathbb{R}^n \cap B(0, \frac{r}{k})$, and its differential is given by*

$$\begin{aligned}
D\psi^x_{n\,(\tau^n)}\big(\sigma^n\big) = {}& D\phi^{X_n}_{\tau_n\ (\psi_{n-1}(\tau^{n-1}))} \circ \cdots \circ D\phi^{X_1}_{\tau_1\ (x)}\\
&\times \left[\sum_{p=1}^{n} \sigma_p D\phi^{X_1}_{-\tau_1\,(\psi_1(\tau^1))} \circ \cdots \circ D\phi^{X_p}_{-\tau_p\,(\psi_p(\tau^p))}\big(X_p\big(\psi_p\big(\tau^p\big)\big)\big)\right].
\end{aligned}$$

Moreover, we have

$$\big\|D\psi^x_{n\,(\tau^n)}\big\| \leqslant k e^{2k\|\tau\|_1}$$

for all $x \in V$ and any $n \in B$.

The proof of this lemma is an elementary calculus by induction. For more details see [11, Chapter 5].

Afterwards, we will simply note, for any fixed $x \in V$:

$$D\psi_{n(\tau^n)}(\sigma^n) = D\phi_{t_n}^{X_n} \circ \cdots \circ D\phi_{\tau_1}^{X_1}\left[\sum_{p=1}^{n} \sigma_p D\phi_{-\tau_1}^{X_1} \circ \cdots \circ D\phi_{-\tau_p}^{X_p}\big(X_p\big(\psi_p\big(\tau^p\big)\big)\big)\right].$$

We now define the following map:

$$\begin{aligned} \triangle\psi_n^x(\tau^n) &= D\phi_{\tau_n}^{X_n} \circ \cdots \circ D\phi_{\tau_1}^{X_1}(x), \\ \triangle\hat{\psi}_n^x(\tau^n) &= D\phi_{\tau_1}^{X_1} \circ \cdots \circ D\phi_{\tau_n}^{X_n}(x). \end{aligned} \tag{27}$$

For these maps in the same way, we obtain:

Lemma 6.3. *For any fixed $x \in V$, for each $n \in \mathbb{N}$ the maps $\triangle\psi_n^x$ and $\triangle\hat{\psi}_n^x$ are differentiable on $\mathbb{R}^n \cap B(0, \frac{r}{k})$.*

Now we are in situation to prove part 1 *of Theorem* 2.

Let $x_0 \in V$ and $r > 0$ be such that $B_f(x_0, 2r) \subset V$ and fix $\tau = (\tau_\alpha)_{\alpha\in A} \in \mathbb{R}^A$ such that $\tau \in B(0, \frac{r}{k}) \subset \mathbb{R}^A$. We fix some countable subset $B \subset A$ which contains the set of indexes α such that $\tau_\alpha \neq 0$. As before the ordered set B can be written $B = \{\beta_i,\ i \in \mathbb{N}\}$ and each τ_{β_i} with $\beta_i \in B$ will be denoted τ_i. With these notations, we set

$$T = \sum_{i\in\mathbb{N}} |\tau_i| = \sum_{\alpha\in A} |\tau_\alpha| = \|\tau\|_1.$$

• Now we use Theorem 1 with the following adaptations: $I = \mathbb{R}$, $u = \Gamma^\tau$, $t_0 = 0$, δ a real number large enough and $T_0 = T$.

From Lemma 6.1 we have $\Gamma^\tau \in L_b^1(\mathbb{R})$, with $\|\Gamma^\tau\|_\infty = 1$. As $T < \frac{r}{k}$, if we set $I_0 = [-T, T]$ and $U_0 = B_0 = B(x_0, r - kT)$, we get a flow Φ_{Γ^τ}, defined on $I_0 \times U_0$. In particular, from Theorem 1 the map $\phi_\tau^\xi = \Phi_{\Gamma^\tau}(T,)$ is a C^s diffeomorphism, and moreover, by construction, we get

$$\phi_\tau^\xi(x) = \lim_{n\to\infty} \phi_{\tau_n}^{X_n} \circ \cdots \circ \phi_{\tau_1}^{X_1}(x) = \lim_{n\to\infty} \psi_n^x(\tau^n).$$

The same argument can be used to obtain the result concerning $\hat{\psi}_\tau^\xi$.

