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Graded hypothesis theories

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Abstract

Reasoning about expert domains often involves imperfect knowledge. In such cases any piece of information may prove to be useful. This requires considering uncertain beliefs and sometimes, when the expected information is missing, making additional assumptions. This paper presents a qualitative approach, which allows for the simultaneous representation of uncertain and/or incomplete information. A running example illustrates the need for such a representation. The proposed framework is a multimodal logic in which uncertainty is represented by means of a set of partially ordered symbolic grades, expressed as modal operators. Assumptions are formulated as hypotheses in Siegel and Schwind's hypothesis theory. We show that such hypotheses may be interpreted as constraints on the set of possible beliefs. We thus obtain a very natural integration of multimodal graded logic and hypothesis theory. We also study various properties of this formalism in presence of partial inconsistency.

1. Motivations

To achieve successful reasoning under imperfect information, it is generally necessary to take advantage of any kind of available knowledge. This includes true facts as well as less certain pieces of information, which may be considered as uncertain beliefs. Generally, such beliefs do not all have the same strength. For instance, one might have more confidence into the fact that a good student will succeed his exams than into the fact that a drunken person will safely drive back home. When using such beliefs in deductions, the characterization of the strength of the derived conclusions is an important issue. Many approaches for handling uncertain beliefs are based on numerical settings, e.g., Probability theory [31, 32], Dempster–Shafer theory [39] and Possibility theory [45, 14]. However, it is not always easy to give precise estimations of certainty degrees. In such cases, instead of giving precise values, experts often prefer using qualitative values to express that a statement is considered as more certain than

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another. Several attempts have proposed qualitative views of numerical settings (e.g., [15, 17, 19, 44]). However, most of these approaches assume a total ordering among the set of grades. But there might be circumstances under which experts do not want to compare uncertain beliefs, either because it does not make sense, or because not enough information is available. A possible alternative is to express uncertain beliefs by means of a set of partially ordered qualitative grades, as already investigated in [16, 6, 5, 1].

Another concern, when reasoning under incomplete information, is the case of missing knowledge. Obtaining satisfactory conclusions often requires to make additional assumptions. For instance, a company may reasonably expect to receive some mail every working day. Such a rule is known to have exceptions (e.g. when the post is on strike), but if there is no particular evidence supporting such an exception, we generally assume that the situation is not exceptional, and perform the deduction *by default*. It is generally assumed that the derived conclusions only hold as far as they do not contradict the facts and assumptions from which they are drawn. Many approaches have been proposed for modelling such kind of reasoning under incomplete information, ranging from model preference logics (circumscription [29], conditional logics [11, 22, 26]), to fix point logics (default logic [2, 35, 36], modal nonmonotonic approaches [30, 28, 27]). As a further development of the logic of supposition [3], Siegel [40] has introduced a modal formalism with a syntactic notion of hypothesis, which has been further developed in [41, 38] .

Depending on the way incomplete information is formalized, i.e. by means of weighted formulas or default assumptions, one may or not achieve defeasible reasoning. The two kinds of imperfections mentioned above are orthogonal but not mutually exclusive. For instance, in summer, a researcher might expect to receive some mail every day but, because of the general decrease of activity, even under normal circumstances there is still some possibility for him to find an empty mailbox. In such cases one would like to be able to derive such a conclusion, assuming the situation is not exceptional, but associated to some grade reflecting the level of certainty of the conclusion. Another motivation for using simultaneously these two kinds of incomplete knowledge is that it is generally not possible to consider all possible exceptions of a given default. In such a case, associating a default with some grade may be a way to express that even if none of the explicitly mentioned exceptional cases is encountered, there are still some other (omitted) cases where the conclusion may not hold. Therefore, the default conclusion may not be considered as certain.

This paper presents a framework in which it is possible to handle incomplete information in two ways. Grades corresponding to various levels of support for beliefs are represented by modal operators as in [7, 8]. Assumptions are represented as hypotheses in the spirit of Siegel and Schwind's hypothesis theory [40, 41, 38]. We show that by considering hypotheses as constraints on the set of possible beliefs, we obtain a very natural integration of multimodal graded logic and hypothesis theory. In this new framework, called *graded hypothesis theory*, the use of one or the other way of representing imperfect knowledge is related to the status of the derived conclusion, leading to either defeasible or non-defeasible graded deductions.

The remainder of the paper is organized as follows: in the next section, we introduce the multimodal graded logic and we prove its completeness and soundness. In the third section, graded hypothesis theory is introduced and its relationship to graded default logic is established. In the fourth section, we show how partial inconsistency can be handled in graded hypothesis theory. In Section 5 related work is discussed.

2. The multimodal graded logic $\mathscr{H}_\Gamma$

2.1. Expressing uncertain beliefs

Representing uncertain beliefs requires putting together two components, one for characterizing what is believed and another for characterizing how much it is believed. In multimodal graded logic [7, 8] uncertain beliefs are expressed by means of modal formulas of the form $[\alpha]f$. The formula f denotes what is believed and may be any formula of the language. The modal operator $[\alpha]$ is used to express that the formula f is supported with at least some grade α.

When reasoning under uncertainty any piece of information may prove to be useful and it often happens that a given formula is supported in different ways. For instance, it might be supported by different sources of evidence or it might be obtained as the conclusion of different deductions. In a graded theory, a formula may thus be supported with different grades. For this reason, a grade cannot be used to represent the very certainty degree of a formula f. We assume that such a certainty degree exists but, since our knowledge is imperfect, we rather consider that grades only represent lower bounds of it (principle 1).

Given a knowledge base and a formula f, our aim is then to obtain as much information (i.e. support) as possible concerning f. Whenever a formula f is supported in different ways, with different grades $\alpha_1, \ldots, \alpha_n$, all these grades are lower bounds of the certainty degree of f. As a consequence the least upper bound of all these grades, if it exists, must also be a lower bound of the certainty degree of f. Such a value may be used as a way to summarize all the amount of support concerning the formula f (principle 2).

Another important concern, when considering uncertain beliefs, is the way levels of support are combined during deductions. It is generally considered as a good knowledge representation principle that a belief obtained as the conclusion of some deduction should not be more supported than each of the beliefs used in the deduction (principle 3). When considering qualitative values for grades, this leads to consider the greatest lower bound of the grades used in the deduction.

Expressing the initial set of beliefs requires associating grades with formulas and expressing that some formulas are more supported than others, i.e. that some grades are smaller than others. Therefore, we start from a finite and partially ordered set $(\Gamma_0, \leqslant)$. We assume the existence of a greatest (resp., smallest) element of Γ_0 represented by $\top$ (resp., $\bot$). It is further assumed that these two elements correspond, respectively, to the highest possible level of support (i.e. total certainty) and the lowest possible level of support (i.e. empty support). However, because of the above-mentioned principles 2 and 3, we need some richer structure in order to deal with least upper bounds and greatest lower bounds of sets of grades. Therefore, in the following we use as the set of grades the free distributive lattice $(\Gamma, \wedge, \vee, \preccurlyeq)$ induced by $(\Gamma_0, \preccurlyeq)$ [4], for which $\preccurlyeq$ denotes the extended partial order relation on the extended set of grades Γ and $\wedge$ (resp. $\vee$) is called the *meet* (resp. *join*) operator. For any grades $\alpha, \beta \in \Gamma$, the expressions $\alpha \vee \beta$ and $\alpha \wedge \beta$ correspond, respectively, to the least upper bound (*lub*) and the greatest lower bound (*glb*) of α and β. Note that since Γ_0 is supposed to be finite, this is also the case for Γ.

In the following we also denote by $(\Gamma_\wedge, \wedge, \preccurlyeq)$ the lower semi-lattice of Γ. This corresponds to elements of Γ that can be described by expression which do not use the operator $\vee$ [4]. An important property of distributive lattices is that any grade may be described by an expression under disjunctive normal form. Formally this means that $\forall \alpha \in \Gamma$, $\exists \alpha_1, \ldots, \alpha_p \in \Gamma_\wedge$ such that $\alpha = \alpha_1 \vee \cdots \vee \alpha_p$ and $\forall i, j$, if $i \neq j$ then α_i and α_j are not comparable.

2.2. The language of graded hypothesis theory

Let $(\Gamma, \wedge, \vee, \preccurlyeq)$ be a finite distributive lattice of grades. The language $\mathscr{S}(\mathscr{H}_\Gamma)$ is a multimodal extension of classical first-order logic, containing, for each grade α of Γ, a modal operator $[\alpha]$. $\mathscr{S}(\mathscr{H}_\Gamma)$ consists of a set of individual variables $x, y, z, \ldots$, a finite set of function symbols and a finite set of predicate symbols, including the zero-place predicates true and false. Terms and formulas are defined as usual and if p is a formula and α is a grade of Γ, then $[\alpha]p$ is a formula. We call atomic formula or atom any positive literal.

2.3. Axiomatic characterization of the multimodal logic $\mathscr{H}_\Gamma$

We now introduce the system Σ_Γ, which is based on the modal system K, as a syntactical characterization of our multimodal logic $\mathscr{H}_\Gamma$.

Axiom schemes:

(C) Classical axioms of predicate logic
(K) $[\alpha](A \rightarrow B) \rightarrow ([\alpha]A \rightarrow [\alpha]B)$
($\mathrm{D}_\top$) $\neg[\top]$false
(A_1) $([\alpha]A \wedge [\beta]A) \rightarrow [\alpha \vee \beta]A$
(A_2) $[\alpha]A \rightarrow [\beta]A \quad \forall \alpha, \beta \in \Gamma$ such that $\beta \prec \alpha$
(A_3) $(\forall x[\alpha]A) \rightarrow [\alpha](\forall x A)$

Inference rules:

(MP) $\dfrac{\vdash_\Sigma A \vdash_\Sigma A \rightarrow B}{\vdash_\Sigma B}$ (modus ponens rule)

(NR) $\dfrac{\vdash_\Sigma A}{\vdash_\Sigma [\top]A}$ (necessitation rule)

(GR) $\dfrac{\vdash_\Sigma A \to B}{\vdash_\Sigma A \to \forall x B}$ provided x is not free in A (generalization rule)

The symbols α and β refer to any element of Γ, and symbols A and B refer to any formulas. Theorems p of Σ_Γ are denoted by $\vdash_\Sigma p$. As in [10] we say that a formula p is *deducible* or *derivable* from a set S of formulas (written $S \vdash_\Sigma p$) in Σ_Γ if and only if we can find a finite subset $\{q_1, \ldots, q_n\} \subseteq S$ such that $(q_1 \wedge \cdots \wedge q_n) \to p$ is a theorem of Σ_Γ. We denote by $\mathrm{Th}_\Sigma(S)$ the set of formulas that may be derived from S. In the following, if S is finite, we shall also denote by $S_\wedge$ the conjunction of all formulas of S.

In Σ_Γ, the partial order on Γ is expressed by means of the weakening axiom schemes A_2 (strict ordering is sufficient since $p \to p$ is a theorem of classical logic). This is a direct expression of principle 1. The axiom scheme A_1 expresses our second basic principle which states that, if there are two ways to deduce a given proposition with different grades, the level of support for this proposition should be at least as high as any of those grades. More generally, in the case where there are several derivations of a same formula with different grades, A_1 will be used to obtain the *best*-possible grade, i.e. the *greatest lower bound* of the certainty degree of this formula.

The parameterized modal operators $[\alpha]$ correspond to necessity operators in the system K [10], except $[\top]$ which corresponds to the necessity operator of the system D. This comes from the axiom $D_\top$, which expresses that any theory from which it is possible to derive the contradiction with total certainty is inconsistent. Using $D_\top$, it is possible to derive $[\top]A \to \neg[\top]\neg A$, expressing that the accessibility relation corresponding to $\top$ is serial.

The notion of consistency in Σ_Γ corresponds to the usual notion of consistency in modal logic.

Definition 2.1. A set S of formulas is said to be *inconsistent* if and only if the formula "false" is deducible from S. Otherwise it is said to be *consistent*.

In particular, the following property holds.

Property 2.1. *A set S of formulas is* inconsistent *if and only if it has a finite subset $\{q_1, \ldots, q_n\}$ such that*

$$\vdash_\Sigma \neg q_1 \vee \cdots \vee \neg q_n.$$

It should be pointed out that the consistency of a set S of formulas does not imply the consistency of the set of beliefs expressed by S. Particularly, S may be consistent while allowing the formula $[\alpha]$false (with $\alpha \neq \top$) to be derivable from S. This is rather satisfactory from the knowledge representation point of view, since with uncertain knowledge we often have to deal with conflicting pieces of information [12, 25]. We call such theories *partially inconsistent* theories [9].

Definition 2.2. A set S of formulas is α-*inconsistent* if and only if the formula $[\alpha]$false is deducible from S. Otherwise it is called α-*consistent*.

It is easy to see that the following formulas are derivable in Σ_Γ.

Property 2.2.

(a) *from* $\vdash_\Sigma A_1 \wedge \cdots \wedge A_n \to A$ *we may infer* $\vdash_\Sigma \sigma A_1 \wedge \cdots \wedge \sigma A_n \to \sigma A$, (RK$_m$) *where* σ *is any sequence of operators* $[\alpha], [\beta], [\gamma], \ldots$

(b) $\vdash_\Sigma \sigma(A \wedge B) \leftrightarrow \sigma A \wedge \sigma B$, *where* σ *is any sequence of operators* $[\alpha], [\beta], [\gamma], \ldots$

(c) $\vdash_\Sigma \neg[\alpha]\neg(A \wedge B) \to \neg[\alpha]\neg A$

(d) $\vdash_\Sigma [\alpha](A \to B) \wedge \neg[\alpha]\neg A \to \neg[\alpha]\neg B$

(e) $\vdash_\Sigma \neg[\alpha]\neg(A \to B) \wedge [\alpha]A \to \neg[\alpha]\neg B$

(f) $\vdash_\Sigma [\alpha \vee \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$

(g) $\vdash_\Sigma [\alpha]A \wedge [\beta](A \to B) \to [\alpha \wedge \beta]B$ (A$_{\text{gmp}}$)

The last axiom scheme (A$_{\text{gmp}}$) clearly expresses our third basic principle and is related to the *graded modus ponens* rule introduced in [16]. It allows for an easy formalization of deductions in theories with uncertain knowledge.

As an immediate consequence of Property 2.2(f) we have:

Property 2.3 (Chatalic and Froidevaux [8]). *Let S be a set of graded formulas over Γ, let $\alpha \in \Gamma$ and $\alpha_1 \vee \cdots \vee \alpha_p$ be its disjunctive normal form, where $\forall i = 1 \ldots p$, $\alpha_i \in \Gamma$. Then $S \vdash_\Sigma [\alpha]A$ iff $\forall i = 1 \ldots p$, $S \vdash_\Sigma [\alpha_i]A$.*

2.4. Examples

We now consider several examples of deductions in Σ_Γ, in which uncertain beliefs are expressed by formulas of the form $[\alpha]f$, with $\bot \prec \alpha \prec \top$ and certain beliefs by formulas of the form $[\top]f$. Note that using a formula like $[\bot]f$ would amount to giving an empty support to f. In general, this is not very informative concerning f. However, there are some cases where such a formula may be used to express some kind of preference for f over $\neg f$, without giving real support to f. We shall see in the next section that this may be of some interest in order to block the addition of hypotheses.

Example 2.1. It seems reasonable to believe that a person who has a paper accepted for a presentation at a conference will be attending this conference. We might also consider that a person who is very active in the field of this conference will be attending the conference. Suppose we believe that John is an active researcher in the field of a conference and that he submitted a paper, at least one review of which is known to be very good. Therefore, we also have some reason to believe that his paper will be accepted.

$$S = \{[\alpha]\text{active}(\text{john}, \text{field}(c)),\ [\beta]\text{accepted\_paper}(\text{john}, c),$$
$$[\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y)),$$
$$[\delta](\text{active}(X, \text{field}(Y)) \rightarrow \text{attend}(X, Y))\}.$$

Let us assume we cannot compare α, γ, β and δ but that $\perp \prec \alpha, \gamma, \beta, \delta$. Then using ($A_{gmp}$) we have:

(1) $\vdash_\Sigma$ $([\alpha]\text{active}(\text{john}, \text{field}(c)) \wedge [\delta](\text{active}(X, \text{field}(Y)) \rightarrow \text{attend}(X, Y))) \rightarrow [\alpha \wedge \delta]\text{attend}(\text{john}, c)$ $(C + A_{gmp})$

(2) $\vdash_\Sigma$ $([\beta]\text{accepted\_paper}(\text{john}, c) \wedge [\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y))) \rightarrow [\beta \wedge \gamma]\text{attend}(\text{john}, c)$ $(C + A_{gmp})$

and thus using (A_1) and properties of classical logic:

(3) $\vdash_\Sigma S_\wedge \rightarrow [(\alpha\wedge\delta)\vee(\beta\wedge\gamma)]\text{attend}(\text{john}, c)$ i.e. $S_\wedge \vdash_\Sigma [(\alpha\wedge\delta)\vee(\beta\wedge\gamma)]\text{attend}(\text{john}, c)$. Thus we may derive the fact attend(john, c) with the grade $(\alpha \wedge \delta) \vee (\beta \wedge \gamma)$ and this is the greatest grade that can be obtained for attend(john, c). This example illustrates the need for principle 3. Principle 2 makes it possible to reinforce that belief, which is supported by two independent facts: he is active in the field and he has probably an accepted paper.

The second example justifies the need for the distributivity property.

Example 2.2. Let us suppose now that Paul has submitted two papers which are known to have good chances to be accepted. Then, we have two different reasons to believe that Paul will have a paper accepted, with different degrees α and β. Thus we have

$$S = \{[\alpha]\text{accepted\_paper}(\text{paul}, c),\ [\beta]\text{accepted\_paper}(\text{paul}, c),$$
$$[\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y))\}.$$

We still assume that α, β, γ are not comparable. We consider the following two graded deductions:

(a) 1 $\vdash_\Sigma$ $([\alpha]\text{accepted\_paper}(\text{paul}, c) \wedge [\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y)))$ $\rightarrow [\alpha \wedge \gamma]\text{attend}(\text{paul}, c)$ (A_{gmp})

2 $\vdash_\Sigma$ $([\beta]\text{accepted\_paper}(\text{paul}, c) \wedge [\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y)))$ $\rightarrow [\beta \wedge \gamma]\text{attend}(\text{paul}, c)$ (A_{gmp})

hence $S_\wedge \vdash [(\alpha \wedge \gamma) \vee (\beta \wedge \gamma)]\text{attend}(\text{paul}, c)$ (A_1 and m.p.).

(b) 1 $\vdash_\Sigma$ $([\alpha]\text{accepted\_paper}(\text{paul}, c), \wedge [\beta]\text{accepted\_paper}(\text{paul}, c)) \rightarrow [\alpha \vee \beta]\text{accepted\_paper}(\text{paul}, c)$ (A_1)

2 $\vdash_\Sigma$ $([\alpha\vee\beta]\text{accepted\_paper}(\text{paul}, c)\wedge[\gamma](\text{accepted\_paper}(X, Y) \rightarrow \text{attend}(X, Y)))$ $\rightarrow [(\alpha \vee \beta) \wedge \gamma]\text{attend}(\text{paul}, c)$ (A_{gmp})

hence $S_\wedge \vdash_\Sigma [(\alpha \vee \beta) \wedge \gamma]\text{attend}(\text{paul}, c)$.

But since the lattice is distributive, we have: $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge \gamma$. More generally, the distributivity property makes it possible to apply the inference rules in any order.

The third example motivates the use of a greatest lower bound for noncomparable grades.

Example 2.3. In this example, John must go to the theatre and is afraid of possible traffic jams which would probably cause him to arrive late at the theatre. Moreover, he has a lot of work and thus little chance to finish his work early. But if he still manages to finish early, he might go to the restaurant before going to the theatre. This may be formalized by

$$S = \{[\alpha]\text{traffic\_jams}, [\delta]\text{finish\_early(john)}, [\beta](\text{traffic\_jams} \rightarrow \text{late\_theatre(john)}),$$
$$[\gamma](\text{finish\_early(john)} \rightarrow \text{restaurant(john)})\}.$$

This time we assume that $\gamma \preccurlyeq \alpha$ and $\delta \preccurlyeq \beta$ and nothing else. We obtain:

(1) $\vdash_\Sigma ([\alpha]\text{traffic\_jams} \wedge [\beta](\text{traffic\_jams} \rightarrow \text{late\_theatre(john)})) \rightarrow [\alpha \wedge \beta]\text{late\_theatre(john)}$ (A_{gmp})

(2) $\vdash_\Sigma ([\delta]\text{finish\_early(john)} \wedge [\gamma](\text{finish\_early(john)} \rightarrow \text{restaurant(john)})) \rightarrow [\delta \wedge \gamma]\text{restaurant(john)}$ (A_{gmp})

hence $S_\wedge \vdash_\Sigma [\alpha \wedge \beta]\text{late\_theatre(john)} \wedge [\delta \wedge \gamma]\text{restaurant(john)}$.

Notice that α and β (resp. γ and δ) are not comparable, but that we are still able to compare the greatest lower bounds for *late_theatre* and *restaurant*. And since $\delta \wedge \gamma \prec \alpha \wedge \beta$, we have more confidence into the fact that John will be late at the theatre than into the fact that he will go to the restaurant.

The last example illustrates the use of more complex modal formulas involving nested beliefs, to express mutual beliefs in a multiagent context.