• Now we prove that the inverse map of ϕ_τ^ξ is $\hat{\phi}_\tau^\xi$.

$$\begin{aligned} \big\|\phi_\tau^\xi\big(\hat{\phi}_\tau^\xi(x)\big) - x\big\| &\leqslant \big\|\phi_\tau^\xi\big(\hat{\phi}_\tau^\xi(x)\big) - \psi_n^{\hat{\phi}_\tau^\xi(x)}(\tau^n)\big\| + \big\|\psi_n^{\hat{\phi}_\tau^\xi(x)}(\tau^n) - x\big\| \\ &\leqslant \big\|\phi_\tau^\xi\big(\hat{\phi}_\tau^\xi(x)\big) - \psi_n^{\hat{\phi}_\tau^\xi(x)}(\tau^n)\big\| + \big\|\psi_n^{\hat{\phi}_\tau^\xi(x)}(\tau^n) - \psi_n^{\hat{\psi}_n^x(-\tau^n)}(\tau^n)\big\|. \end{aligned}$$

At first, we have

$$\lim_{n\to\infty} \big\|\phi_\tau^\xi\big(\hat{\phi}_\tau^\xi(x)\big) - \psi_n^{\hat{\phi}_\tau^\xi(x)}(\tau^n)\big\| = 0.$$

So, it remains to show that the second term in the previous majoration converges to 0 when $n \to \infty$.

The map $x \mapsto \psi_n^x(\tau^n)$ is of class C^1 and its differential at x is noting but $\triangle\Psi_n(\tau^n)$. So we have

$$\big\|\triangle\Psi_n^x(\tau^n)\big\| \leqslant e^{kT}.$$

So we obtain

$$\left\|\psi_n^{\hat{\phi}_\tau^\xi(x)}\left(\tau^n\right)-\psi_n^{\hat{\psi}_n^x(-\tau^n)}\left(\tau^n\right)\right\| \leqslant e^{kT}\left(\hat{\phi}_\tau^\xi(x)-\hat{\psi}_n^x\left(-\tau^n\right)\right).$$

Finally, we get

$$\lim_{n\to\infty}\left\|\psi_n^{\hat{\phi}_\tau^\xi(x)}\left(\tau^n\right)-\psi_n^{\hat{\psi}_n^x(-\tau^n)}\left(\tau^n\right)\right\|=0$$

which ends the proof of part 1 of Theorem 1.

6.3. *Proof of the second part of Theorem 2*

For any fixed $x\in U_0$, we introduce the following notations:

$$\begin{aligned}\psi_B^x(\tau)&=\lim_{n\to\infty}\psi_n^x\left(\tau^n\right)=\phi_\tau^\xi(x),\\ \triangle\psi_B^x(\tau)&=\lim_{n\to\infty}\triangle\psi_n^x\left(\tau^n\right)=D_2\Phi_{\Gamma^\tau}(T,x).\end{aligned}\tag{28}$$

As a consequence of Lemma 6.2 and Lemma 6.3 we get:

Lemma 6.4. *ψ_B^x and $\triangle\psi_B^x$ are continuous maps on $\mathbb{R}^B\cap B(0,\frac{r}{k})$.*

For each $\alpha\notin B$ we can remark that $t_\alpha=0$ and so $\phi_{t_\alpha}^{X_\alpha}=Id$ and $D\phi_{t_\alpha}^{X_\alpha}=Id$. So the previous limits (28) can be seen as an uncountable composition of maps of type $(\phi_\alpha)_{\alpha\in A}$, evaluated at x, where only a countable subset of them are not equal to the identity.