Example 2.4. Some agent John expresses his degree of confidence into the point of view of agent Mike: John is almost certain (α_J) that if Mike is quite certain (β_M) that Tom will not come tonight, then Mike thinks that it is highly likely (γ_M) that Mary will not come. Moreover, John is rather convinced (α'_J) that Mike is quite certain that Tom will not come tonight.

We assume that $\alpha'_J \preccurlyeq \alpha_J$.

We obtain the set of formulas

$$S = \{[\alpha_J]([\beta_M]\neg\text{coming(tom)} \rightarrow [\gamma_M]\neg\text{coming(mary)}),\ [\alpha'_J][\beta_M]\neg\text{coming(tom)}\}.$$

Since $\alpha'_J \preccurlyeq \alpha_J$ by axiom A_2 we get

$$\vdash [\alpha_J]([\beta_M]\neg\text{coming(tom)} \rightarrow [\gamma_M]\neg\text{coming(mary)}) \rightarrow [\alpha'_J]([\beta_M]\neg\text{coming(tom)}$$
$$\rightarrow [\gamma_M]\neg\text{coming(mary)}).$$

Then using (K) we have

$$\vdash S \rightarrow ([\alpha'_J][\beta_M]\neg\text{coming(tom)} \rightarrow [\alpha'_J][\gamma_M]\neg\text{coming(mary)}).$$

Hence,

$$S \to [\alpha'_J][\gamma_M]\neg \text{coming(mary)}.$$

In conclusion, John is rather convinced that Mike thinks that it is highly likely that Mary will not come.

2.5. Semantical characterization of $\mathscr{H}_\Gamma$

In this section we define the meaning of formulas in terms of possible worlds semantics [23] involving families of accessibility relations. With each grade $\alpha \in \Gamma$, we associate an accessibility relation R_α. Our intuition is that the higher is the grade associated with a formula, the more constraining the corresponding accessibility relation should be. We express this idea by the fact that if $\alpha \preccurlyeq \beta$ then $R_\alpha \subset R_\beta$. In such a case, from a given possible world w, if there are more possible worlds accessible through R_β than through R_α, it will be more difficult to satisfy the formula $[\beta]p$ than the formula $[\alpha]p$ in w.

2.5.1. Graded interpretations

We first introduce the notion of $\mathscr{H}_\Gamma$-structures as follows.

Definition 2.3. Let $(\Gamma, \wedge, \vee, \preccurlyeq)$ be a distributive lattice of grades. A *$\mathscr{H}_\Gamma$-structure* is defined as a triple $S = \langle W, \mathscr{A}(R_\alpha)_{\alpha\in\Gamma}\rangle$ such that:

- W is a nonempty set of possible worlds.
- $\mathscr{A}$ is an assignment function mapping every possible world w into a classical first-order structure $\mathscr{A}(w) = \langle O, F_w, P_w\rangle$ such that O is a domain of objects, F_w is a set of operations on O and P_w is a set of relations on O.
- $(R_\alpha)_{\alpha\in\Gamma}$ is a family of binary accessibility relations (i.e. subsets of $W \times W$)) verifying:
  - (i) $\forall\alpha, \beta \in \Gamma$, if $\alpha \preccurlyeq \beta$ then $R_\alpha \subseteq R_\beta$
  - (ii) $\forall\alpha, \beta \in \Gamma$, $R_{\alpha\vee\beta} \subseteq R_\alpha \cup R_\beta$
  - (iii) $R_\top$ is serial (i.e., $\forall w \in W$, $\exists w' \in W$ such that $R_\top(w, w')$).

The notion of Γ-interpretation on a $\mathscr{H}_\Gamma$-structure is defined as usual for first-order formulas. In particular, a formula of $\mathscr{S}(\mathscr{H}_\Gamma)$ of the form $[\alpha]A$ is true at a world w for an interpretation I iff for all $w' \in W$, $R_\alpha(w, w')$ implies $I, w' \models A$. An open formula of $\mathscr{S}(\mathscr{H}_\Gamma)$ is said to be true at a world w for the interpretation I iff every of its closed instances is true at w for I. An immediate consequence for the previous definition is that $\forall\alpha, \beta \in \Gamma$, $R_{\alpha\vee\beta} = R_\alpha \cup R_\beta$.

The notions of satisfiability and validity are then defined as usual in modal logic [10, 21].

Definition 2.4.

- A formula A is *satisfiable* iff there exists a Γ-interpretation I based on a $\mathscr{H}_\Gamma$-structure $S = \langle W, \mathscr{A}(R_\alpha)_{\alpha\in\Gamma}\rangle$ and a world $w \in W$, such that there exists some instance of A which is true in w.
- Let I be a Γ-interpretation on some $\mathscr{H}_\Gamma$-structure $S = \langle W, \mathscr{A}(R_\alpha)_{\alpha\in\Gamma}\rangle$ and let $w \in W$. A formula A is *valid in I at the world w* iff every instance of A is true in w.
- A formula A is *valid in I* (written $I \models A$) iff it is valid in any world of W. Then we say that I is a *Γ-model* of A.
- A formula A is *valid* (written $\models A$) iff every Γ-interpretation is a Γ-model of A.

The previous definitions may be extended to sets of formulas. In the following, when there is no ambiguity, the interpretation I will not be mentioned any more and we shall merely write $w \models f$ instead of $I, w \models f$.

2.5.2. Soundness and completeness of Σ_Γ

The system can be proven to be sound and complete with respect to the semantics defined above.

Theorem 2.1. $\vdash_\Sigma A$ *if and only if* $\models A$.

The core of the proof may be found in the appendix.

3. Graded hypothesis theories

The multimodal logic presented so far is monotonic and gives the possibility to characterize levels of certainty associated to initial or derived formulas. In this section we introduce the notion of hypothesis to allow for making additional assumptions in the case of incomplete information. Since this notion is in the spirit of Siegel and Schwind's hypothesis theory [41, 38], we recall the main ideas and properties of this formalism.

Hypothesis theory is a modal logic-based formalism for nonmonotonic reasoning comprising two modal operators **L** and **H**. The first one is used for expressing what is known and the second one is used for expressing what can be hypothesized. Although there is some relation between these two modal operators, it should be stressed that **H** is not the dual operator of **L**. Actually, **L** is a necessity operator based on the modal system T, including the axiom $\mathbf{L}p \to p$ (what is known is true). The relationship between **L** and **H** is given by $\mathbf{H}p \to \neg\mathbf{L}\neg p$ which means that what is hypothesized, cannot be known to be false or, by contraposition, that if q is known then $\neg q$ cannot be hypothesized. It should be pointed out that the converse of this axiom ($\neg\mathbf{L}\neg p \to \mathbf{H}p$) does not hold: it can be the case that $\neg p$ is not known and p is not hypothesized. This is the reason why **H** cannot be defined as the dual operator of **L** but is defined as a dual operator of another – weaker – modal operator. In [38], a complete axiomatization for this system is given, with $\mathbf{H}$ being defined as $\neg[\mathbf{H}]\neg$, where $[\mathbf{H}]$ denotes a necessity modal operator based on the modal system K. The relationship between $\mathbf{L}$ and $[\mathbf{H}]$ is then expressed by the axiom $\mathbf{L}p \rightarrow [\mathbf{H}]p$ (which is equivalent to $\mathbf{H}p \rightarrow \neg\mathbf{L}\neg p$).

The main idea of Siegel and Schwind's hypothesis theory is to try to augment a set S of formulas by adding hypotheses of a set Hyp, while preserving consistency. Intuitively, S represents our basic knowledge, i.e. formulas that are taken for granted, while Hyp represents a set of possible assumptions that may be stated in the case of incomplete knowledge, provided they remain consistent with S. Hypotheses of Hyp are formulas of the form $\mathbf{H}p$ where p is a ground formula (i.e. without free variables). The differences in the way these two sets of formulas are used is formalized by the notion of *extension*, defined as a maximal consistent subset of $F \cup \mathrm{Hyp}$ containing F. Adding hypotheses or assumptions in this way is a common approach to nonmonotonicity and is very similar to Poole's default extensions [34]. Hypothesis theories enjoy two important properties: the existence of at least one extension and the compactness property (which relies on the compactness of the underlying modal system).

The present approach is similar to the one of [38], concerning the notion of extension but differs essentially on two points. First, we use a family of modal operators $[\alpha]$ to denote what is believed instead of a single modal operator $\mathbf{L}$ to characterize what is known. The second difference is that since [38] is concerned with what is known, the necessity operator $\mathbf{L}$ follows the rule of the system T [10]. This is justified by the fact that what is known should be consistent with the true facts. In our approach, since we agree to have partially inconsistent beliefs, our parameterized modal operators $[\alpha]$ (resp. $[\top]$) follow the rules of the modal system K (resp. D). Moreover in our approach, hypotheses are given a clear interpretation, with respect to uncertain beliefs, which does not require to introduce a specific modal operator $[\mathbf{H}]$ since this role may be played by any $[\alpha]$ for $\alpha \neq \top$.

3.1. Hypotheses as constraints on sets of beliefs

Let us consider a graded formula of the form $\neg[\alpha]\neg p$. In a consistent theory, the presence of such a formula expresses that the formula $\neg p$ is not supported with the grade α. Since from $\neg[\alpha]\neg p$, we can conclude $\neg[\beta]\neg p$, whenever $\alpha \preccurlyeq \beta$, this means that $\neg p$ is also not supported by any degree β greater than α. This does not imply that the formula $\neg p$ is not supported at all, but that if $\neg p$ is supported, it must be with some grade β such that $\alpha \not\preccurlyeq \beta$. Thus the presence of such a formula $\neg[\alpha]\neg p$ can be considered as a form of constraint on the set of levels of possible beliefs concerning $\neg p$. The lower is the grade α the stronger is the constraint.

The extreme case is the situation where $\alpha = \bot$. Since $\bot$ is a universal lower bound for Γ, $\bot \preccurlyeq \beta$ for all $\beta \in \Gamma$. Since from $\neg[\bot]\neg p$, we can conclude $\neg[\beta]\neg p$, for any β, this means that $\neg p$ is not supported at all. Otherwise from $[\beta]\neg p$ it would be possible to derive $[\bot]\neg p$ by weakening and this would lead to an inconsistency. Let us notice that although $\neg[\bot]\neg p$ does not express any explicit belief in favor of the formula p, it rejects any possible belief supporting $\neg p$. This may be considered as a way of *assuming* the formula although not giving any explicit support in it. Such a formula is at the same time weaker and stronger than a formula like $[\perp]p$. It is weaker in the sense that it does not express any explicit support in p, while $[\perp]p$ does (even if it corresponds to an empty support). But it is much stronger in the sense that it prevents any possible support in $\neg p$, while $[\perp]p$ does not. With $[\perp]p$ we may consistently support $[\alpha]\neg p$, which would just lead to a partial inconsistency where the level of inconsistency would be $\perp$ (i.e. the lowest possible one). Thus, the presence of a formula of the form $[\perp]p$ might rather be considered as a way to express that, although there is no strict evidence in favor of p, we have some *preference* for the formula p over $\neg p$.

Clearly, a formula like $\neg[\perp]\neg p$ is closely related to the hypothesis $\mathbf{H}p$ in hypothesis theory [38]. Adding a formula $\neg[\perp]\neg p$ to a multimodal graded theory obviously amounts to adding a hypothesis in hypothesis theory. It can be considered as a *constraint* that is added to an initial theory and that rejects any belief supporting the proposition $\neg p$. Let us notice that while the original hypothesis theory requires the introduction of a specific modal operator $[\mathbf{H}]$ to express a hypothesis $\mathbf{H}p$ as $\neg[\mathbf{H}]\neg p$, this is not necessary in our approach. Indeed, thanks to the axiom scheme (A_2) and the fact that $\perp$ is a universal lower bound of Γ, for all grades $\alpha \in \Gamma$, we have $\neg[\perp]\neg p \rightarrow \neg[\alpha]\neg p$. Moreover there is no reason to restrict to the case of hypotheses of the form $\neg[\perp]\neg p$. We might also consider to have weaker constraints of the general form $\neg[\alpha]\neg p$, expressing that if the formula $\neg p$ is ever supported it cannot be with a grade β such that $\alpha \preccurlyeq \beta$. In the following, to keep close with Siegel and Schwind's original notations, we shall also use the notation $\langle H_\alpha\rangle p$ to represent the formula $\neg[\alpha]\neg p$, and we simply use $\langle H\rangle$ in place of $\langle H_\perp\rangle$. From the semantical point of view, a Γ-interpretation is thus a model of a hypothesis $\langle H_\alpha\rangle p$ if from any possible world, there exists at least one possible world, accessible by R_α, in which p is true. This seems intuitively satisfactory. Moreover, the smaller is the grade, the smaller is the corresponding accessibility relation and thus, the more difficult it is to find such a world. This corresponds to the fact that the smaller the grade of the hypothesis, the stronger is the constraint corresponding to the hypothesis.

3.2. Formalizing graded hypothesis theories

In the rest of this section we show how multimodal graded logic may be extended in order to incorporate this notion of hypothesis and show that most results of hypothesis theory are preserved in their graded version.

Definition 3.1. A *graded hypothesis theory* is a pair $\mathrm{HT} = (S, \mathrm{Hyp})$ where:
- S is a set of formulas of $\mathscr{S}(\mathscr{H}_\Gamma)$
- Hyp is a set of hypothesis $\langle H_\alpha\rangle f = \neg[\alpha]\neg f$, with $\alpha \in \Gamma$, and $f \in \mathscr{S}(\mathscr{H}_\Gamma)$.

Given a graded hypothesis theory (S, Hyp), we are interested in the characterization of sets of hypotheses from Hyp that are consistent with S. An extension is thus obtained by successive additions to S of new hypotheses from Hyp, while preserving consistency, until no more hypothesis may be added.

Definition 3.2. An *extension* of a graded hypothesis theory $\mathrm{HT} = (S, \mathrm{Hyp})$ is a set

$$E = \mathrm{Th}_{\Sigma}(S \cup H) \text{ where } H \subseteq \mathrm{Hyp} \text{ is a maximal subset of Hyp consistent with } S.$$

Example 3.1. Let us consider a given student, supposed to be a good student. We also think that, if this student is good and if it is possible to make the hypothesis that he has worked hard to prepare his exams, then he is likely to pass his exams. This can be expressed by the hypothesis theory $\mathrm{HT} = (S, \mathrm{Hyp})$ defined as follows:

$$S = \{([\top]\text{good\_student} \wedge \langle H\rangle\text{worked\_hard}) \to [\alpha]\text{pass\_exams}, [\top]\text{good\_student}\}$$

$$\mathrm{Hyp} = \{\langle H\rangle\text{worked\_hard}\}.$$

Since nothing contradicts the hypothesis $\langle H\rangle$worked_hard, the theory HT has exactly one extension E which contains the formula $[\alpha]$pass_exams.

Now, let us suppose we have also heard from another person that this student spent all the days before the exam at the swimming pool. We might then reasonably believe (but we cannot be certain of that) that he did not work hard. This can be expressed by adding to S the formula $[\beta]\neg$worked_hard. But from $[\beta]\neg$worked_hard, it is possible to derive $[\bot]\neg$worked_hard (since $\bot \preccurlyeq \beta$) and recall that $\langle H\rangle$worked_hard $\equiv$ $\neg[\bot]\neg$worked_hard. Therefore, it is no more possible to make the hypothesis $\langle H\rangle$ worked_hard, since this would lead to an inconsistency. Thus the only possible extension of HT is reduced to the set of theorems of S.

As it can be seen in this example, adding a new formula f to the set S may have an incidence on the set of possible extensions. It may cause some hypothesis, that was previously in some extension of S, to be no more consistent with the new set of beliefs $S \cup \{f\}$. This exhibits the nonmonotonic behavior of this approach.

In standard hypothesis theory [41, 38], it is possible to have a hypothesis theory which admits one extension containing both hypotheses Hp and $H\neg p$. Hypothesis theories behave in this respect like default theories, where it is possible to justify some facts by the impossibility to derive a fact p and to justify other facts – in the same extension – by the impossibility to derive $\neg p$. Graded hypothesis theories behave in the same way. This can be illustrated by the following example.

Example 3.2. We believe that serious people usually work hard. On the other hand, we think that usually they sleep during the night. We have two hypotheses, namely that Mary is serious and also that she does not always sleep the night. This may be formalized by a hypothesis theory of the form $\mathrm{HT} = (S, \mathrm{Hyp})$:

$$S = \{[\beta](\text{serious}(x) \to \text{sleeps}(x))\}$$

$$\mathrm{Hyp} = \{\langle H_\alpha\rangle\text{serious(Mary)},\ \langle H_\beta\rangle\neg\,\text{sleeps(Mary)}\}.$$

Let us suppose $\alpha \preccurlyeq \beta$. HT admits one extension containing $\langle H_\beta \rangle$serious(Mary), because it is consistent to add $\langle H_\alpha \rangle$serious(Mary), as well as $\langle H_\beta \rangle \neg$serious(Mary), the latter being derivable by contraposition from S and $\langle H_\beta \rangle \neg$sleeps(Mary).

The next example illustrates the need for different levels of hypotheses.

Example 3.3. Tom is about having lunch at the cafeteria when Bob comes saying that it is not open today. However, Bob is often joking and Tom gives only little support to the fact that the cafeteria might be closed. As far as there is not enough support into the fact that the cafeteria is closed. Tom still thinks preferable to go to the cafeteria and check by himself. This may be formalized by a hypothesis theory of the form:

$$S = \{\langle H_\alpha \rangle \text{open} \rightarrow [\top]\text{go\_to\_cafeteria},\ [\beta]\neg\text{open}\}$$

$$\text{Hyp} = \{\langle H_\alpha \rangle \text{open}\}.$$

Let us suppose $\beta \preccurlyeq \alpha$. Then it is strongly consistent to add the hypothesis $\langle H_\alpha \rangle$open to S. The extension will then contain formulas of the form $[\gamma]\neg$open, for any $\gamma \preccurlyeq \beta$, and $\neg[\delta]\neg$open, for any $\alpha \preccurlyeq \delta$. But this does not entail any inconsistency (neither strong nor partial inconsistency) and thus the extension contains $[\top]$go_to_cafeteria.

Let us suppose that Tom meets another person (more confident) who also tells him that the cafeteria is probably closed today. Then we have $S = \{\langle H_\alpha \rangle \text{open} \rightarrow [\top]\text{go\_to\_cafeteria}, [\beta]\neg\text{open}, [\lambda]\neg\text{open}\}$ from which we may derive $[\beta \vee \lambda]\neg$open. If $\alpha \preccurlyeq \beta \vee \lambda$ then it is no more possible to consistently add the hypothesis $\langle H_\alpha \rangle$open.

As for hypothesis theories, we may state a sufficient condition for the existence of extensions.

Theorem 3.1. *Let* HT $= (S, \text{Hyp})$ *be a hypothesis theory. If S is consistent then* HT *has an extension.*

The proof is analogous to the one in [41]. As pointed out in [41], this implies that extending the set Hyp of hypotheses of a graded hypothesis theory can augment the content of extensions, can yield more extensions but will never eliminate any of the previous extensions.

Other results of [41] still valid in the graded version are the compactness property and the fixpoint characterization of extensions that follow from the definition.

Theorem 3.2. *Let* HT$=(S, \text{Hyp})$ *be a graded hypothesis theory and let f be a formula of $\mathscr{S}(\mathscr{H}_\Gamma)$.*

(1) *let $E = \text{Th}_\Sigma(S \cup H)$ be an extension of* HT.
Then $f \in E$ iff $\exists\{h_1, \ldots, h_n\} \subseteq H$ such that $f \in \text{Th}_\Sigma(S \cup \{h_1, \ldots, h_n\})$.

(2) *f is a theorem of some extension E of* HT *iff $\exists\{h_1, \ldots, h_n\} \subseteq \text{Hyp}$ such that $f \in \text{Th}_\Sigma(S \cup \{h_1, \ldots, h_n\})$ and $S \cup \{h_1, \ldots, h_n\}$ is consistent.*

The proof is analogous to the one in [41] except the fact that it relies on the compactness of K.

Theorem 3.3. *Let* HT $=(S, \mathrm{Hyp})$ *be a graded hypothesis theory and E an extension of* HT. *Then E is a solution of the recursive equation*:

$$E = \mathrm{Th}_{\Sigma}(S \cup \{h \in \mathrm{Hyp} : \neg h \notin E\})$$

Corollary 3.1. *If E is an extension of S in* (S, Hyp), *then for all* $h \in \mathrm{Hyp}$, *either* $h \in E$ *or* $\neg h \in E$.

Again the proofs are direct adaptations of those in [41, 38].

3.3. Relation with graded default logic

Another formalism has been introduced in [6, 8] that makes it possible to handle simultaneously uncertain and incomplete knowledge. A graded formula is then represented by a pair $(p\,\alpha)$, where p is a classical propositional formula and α is a grade on the lattice Γ. In order to avoid ambiguities we shall refer this former work as the *classical graded logic* approach, as opposed to the *multimodal graded logic* approach. Nonmonotonic behaviour is achieved by classical defaults [35, 2], which are associated with grades, in a similar way, and by applying the principle of graded modus ponens to default inference. In this part we exhibit some correspondences between both approaches for the propositional case.