Notations 6.5. Given any $\alpha\in A$ we set $\tau^\alpha=(t_{\alpha'})$ with $\alpha'\in A$, $\alpha'\leqslant\alpha$.
On the other hand for any $\alpha\in A$ we consider the set

$$B_\alpha=\{\beta_i \text{ such that } \beta_i\leqslant\alpha\}.$$

Considering the family of local diffeomorphisms associated to the family ξ of vector fields we denote by

$$\begin{aligned}\psi_\alpha^x\left(\tau^\alpha\right)&=\begin{cases}\psi_n^x(\tau^n) & \text{if } B_\alpha=\{\beta_1,\dots,\beta_n\},\\ \psi_B^x & \text{if } B_\alpha=B,\end{cases}\\ \triangle\psi_\alpha^x\left(\tau^\alpha\right)&=\begin{cases}\triangle\psi_n^x(\tau^n) & \text{if } B_\alpha=\{\beta_1,\dots,\beta_n\},\\ \triangle\psi_B^x & \text{if } B_\alpha=B,\end{cases}\\ \triangle\hat{\psi}_\alpha^x\left(\tau^\alpha\right)&=\begin{cases}\triangle\hat{\psi}_n^x(\tau^n) & \text{if } B_\alpha=\{\beta_1,\dots,\beta_n\},\\ \triangle\hat{\psi}_B^x & \text{if } B_\alpha=B,\end{cases}\\ \psi^x(\tau)&=\psi_B^x(\tau)=\phi_\tau^\xi \quad\text{and}\quad \triangle\psi^x(\tau)=\triangle\psi_B^x(\tau)=D_2\Phi_{\Gamma^\tau}(T,x)=D\psi_\tau^\xi(x).\end{aligned}$$

Given any $\sigma=(\sigma_\alpha)_{\alpha\in A}\in\mathbb{R}^A$, by taking for B any countable set which contains the (countable) sets $\{\alpha$ such that $\tau_\alpha\neq 0\}$ and $\{\alpha$ such that $\sigma_\alpha\neq 0\}$, from Lemma 6.4 and Notations 6.5 we get:

Lemma 6.6. *The map ψ^x and $\triangle\psi^x$ are continuous on $B(0,\frac{r}{k})$.*

Now we can prove part 2 *of Theorem* 2.

We begin by proving that ψ^x is a C^1 map. We will use the following result of [5, p. 426]:

Proposition 6.7. *Let X and Y be two Banach spaces, $U \subset X$ an open set and D a dense vector subspace of X. Consider a continuous map $f : U \to Y$ such that, for all $(x, v) \in U \times X$, the derivative f at x in the direction v denoted by $\partial_v f(x)$ exists. Moreover, assume that there exists a continuous map $L : U \to L(X, Y)$ such that, for any $(x, v) \in U \cap D \times D$, we have $\partial_v f(x) = L(x)(v)$. Then $f \in \mathcal{C}^1(U, Y)$ and $Df = L$.*

We apply this result to the sets:

$X = \mathbb{R}^A$, $U = B(0, \frac{r}{k})$ and $Y = E$;
$D = \text{span}\{e_\alpha,\ \alpha \in A\}$ where $e_\alpha = (\delta^\alpha_\beta)_{\beta \in A}$, where $\delta^\alpha_\beta = 1$ if $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ for $\alpha \neq \beta$;
(in fact, $\{e_\alpha,\ \alpha \in A\}$ is the canonical basis of $\mathbb{R}^A$);
the map f is the map ψ^x on $B(0, \frac{r}{k}) \subset \mathbb{R}^A$;
L is defined in the following way:

for $\sigma = (\sigma_\alpha)_{\alpha \in A} \in B(0, \frac{r}{k}) \subset \mathbb{R}^A$: $L(\tau)(\sigma) = \triangle\psi^x(\tau)[\sum_{\alpha \in A} \sigma_\alpha \triangle\hat{\psi}^x(-\tau^\alpha)[X_\alpha(\psi_\alpha(\tau^\alpha)))]$.

- It is clear that D is a dense set in $\mathbb{R}^A$.
- The continuity of ψ^x follows from Lemma 6.6.
- Now we prove that $\forall \tau \in B(0, \frac{r}{k}) \cap D$, $\forall \sigma \in D$, $\partial_\tau \psi^x(\sigma) = A(\tau)(\sigma)$.