The axiom system of classical graded logic (denoted here by Ξ_Γ) is composed of classical axioms schemes of propositional logic (graded by $\top$) and three inference rules denoted, respectively, by

$$\frac{(p\,\alpha) \quad (p \to q\,\beta)}{(q\,\alpha \wedge \beta)}\ (\mathrm{MPG}) \qquad \frac{(p\,\alpha)}{(p\,\beta)\forall\beta \prec \alpha}\ (\mathrm{WR})$$

graded modus ponens rule weakening rule

$$\frac{(p\,\alpha) \quad (p\,\beta)}{(p\,\alpha \vee \beta)}\ (\mathrm{SR})$$

strengthening rule

It has been shown in [5] that there are some correspondences between classical graded logic and *multimodal graded logic*, as well as between graded default theories and graded hypothesis theories.

The idea is to define a mapping Θ, which transforms a classical graded theory S into a corresponding multimodal graded theory MS, by translating every graded proposition $(p\,\alpha)$ of S into the corresponding modal formula $[\alpha]p$. Then, if we denote the derivability in Ξ_Γ by $\mid\!\sim_{\Xi}$ it has been established that:

Theorem 3.4 (Chatalic [5]). *Let S be a set of classical graded formulas and $MS = \Theta(S)$ its translation into multimodal graded logic. Then*

$$S \mathrel{|\!\!\sim}_{\Xi} (p\,\alpha) \text{ iff } MS \vdash_{\Sigma} [\alpha]p.$$

In [41] a correspondence between Reiter's default logic and hypothesis theories is established. Given the existing link between classical graded logic and multimodal graded logic, this suggests a similar correspondence between graded default theories and graded hypothesis theories.

Recall that in Reiter's approach [35] a default d is a specific nonmonotonic inference rule of the form $(p\colon q/r)$, where p, q and r are elements of P; p is called the *prerequisite* of d, q its *justification* and r its *consequent*. We use Pre(D), Just(D) and Cons(D) to denote, respectively, the set of prerequisites, justifications and consequents of a set of defaults D. A *graded default* is defined as a pair $(d\,\alpha)$, where d is a classical default and α a grade. As for a classical default, p and q are called, respectively, the prerequisite and the justification of d. A *graded default theory* Δ is then defined as a pair (W,D), where W is a set of graded formulas and D is a set of graded defaults.

The approach followed by [6] consists in generalizing the principle of graded inference (principle 2) to the case of default inference. A default $(p\colon q/r\,\beta)$ may be triggered if its prerequisite p is believed with some grade α and if nothing contradicts q (i.e. if $\neg q$ is not believed at all). Then r is inferred with the grade $\alpha \wedge \beta$. In contrast with classical defaults, if the prerequisite of a graded default is supported with different grades, this default may produce different consequents according to the different grades supporting its prerequisite. Extensions of a graded default theory are then characterized by the following fixpoint definition:

Definition 3.3. Let $\Delta = (W,D)$ be a graded default theory. Let E be a set of graded formulas. The sequence $(E_i)_{i\geqslant 0}$ is defined as follows:

$E_0 = W$ and for $i \geqslant 0$,

$$E_{i+l} = \mathrm{Th}_{\Sigma}(E_i) \cup \{(r\,\alpha \wedge \beta)(p\colon q/r\,\beta) \in D, (p\,\alpha) \in E_i \text{ and } \neg q \notin \bar{E}\}.$$

E is *a graded extension* for Δ iff $E = \bigcup_{i\geqslant 0} E_i$.

In this definition $\bar{S}$ denotes the *support* of a set of graded formulas S, i.e. the set of formulas of S without their grades. It has been shown in [6] that graded extensions are closely related to classical extensions of a non-graded default theory. In particular, the support of each extension of Δ corresponds to an extension of the support of Δ and conversely.

One important result reported in [41] is the relationship between Reiter's default logic and hypothesis theory. Indeed default theories may be embedded into hypothesis theories with a simple criterium for the existence of extensions. A default $(p\colon q/r)$ is translated into the modal formal $\mathbf{L}p \wedge \mathbf{H}q \rightarrow \mathbf{L}r$. A default theory (W,D) is translated into the hypothesis theory $(W^{\mathbf{L}}, \mathrm{Hyp})$, where $W^{\mathbf{L}} = \{\mathbf{L}w: w \in W\} \cup \{\mathbf{L}p \wedge \mathbf{H}q \rightarrow \mathbf{L}r: (p: q/r) \in D\}$ and $\mathrm{Hyp} = \{\mathbf{H}q: (p: q/r) \in D\}$. Given a hypothesis theory HT which is a translation of a default theory Δ, it has been shown that Δ has an extension iff HT has an extension which contains $\mathbf{L}\neg p$ whenever it contains $\neg\mathbf{H}p$.

Concerning the correspondence between graded default theories and graded hypothesis theories, one has first to notice a little difference between the sets of grades used in both approaches. Graded default theories Δ are characterized with only strictly "positive" grades, i.e. grades of $\Gamma^+ = \Gamma \backslash \{\perp\}$, while graded hypothesis theories use the full range Γ. One way to overcome this technical problem is to consider that although the set of grades used in Δ does not contain $\perp$, we still express Δ as a graded default theory based on a set of grades Γ [Ch 6]. To avoid ambiguities, we rather denote by Δ_+, the initial graded default theory based on the set of grades Γ^+ and by Δ the same graded default theory described on the whole set of grades Γ. Then, it may be shown that as far as $\perp$ is not used in the characterization of a set of formulas S, a formula p is derivable from S with the grade $\perp$ if and only if it is derivable from S with some grade $\alpha \in \Gamma^+$. More generally, it may be shown that:

Property 3.1 (Chatalic [5]). *Let S be a set of graded formulas such that $\forall (p\,\alpha) \in S$, $\alpha \in \Gamma^+$,*

$$\mathrm{Th}_{\Xi\Gamma^+}(S) = \mathrm{Th}_{\Xi\Gamma}(S)^+, \text{ where } \mathrm{Th}_{\Xi\Gamma}(S)^+ = \{(p\,\alpha) \in \mathrm{Th}_{\Xi\Gamma}(S): \alpha \in \Gamma^+\}.$$

The mapping Θ may now be extended in order to translate any graded default theory $\Delta = (W, D)$ into a corresponding multimodal graded theory $\Theta(\Delta) = (S, \mathrm{Hyp})$ defined as

$$S = \Theta(W) \cup \Theta(D), \text{ where } \begin{cases} \Theta(W) = \{[\alpha]p: (p\,\alpha) \in W\} \\ \Theta(D) = \{[\gamma]p \wedge \langle H\rangle q \rightarrow [\gamma]r \Big/ \left(\begin{matrix} p: q \\ r \end{matrix}\ \alpha\right) \in D \\ \qquad \text{and } \perp \preccurlyeq \gamma \preccurlyeq \alpha\} \end{cases}$$

$$\mathrm{Hyp} = \{\langle H\rangle p: p \in \mathrm{just}(D)\}$$

In this mapping, to preserve the principle of graded inference, a graded default $(p: q/r\,\alpha)$ is translated into a set of formulas of the form $[\gamma]p \wedge \langle H\rangle q \rightarrow [\gamma]r$ with $\gamma \preccurlyeq \alpha$. For convenience, we may represent such a set of formulas by an axiom scheme of the form $[x]p \wedge \langle H\rangle q \rightarrow [x \wedge \alpha]r$. Note that the formula $[\alpha]p \wedge \langle H\rangle q \rightarrow [\alpha]r$ alone would not be sufficient since in such a case it would be impossible to infer anything from $[\beta]p$ if $\alpha \not\preccurlyeq \beta$. The set of hypotheses of the translation corresponds to the set of justifications of the defaults of D.

Now we are able to state the correspondence theorems. Roughly, the idea is that to any consistent extension E of a graded default theory Δ_+, corresponds some extension E' of $\Theta(\Delta_+)$ such that the set of hypotheses characterizing the extension E' corresponds precisely to the set of justifications of the defaults compatible with E.

Theorem 3.5 (Chatalic [5]). *Let $\Delta_+ = (W, D)$ be a graded default theory and* HT $=$ (S, Hyp) *be its translation into hypothesis theory. Let E_+ be a consistent extension of Δ_+, and let*

$$H(E_+) = \{\langle H \rangle p / p \in \mathrm{Just}(D) \textit{ and } \forall \gamma \in \Gamma^+,\ (\neg p\, \gamma) \notin E_+\},$$

(1) $E' = \mathrm{Th}_\Sigma(S \cup H(E_+))$ *is an extension of* HT,
(2) $E_+ = \{(p\, \alpha) \colon \alpha \in \Gamma^+, [\alpha] p \in E'$ *and p is a non-modal formula*$\}$,
(3) *If* $\neg\langle H \rangle p \in E'$ *then* $\exists \alpha \in \Gamma^+$ *such that* $[\alpha] \neg p \in E'$.

The latter point is an extension to the graded case of the property introduced in [41] and mentioned above, which establishes the link between extensions of hypothesis theories and of default theories.

Theorem 3.6 (Chatalic [6]). *Let $\Delta_+ = (W, D)$ be a default theory and* HT $= (S, \mathrm{Hyp})$ *be its translation.*

Let E' be an extension of HT *such that if* $\neg\langle H \rangle q \in E'$ *then* $\exists \gamma \in \Gamma^+$, $[\gamma] \neg q \in E'$. *Then there exists an extension E_+ of Δ_+, such that*:

$$E_+ = \{(p\, \alpha) \colon \alpha \in \Gamma^+,\ [\alpha] p \in E' \textit{ and } p \textit{ is a non-modal formula}\}.$$

3.4. A more complex example

In this section we consider a more comprehensive example which illustrates the use of weaker constraints as hypotheses.

Example 3.4. Mr. Johnson has been murdered today between 10 and 11 a.m. The police is investigating. The few clues gathered at the moment are the following ones:

- *Any person having a good motive is suspected, unless he/she has an alibi.*
- *Any person that has been seen in the area of Mr. Johnson's House at the time of the crime is also suspected by default.*
- *Mr. Angel had a strong argument with Mr. Johnson two days ago concerning money problems.*
- *The police considers as plausible that quarrels about money might be a good motive.*
- *Mrs. Angel certifies her husband spent the whole day with her at his home which would give him an alibi.*
- *Joan (the cleaning lady) has been working at Mr. Johnson's house all the morning.*
- *The postman has seen Mr. Johnson alive just before 10 a.m.*

This example involves different kinds of beliefs:

- Certain beliefs:
  - the fact that Mr. Angel had a strong argument with Mr. Johnson,
  - the fact that Joan was in the area of the crime at that time,
  - and also the fact that if Mr. Angel was at home at that time then he has an alibi.

- Uncertain beliefs:
  - the fact that a quarrel might be considered as a good motive for such a crime,
  - the fact that Mr. Angel was at home at that time,
  - the fact that the postman might still have been in the area at that time.
- Defaults:
  - the fact that a person with a motive is suspected unless he/she has an alibi,
  - the fact that a person in the area of Mr. Johnson's House at the time of the crime is also suspected.

All these elements are not equally supported. For instance we will probably give more credit to the fact that a quarrel is a good motive, than to the fact that the postman was in the area at the time of the crime. Concerning defaults, one may agree that the fact of being suspected will be more supported in the case where the person has a motive, than in the case where he/she just was in the area at that time. Eventually, since this has been reported by his wife, the fact that Mr. Angel was at home should be less supported than any other fact.

This may be formalized by the graded hypothesis theory $\mathrm{HT} = (S, \mathrm{Hyp})$, where

$$\begin{aligned} S = \{ & [\top]\mathrm{quarrel(angel)}, [\top]\mathrm{in\_area(joan)}, [\top](\mathrm{at\_home(angel)} \rightarrow \mathrm{alibi(angel)}), \\ & [\alpha](\mathrm{quarrel}(X) \rightarrow \mathrm{motive}(X)), [\beta]\mathrm{in\_area(postman)}, [\omega]\mathrm{at\_home(angel)}, \\ & [x]\mathrm{motive}(X) \wedge \langle H\rangle \neg\, \mathrm{alibi}(X) \rightarrow [\delta \wedge x]\mathrm{suspect}(X), [x]\mathrm{in\_area}(X) \wedge \\ & \langle H\rangle \mathrm{suspect}(X) \rightarrow [\gamma \wedge x]\mathrm{suspect}(X)\} \end{aligned}$$

with the initial partial ordering $\perp \prec \omega \prec \beta \prec \alpha \prec \top$ and $\omega \prec \gamma \prec \delta \prec \top$ and $\mathrm{Hyp} = \{\langle H\rangle \neg\, \mathrm{alibi(angel)}, \langle H\rangle \mathrm{suspect(joan)}, \langle H\rangle \mathrm{suspect(postman)}\}$.

From this theory, it is possible to derive $[\alpha]\mathrm{motive(angel)}$ as well as $[\omega]\mathrm{alibi(angel)}$. From the latter we may infer by weakening $[\perp]\mathrm{alibi(angel)}$. This prevents us from adding the hypothesis $\langle H\rangle \neg\, \mathrm{alibi(angel)}$. Using the second default it is possible to derive $[\gamma]\mathrm{suspect(joan)}$ as well as $[\beta \wedge \gamma]\mathrm{suspect(postman)}$.

This conclusion might not correspond exactly to the expected result. Actually, the reason why Mr. Angel is not suspected is that he is believed to have an alibi. But the level of support of this alibi is likely to be weak in comparison with the other grades of the lattice. The rule $[x]\mathrm{motive}(X) \wedge \langle H\rangle \neg\, \mathrm{alibi}(X) \rightarrow [\delta \wedge x]\mathrm{suspect}(X)$ is blocked because the hypothesis expressed in the condition is very strong. Using $\langle H\rangle \neg\, \mathrm{alibi}(X)$ means that there must be absolutely no evidence supporting the fact that X has an alibi. In the present situation, one would clearly prefer to still apply this default, despite the weak evidence supporting Angel's alibi.

A possible way to relax the constraint expressed by such a hypothesis is to consider a weaker form of hypotheses like $\langle H_\varepsilon\rangle \neg\, \mathrm{alibi}(X)$. Such a constraint, means that $\mathrm{alibi}(X)$ should not be supported with a grade greater than ε. Thus, this is still consistent with some evidence supporting $\mathrm{alibi}(X)$ provided the corresponding level of support is not greater than ε.

Let us replace the previous rule by $[x]\text{motive}(X) \wedge \langle H_\varepsilon \rangle \neg \text{alibi}(X) \to [\delta \wedge x]\text{suspect}(X)$ such that $\varepsilon \not\preccurlyeq \omega$. Then it is now consistent to make the hypothesis $\langle H_\varepsilon \rangle \neg \text{alibi(angel)}$ and we may derive $[\delta \wedge \alpha]\text{suspect(angel)}$ as an additional conclusion. Let us notice that the corresponding grade is greater than the level of support for the suspicion of Joan and of the postman, which seems to be closer to our expectations.

Through this example, we have seen that the use of weaker hypotheses may be a way to express some form of tolerance on default justifications. It should be noticed that such a tolerance is expressed locally and may be adapted according to each such rule.

4. Partial inconsistency

Working under incomplete information often leads to gather evidence from different sources. It sometimes happens that several sources do not agree on all the facts, which results in partial inconsistency of the gathered beliefs. As mentioned in Section 2, this form of inconsistency is supported in some way by the logic, $\mathscr{H}_\Gamma$, since it is possible to have consistent theories (in the usual sense) involving partially inconsistent beliefs. In this section we study how this notion of partial inconsistency interacts with the notion of hypotheses.

4.1. Several examples involving partial inconsistency

Recall that a theory S is called α-inconsistent whenever $S \vdash_\Sigma [\alpha]\text{false}$. But in Σ_Γ, for any formula f, false $\to f$ is a theorem and thus $[\alpha]\text{false} \to [\alpha]f$ (by RK). As a consequence, in an α-inconsistent theory S any formula of the form $[\alpha]f$ may be trivially derived. In such a theory it is thus (strongly) inconsistent to add any hypotheses of the form $\langle H_\beta \rangle f$ with $\beta \preccurlyeq \alpha$. As a matter of fact, this would amount to adding the formula $\neg[\beta]\neg f$. But if the theory is α-inconsistent then $[\alpha]\neg f$ is trivially derivable. By weakening, this entails $[\beta]\neg f$ and thus a contradiction.

Example 4.1. We extend Example 3.3 by considering that it is generally the case that when somebody's telephone answering machine is on, this person is not at home. Let us suppose that it has been reported that Mr. Angel's answering machine was on at the time of the crime. We now have the theory:

$$\begin{aligned}
S = \{&[\top]\text{quarrel(angel)},\ [\top]\text{in_area(joan)}, [\top]\text{phone_answ_on(angel)},\\
&[\beta]\text{in_area(postman)}, [\omega]\text{at_home(angel)},\\
&[\top](\text{at_home(angel)} \to \text{alibi(angel)}),\\
&[\alpha](\text{quarrel}(X) \to \text{motive}(X)),\\
&[\rho](\text{phone_answ_on}(X) \to \neg\,\text{at_home}(X)).\\
&[x]\text{motive}(X) \wedge \langle H_\varepsilon \rangle \neg\,\text{alibi}(X) \to [\delta \wedge x]\text{suspect}(X),\\
&[x]\text{in_area}(X) \wedge \langle H \rangle \text{suspect}(X) \to [\gamma \wedge x]\text{suspect}(X)\}\\
\text{Hyp} = \{&\langle H_\varepsilon \rangle \neg\,\text{alibi(angle)}, \langle H \rangle \text{suspect(joan)}, \langle H \rangle \text{suspect(postman)}\}.
\end{aligned}$$

Recall that $\bot \prec \omega \prec \beta \prec \alpha \prec \top$ and $\omega \prec \gamma \prec \delta \prec \top$. We choose ρ such that $\beta \prec \rho \prec \top$ and $\gamma \prec \rho$ and ε such that $\varepsilon \not\preccurlyeq \omega$.

From this theory we may now derive $[\rho]\neg$at_home(angel) and $[\omega]$at_home(angel) and thus $[\omega]$false. It is thus possible to derive $[\omega]\neg$suspect(joan) and thus by weakening $[\bot]\neg$suspect(joan). Similarly we may derive $[\bot]\neg$suspect(postman). Therefore, it is no more possible to add the hypotheses $\langle H\rangle$suspect(joan) and $\langle H\rangle$suspect(postman). Conversely, as far as $\varepsilon \not\preccurlyeq \omega$, is not possible to derive $[\varepsilon]\neg$alibi(angel) thus it is still consistent to add $\langle H_\varepsilon\rangle\neg$alibi(angel). Thus in this example we obtain one extension which is ω-inconsistent.

We may notice that in the previous example, the two hypotheses that have been rejected are not directly related with the cause of partial inconsistency.

More generally, a side-effect of α-inconsistency is that it blocks the addition of any hypotheses $\langle H_\beta\rangle f$ such that $\beta \preccurlyeq \alpha$. Particularly, this is the case for any hypothesis of the form $\langle H\rangle f$. This happens whatever the hypothesis may be, even if it has nothing to do with the cause of partial inconsistency.

Again, one possible way to overcome this blocking effect is to relax the constraints underlying such hypotheses. For instance if we use hypotheses of the form $\langle H_\varepsilon\rangle$ such that $\varepsilon \not\preccurlyeq \alpha$, instead of $\langle H\rangle$ in the formula $[x]$in_area$(X) \wedge \langle H\rangle$suspect$(X) \rightarrow [\gamma \wedge x]$suspect$(X)$, it becomes consistent to make the hypotheses $\langle H_\varepsilon\rangle$suspect(joan) and $\langle H_\varepsilon\rangle$suspect(postman).

In the previous example, the partial inconsistency came from the set S. But partial inconsistency may also result from the addition of some hypothesis by the following example.

Example 4.2. Let us consider the variant of Example 4.1 in which we consider as a possible exception to the rule concerning the phone answering machine, the case where the owner of the machine is sleeping. The hypothesis theory considered is now:

$$
\begin{aligned}
S = \{&[\top]\text{quarrel(angel)},\ [\top]\text{in\_area(joan)},\ [\top]\text{phone\_answ\_on(angel)},\\
&[\beta]\text{in\_area(postman)}, [\omega]\text{at\_home(angel)},\\
&[\top](\text{at\_home(angel)} \rightarrow \text{alibi(angel)}),\\
&[\alpha](\text{quarrel}(X) \rightarrow \text{motive}(X)),\\
&[x]\text{phone\_answ\_on}(X) \wedge \langle H\rangle \neg\,\text{sleeping}(X) \rightarrow [\rho \wedge x]\neg\,\text{at\_home}(X),\\
&[x]\text{motive}(X) \wedge \langle H_\varepsilon\rangle \neg\,\text{alibi}(X) \rightarrow [\delta \wedge x]\text{suspect}(X),\\
&[x]\text{in\_area}(X) \wedge \langle H\rangle\text{suspect}(X) \rightarrow [\gamma \wedge x]\text{suspect}(X)\}
\end{aligned}
$$

$$
\begin{aligned}
\text{Hyp} = \{&\langle H_\varepsilon\rangle \neg\,\text{alibi(angel)},\ \langle H\rangle\text{suspect(joan)},\ \langle H\rangle\text{suspect(postman)},\\
&\langle H\rangle \neg\,\text{sleeping(angel)}\}.
\end{aligned}
$$

In this hypothesis theory, the set S is no more partially inconsistent since it is not possible to derive $[\rho]\neg$ at_home(angel) without making the assumption $\langle H\rangle\neg$ sleeping (angel). Thus we may consider adding the assumption $\langle H\rangle\neg$ sleeping(angel). But under this assumption, it becomes possible to derive $[\rho]\neg$ at_home(angel) and thus $[\omega]$false. But this partial inconsistency is incompatible with any hypothesis of the form $\langle H\rangle f$, including $\langle H\rangle\neg$ sleeping(angel) itself. Therefore such a hypothesis cannot belong to any extension. In this case the only extension is strongly consistent and is generated by adding to S the set $\{\langle H_\varepsilon\rangle\neg$ alibi(angel), $\langle H\rangle$suspect(postman), $\langle H\rangle$suspect(joan)$\}$. Again, we could avoid the blocking effect on $\langle H\rangle\neg$ sleeping(angel) by relaxing the hypothesis to $\langle H_\varepsilon\rangle\neg$ sleeping(angel).