Let be $(\tau, \sigma) \in B(0, \frac{r}{k}) \cap D \times D$. So we have

$$\tau = (\tau_{\alpha_i})_{i=1,\dots,p} \quad \text{with } \|\tau\|_1 < \frac{r}{k},$$
$$\sigma = (\sigma_{\beta_j})_{j=1,\dots,q} \quad \text{with } \|\sigma\|_1 < \frac{r}{k}.$$

The family $\{(e_{\alpha_i})_{i=1,\dots,p},\ (e_{\beta_j})_{j=1,\dots,q}\}$ can be put in an ordered family $\{e_{\alpha_l}\}_{l=1,\dots,n}$ with $n \leqslant \inf(p, q)$. So we can consider that τ and σ belong to $\text{span}\{e_{\alpha_1}, \dots, e_{\alpha_n}\}$. For simplicity we denote by τ_i (resp. σ_i) the component of τ (resp. σ) on e_{α_i} and X_i instead of X_{α_i}, for $i = 1, \dots, n$. Now, for any $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, we have

$$\begin{aligned}
&\psi^x(\tau + \lambda\sigma) - \psi^x(\tau) \\
&\quad = \psi^x_n\big(\tau^n + \lambda\sigma^n\big) - \psi_n\big(\tau^n\big) \\
&\quad = \lambda D\phi^{X_n}_{\tau_n} \circ \dots \circ D\phi^{X_1}_{\tau_1}\left[\sum_{p=1}^{n} \sigma_p D\phi^{X_1}_{-\tau_1} \circ \dots \circ D\phi^{X_p}_{-\tau_p}\big(X_p\big(\psi_p\big(\tau^p\big)\big)\big)\right] + o\big(\lambda\sigma^n\big)
\end{aligned}$$

so

$$\partial_\tau \Psi(\sigma) = \lim_{\lambda \to 0} \frac{\Psi(\tau + \lambda\sigma) - \Psi(\tau)}{\lambda} = L(\tau)(\sigma).$$

- The continuity of $\tau \to L(\tau)$:

Now we consider the following map:

$$\mathcal{R} : B(0,\rho) \to \mathcal{L}(\hat{\mathcal{D}}_x),$$
$$\tau \mapsto \mathcal{R}(\tau)$$

defined by $\mathcal{R}(\tau)(\sum_{\alpha\in A}\sigma_\alpha X_\alpha(x)) = \sum_{\alpha\in A}\sigma_\alpha \triangle\hat{\psi}^x_\alpha((-\tau)^\alpha)[X_i(\psi^x_\alpha(\tau^\alpha))]$.

Note that from Lemma 6.2, we have

$$\left\| \mathcal{R}(\tau)\left(\sum_{\alpha\in A}\sigma_\alpha X_\alpha(x)\right)\right\| \leqslant k e^{k\|\tau\|_1}\|\sigma\|_1.$$

So $\mathcal{R}(\tau)$ is a continuous linear map. On the other hand, we can write

$$L(\tau)(\sigma) = \triangle\psi^x(\tau)\circ\mathcal{R}(\tau)\left(\sum_{\alpha\in A}\sigma_\alpha X_\alpha(x)\right). \tag{29}$$

The proof of the following lemma can be found in [11, Chapter 5]:

Lemma 6.8. *The map $\tau \mapsto \mathcal{R}(\tau)$ is continuous on $B(0,\frac{r}{k})$.*

From this lemma and Lemma 6.6, it follows that $\tau \mapsto L(\tau)$ is continuous.

So we obtain that ψ is C^1 on $B(0,\frac{r}{k})$.

To prove that ψ is of class C^s for $s \geqslant 2$, as classically we use the fact that

$$\left(\Phi_{\Gamma^\tau}(t,x), D_2\Phi_{\Gamma^\tau}(T,x),\ldots,D_2^s\Phi_{\Gamma^\tau}(T,x)\right)$$

is the flow of an adapted vector field $\hat{Z}^s$ on an open set of the Banach space $E\times\mathcal{L}(E,E)\times\cdots\times\mathcal{L}^s(E,E)$ and proceed by induction.

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