Formulated in standard default logic, the theory (W,D) corresponding to this example would neither admit an extension because $\neg$ at_home(angel) would have to be inferred by default, which would contradict the fact that at_home(angle) would belong to W. Thus, graded hypothesis theories, which always admit an extension, may give more intuitive results to such theories.

Let us consider another example exhibiting a similar behavior.

Example 4.3. The Reiter's default theory $\Delta = \{W,D\}$ with $W = \{$German(Karl), $\neg$ drink-beer(Karl)$\}$ and $D = \{$German(X): $\neg$ anti-alcoholic(X)/drink-beer(X)$\}$ has no extension. The absence of an extension for this default theory is not very satisfactory. One expects that the theory admits at least one extension containing German(Karl) and $\neg$ drink-beer(Karl)). The corresponding standard hypothesis theory [41] is

$$S=\{\mathbf{L}\ \text{German}(X)\wedge H\neg\,\text{anti-alcoholic}(X)\rightarrow\mathbf{L}\ \text{drink-beer}(X),$$

$$\mathbf{L}\ \text{German(Karl)},\mathbf{L}\neg\,\text{drink-beer(Karl)}\}$$

$$\text{Hyp}=\{H\neg\,\text{anti-alcoholic}(X)\}.$$

Note that this theory admits one extension $\neg H\neg$ anti-alcoholic(Karl) *without* containing $\mathbf{L}$ anti-alcoholic(X). This is a case where the correspondence criterium between extensions in default logic and hypothesis theories is not verified.

Let us consider a graded version of this theory such as

$$S=\{[x]\,\text{German}(X)\wedge\langle H_\varepsilon\rangle\neg\,\text{anti-alcoholic}(X)\rightarrow[\gamma\wedge x]\text{drink-beer}(X),$$

$$[\top]\text{German(Karl)},\ [\top]\neg\,\text{drink-beer(Karl)}\},$$

$$\text{Hyp}=\{\langle H_\varepsilon\rangle\neg\,\text{anti-alcoholic}(X)\},\ \ \varepsilon\not\leqslant\gamma.$$

This theory has one γ-inconsistent extension containing $[\gamma]$drink-beer(Karl) and $[\top]\neg$ drink-beer(Karl). This expresses that Karl surely does not drink beer, but since he is German, the fact that he drinks beer is supported with the graded γ.

More generally, any default theory (W,D) containing a default $(p:q/r)$ the justification of which is not contradicted and such that $\neg r$ belongs to W, does not admit any extension. Represented as the graded hypothesis theory

$$S = \{[x]p \wedge \langle H_\varepsilon\rangle q \rightarrow [\gamma \wedge x]r, [\top]p, [\beta]\neg r\},\ \text{Hyp} = \{\langle H_\varepsilon\rangle q\}.$$

it has a $\gamma \wedge \beta$-inconsistent extension, provided $\varepsilon \not\preccurlyeq \gamma \wedge \beta$.

4.2. Automatic relaxing of the strength of hypotheses

As it has been seen in the previous examples, one possible solution to avoid the blocking effect, when trying to add new hypotheses in the case of partial inconsistency, is to relax the strength to hypotheses by substituting a hypothesis of the form $\langle H\rangle f$ by a hypothesis $\langle H_\varepsilon\rangle f$. In such a case the hypothesis is no more blocked by partial inconsistency, provided the level of inconsistency α is such that $\varepsilon \not\preccurlyeq \alpha$. This requires to have an a priori knowledge of the level of partial inconsistency. It is therefore very difficult to find the right ε.

Another possibility is to consider *hypothesis schemes* of the form $\langle H_x\rangle f$ where x denotes a variable ranging on the set of grades Γ. The idea is then to try to add Γ-instances of such hypothesis schemes obtained by replacing lattice variables by grades of Γ. This amounts to building up extensions of the graded hypothesis theory $\text{HT}' = (S, \text{Hyp}')$ instead of HT, where Hyp' is the set of all Γ-instances of hypotheses of Hyp.

For instance, let us consider the theory $\text{HT} = (\{[\alpha \wedge \beta]\neg p\}, \{\langle H_x\rangle p\})$ with $\Gamma = \{\bot, \alpha \wedge \beta, \alpha, \beta, \alpha \vee \beta, \top\}$ such that α and β are not comparable. Then HT' has one extension containing $[\alpha \wedge \beta]\neg p, \langle H_\alpha\rangle p, \langle H_\beta\rangle p$ but neither $\langle H_{\alpha\wedge\beta}\rangle p$ nor $\langle H_\bot\rangle p$.

Given a hypothesis scheme $\langle H_x\rangle A$, the weakest assumption corresponding to this scheme may be obtained by instantiating x to $\top$, which amounts to saying that $\neg A$ is not certain. By instantiating x with smaller and smaller grades of Γ we make stronger and stronger assumptions. When building an extension, we try to construct maximally consistent subsets of Γ-instances of hypothesis schemes. In fact, because we know that, whenever $\alpha, \beta \in \Gamma$ such that $\alpha \prec \beta$, then $\langle H_\alpha\rangle A \rightarrow \langle H_\beta\rangle A$, this amounts to finding, for each hypothesis scheme of Hyp, minimal grades, such that the corresponding Γ-instances preserve consistency. Let us illustrate this with a more complicated example:

Example 4.4.

$$\begin{aligned} S = \{&[\alpha]p,\ [\beta]\neg q,\ [\omega]\neg s,\\ &[x]p \wedge \langle H_y\rangle \neg q \rightarrow [\delta \wedge x]r,\\ &[x]p \wedge \langle H_z\rangle r \rightarrow [\gamma \wedge x]s\} \end{aligned}$$

$$\text{Hyp} = \{\langle H_y\rangle q, \langle H_z\rangle r\}.$$

Since, $[\alpha]p \in S$, we may consider using both defaults $[x]p \wedge \langle H_y\rangle q \rightarrow [\delta \wedge x]r$ and $[x]p \wedge \langle H_z\rangle r \rightarrow [\gamma \wedge x]s$ to derive new results. From $[\beta]\neg q \in S$ follows a first constraint on possible Γ-instances of $\langle H_y\rangle q$ such that $y \not\preccurlyeq \beta$. Under this condition it is possible to derive $[\delta \wedge \alpha]r$. Concerning the second default, there is nothing a priori restricting possible instances of $\langle H_z\rangle r$ thus we may consider using the second default to derive $[\gamma\wedge\alpha]s$. But since $[\omega]s\in S$, this causes $[\gamma\wedge\alpha\wedge\omega]$false to be derivable. As a consequence, the constraints $y\not\preccurlyeq\gamma\wedge\alpha\wedge\omega$ and $z\not\preccurlyeq\gamma\wedge\alpha\wedge\omega$ must hold as well.

One noticeable advantage of this way of building extensions is that it automatically determines the strongest levels at which a given formula may be assumed while preserving consistency. Note that these levels are not determined independently for each hypothesis scheme. There might exist constraints between several hypotheses. For instance, let us consider the hypothesis theory HT $=(S,\mathrm{Hyp})$ where $S=\{[\alpha]\neg p\vee[\beta]\neg q\}$ and $\mathrm{Hyp}=\{\langle H_y\rangle p,\langle H_z\rangle q\}$. To build up extensions of HT we may consider adding to S the hypothesis $\langle H_\perp\rangle p$. It is consistent with S but implies $\langle H_\alpha\rangle p$ and thus $[\beta]\neg q$. Now to make a hypothesis of the form $\langle H_z\rangle q$ we have to choose Γ-instances of z such that $z\not\preccurlyeq\beta$. Conversely, if we add to S the hypothesis $\langle H_\perp\rangle q$ then we can only add Γ-instances of $\langle H_y\rangle p$ such that $y\not\preccurlyeq\alpha$. We thus obtain two different extensions, one containing $[\alpha]\neg p$ and $\langle H_\perp\rangle q$, and the other one containing $[\beta]\neg q$ and $\langle H_\perp\rangle p$.

4.3. *Extending the class of hypothesis schemes*

The approach sketched in Section 4.2 has also some drawbacks. The first point is that there is no limit to the extent to which constraints are relaxed. The initial motivation for introducing hypotheses of the form $\langle H_\alpha\rangle p$ was the possibility of still making some hypothesis whenever the opposite formula is *weakly* supported. But if lattice variables are allowed to vary on the whole lattice Γ, the only case where all Γ-instances of some hypothesis scheme are rejected is the case where the negation of the assumed formula is certain.

Example 4.5. Let us consider the following graded hypothesis theory:

$$S=\{[\top]p,\ [\beta]\neg q,\ [x]p\wedge\langle H_y\rangle q\rightarrow[\alpha\wedge x]r\}\quad\text{and}\quad\mathrm{Hyp}=\{\langle H_y\rangle q\},$$

with $\Gamma=\{\perp,\alpha,\beta,\delta,\top\}$ and such that $\perp\prec\delta\prec\alpha\prec\beta\prec\top$.

Let us suppose that α denotes a rather strong level of certainty. The only Γ-instance of $\langle H_y\rangle q$ that may be consistently added to S is $\langle H_\top\rangle q$ which expresses that $\neg q$ should not be certain. This is a very weak assumption. Nevertheless, it is used to derive the fact $[\alpha]r$, which here will be strongly supported.

Actually, it is not clear whether such derivations are always appropriate. Let us consider the case of Paul who often has spelling problems when writing reports. When having a problem with some word, he generally checks the spelling with his dictionary, unless he believes that this word is not in the dictionary. This may be formalized by a default like: $[x]$spelling_pb$(W)\wedge\langle H_y\rangle$in_dict$(W)\rightarrow[\alpha\wedge x]$check$(W)$. Let us consider a given word w, which is known to be a problem for Paul (i.e. $[\top]$spelling_pb(w)) and that is believed (but this is not certain) not to be in the dictionary (i.e. $[\beta]\neg$in_dict(w)). In this case we shall probably accept still to use this rule, as long as it is not completely certain that the word is not in the dictionary. Thus we shall use at least the instance of the default obtained by substituting y by $\top$.

Conversely, let us suppose that the first default of Example 4.2 is expressed by $[x]\text{motive}(X) \wedge \langle H_y \rangle \neg \text{alibi}(X) \rightarrow [\delta \wedge x]\text{suspect}(X)$ and suppose we have a strong (but not certain) evidence that Angel has an alibi. In such a case we probably would not want to use an instance of this default to conclude that Angel is suspected.

Notice that although both sets of formulas have the same syntactical structure, we are not expecting the same behavior in both cases. Such a difference is clearly related to our interpretation of these formulas. Intuitively, one is more willing to relax the strength of hypotheses in the first case than in the second case. Actually there might be some cases where we do not want the strength level of hypothesis to be relaxed as much as possible.

One simple way to limit the level of relaxation is to label hypothesis schemes by more general lattice expressions involving constants as well as lattice variables. For instance using a hypothesis scheme of the form $\langle H_{\alpha \wedge x} \rangle f$, we may set up an upper bound to the level of relaxation since whatever the Γ-instance of x, the strength level is such that $\bot \preccurlyeq \alpha \wedge x \preccurlyeq \alpha$. Such a hypothesis scheme thus allows us to perform automatic relaxation of the strength of the hypothesis up to the limit α.

4.4. Restricting the notion of extension to limit partial inconsistency

Another drawback of systematic relaxation is that by unlimited weakening of the strength of hypotheses, more and more defaults become applicable. This may result in an increased level of partial inconsistency, which in turns leads to relax even more the strength level of hypotheses.

Example 4.6. Let us consider the following graded hypothesis theory:

$$S = \{[\top]p,\ [\delta]\neg q,\ [\beta]\neg r,\ [x]p \wedge \langle H_y \rangle q \rightarrow [\alpha \wedge x]r\} \text{ and } \text{Hyp} = \{\langle H_y \rangle q\},$$

with $\Gamma = \{\bot, \alpha, \beta, \delta, \top\}$. For simplicity, let us suppose that Γ is totally ordered: $\bot \prec \delta \prec \alpha \prec \beta \prec \top$.

Since $[\delta]\neg q \in S$, we cannot use an instance of y such that $y \preccurlyeq \delta$. We may consider adding $\langle H_\alpha \rangle q$ and then infer $[\alpha]r$. But because of $[\beta]\neg r$, this produces an α-inconsistent result. As a consequence, $\langle H_y \rangle q$ has to be further relaxed and we just keep $\langle H_\beta \rangle q$.

More generally, we have seen that given a hypothesis scheme $\langle H_x \rangle f$, if the formula f is not believed with full certainty, then it is consistent to add at least the Γ-instance obtained by substituting x by $\top$. This means that unless we are certain of $\neg f$, all defaults based on this hypothesis are applicable when its preconditions are satisfied. Since this should be the same for a majority of hypothesis schemes, we may expect a lot of partial inconsistencies in the extensions resulting of those very weak assumptions. Again, one way to tackle this problem may be to use hypothesis schemes of the form $\langle H_{\alpha \wedge x} \rangle$, which fix an upper bound to the level of relaxation.

Another (complementary) solution could be to require extensions to preserve a certain level of partial consistency. For instance, one may restrict the generation of extensions to those which are ε-consistent.

Definition 4.1. An *ε-extension* of a graded hypothesis theory $\mathrm{HT} = (S, \mathrm{Hyp})$ is a set $E = \mathrm{Th}_\Sigma(S \cup H)$ where H is a maximal subset of the set of Γ-instances of hypothesis schemes of Hyp. such that E is ε-consistent.

One simple solution for enforcing ε-consistency is to look for extensions of the extended hypothesis theory $\mathrm{HT}' = \{S \cup \{\langle H_\varepsilon \rangle \mathrm{true}\}, \mathrm{Hyp}\}$, where $\langle H_\varepsilon \rangle \mathrm{true} = \neg [\varepsilon] \neg \mathrm{true} = \neg [\varepsilon] \mathrm{false}$.

Theorem 4.1. *Let* $\mathrm{HT} = (S, \mathrm{Hyp})$ *be a graded hypothesis theory and let* $E = \mathrm{Th}_\Sigma(S \cup H)$. *Then E is an ε-extension of* HT *iff* $E' = \mathrm{Th}_\Sigma\{S \cup \{\langle H_\varepsilon \rangle \mathrm{true}\} \cup H)$ *is an extension of* $\mathrm{HT}' = (S \cup \{\langle H_\varepsilon \rangle \mathrm{true}\}, \mathrm{Hyp})$.

The proof may be found in the appendix.

5. Related work

The expression *graded modal logic* has also been used by Van der Hoek in [43]. Although this work is also motivated by the representation of uncertain knowledge by means of modal operators, it differs from ours in a fundamental way, since it aims at counting the number of exceptional situations in which some proposition p does not hold. For this, an infinity of necessity operators $[n]$ ($n \in |N$) is introduced, such that $[n]p$ is satisfied by a (possible) world w if and only if there are at most n worlds w' that are reachable from w and that satisfy the formula $\neg p$.

There has been another attempt at translating graded default theories into graded hypothesis theories [33]. However, it is not based on a multimodal approach and it uses the T setting. The basic idea of this work is to encode any grade α of Γ by a chain of modalities of the form $S_\alpha = \Box S \Box$, where S is a sequence of modal operators $\Box$ and $\Diamond$, containing a fixed number of $\Box$ and possibly some $\Diamond$, provided they are not placed at consecutive places in the chain. Such an encoding is made possible by the fact that the logic T contains an infinity of distinct modalities represented by such chains. The author shows that for a given set of grades Γ, it is possible to define a mapping $\phi : \alpha \mapsto S_\alpha$ such that all chains S_α contain the same number n of $\Box$, and such that $\forall \alpha,\ \beta \in \Gamma,\ \alpha \preccurlyeq \beta$ iff $S_\beta p \rightarrow S_\alpha p$ is an axiom of the system T. Uncertain beliefs are then represented by formulas of the form $S_\beta p$. A noticeable drawback of this approach is that it is not incremental. If new grades are introduced, it is necessary to reconstruct the encoding. To perform graded inference in the same way as graded modus ponens does, the author proposes to use the inference rule:

$$\frac{S_\alpha \Box p \qquad S_\alpha \Box (p \rightarrow q)}{S_\alpha \Box q}.$$

A graded default theory $\Delta = (W,D)$, is then translated into the hypothesis theory $\mathrm{HT} = (S, \mathrm{Hyp})$, where

$$S = \{S_\alpha \Box p : (p\ \alpha) \in W\} \cup \{S_\alpha(\Box p \wedge \mathrm{Hq} \rightarrow \Box r) : (p : q/r\ \alpha) \in D\}$$

$$\mathrm{Hyp} = \{\Box^n \mathrm{Hq} : q \in \mathrm{Just}(D)\}.$$

The author justifies the introduction of an extra necessity operator $\Box$ after any chain S_α by technical reasons. In our formalism (apart from the fact that we use the system K), the corresponding translation would require us to replace any chain S_α by a modal operator $[\alpha]$, and to use an extra modal operator of necessity $\Box$. This would give

$$S = \{[\alpha]\Box p/(p\,\alpha) \in W\} \cup \{[\alpha](\Box p \wedge \mathrm{Hq} \rightarrow \Box r : (p : q/r\,\alpha) \in D\}$$

$$\mathrm{Hyp} = \{[\top]\mathrm{Hq} : q \in \mathrm{Just}(D)\}.$$

It has been stressed earlier that the basic principles underlying uncertainty handling in this graded logic approach are those of possibilistic logic [24, 13]. Possibilistic logic is a numerical formalism, in which uncertain formulas are expressed by pairs of the form $(f(N\,\alpha))$ where f denotes a classical formula and $(N\,\alpha)$ expresses that α is a lower bound of the necessity (i.e certainty) degree of f. The set of values which are used to characterize necessity degrees is totally ordered. The syntax of the language is thus simpler since it does not allow for nesting of graded formulas. The semantics is also different since interpretations are considered as fuzzy sets on the set of classical interpretations. Although the approach does not address the problem of building extensions of a given theory by adding new hypotheses, the language makes it also possible to consider theories containing formulas of the form $(f(\Pi\,\alpha))$. Such formulas are used to express that the possibility degree of the formula f is greater than α. The notion of possibility is dual to the notion of necessity and satisfies the condition $\Pi(f) = 1 - N(\neg f)$. Stating that $\Pi(f) \geqslant \alpha$, thus amounts to stating that $N(\neg f) \leqslant 1 - \alpha$. Such a piece of information could be expressed in our graded logic approach by formulas of the form $\neg[\beta]\neg f$ for any grade β such that $1 - \alpha < \beta$ and thus roughly corresponds to the syntactic notion of hypothesis. By using grades ranging onto the interval [0,1], it is easy to define an inverse relation on the set of grades. In our approach it would be more satisfactory to use such inverse values to characterize the strength of hypotheses. Then, the higher the value, the stronger the corresponding hypothesis would be, which seems better from the intuitive point of view. Problems induced by partial inconsistency in possibilistic logic have also been investigated in [24]. Besides the cases of full consistency or inconsistency, two other kinds of inconsistency have been identified, namely *weak inconsistency* and *partial inconsistency*. While the latter corresponds to our own definition of partial inconsistency (i.e. it is possible to derive a formula of the form $[\alpha]$false) the former one has no counterpart in our framework. In fact it corresponds to theories where it is possible to have simultaneously constraints like $N(f) \geqslant \alpha$ and $\Pi(\neg f) \geqslant \beta$. But the latter constraint amounts to saying that $N(f) \leqslant 1 - \beta$ and this gives a contradiction whenever $\alpha > 1 - \beta$. In our approach this would amount to having simultaneously $[\alpha]f$ and $\neg[1-\beta]f$ which, if $\alpha > 1-\beta$, would give a strong contradiction.

6. Conclusion

In this paper we have presented a formalism which allows for the simultaneous handling of uncertain and incomplete information. It is based on a qualitative approach, where the uncertainty is represented by means of partially ordered symbolic grades, interpreted as lower bounds of levels of certainty. However such grades are not intended to be used for expressing preferences or priorities among the set of beliefs. The framework is based on a multimodal logic, in which grades are expressed as modal operators attached to logical formulas and for which a sound and complete first order axiomatization has been provided.

We have shown that Siegel and Schwind's notion of hypothesis may naturally be integrated into this multimodal graded logic. We have extended the framework and shown that most properties of hypothesis theory are preserved in graded hypothesis theory. The proposed semantics differs from the original one of Siegel and Schwind's approach. Modal formulas are interpreted as graded beliefs instead of known formulas. As a consequence, it is not necessary to introduce a new modal operator for hypotheses and these may be given a clear semantics with respect to beliefs. Actually, hypotheses may be considered as constraints restricting the set of possible beliefs. More precisely, the addition of the hypothesis $\langle H\rangle q$ excludes the possibility of having any support in $\neg q$ (i.e. to have $[\alpha]\neg q$, for any grade α). Weaker forms of hypotheses, like $\langle H_\alpha\rangle q$, may also be considered to express relaxed constraints precluding beliefs of the form $[\beta]\neg q$ where $\beta \geqslant \alpha$.

The resulting framework offers two non-exclusive possibilities to express imperfect knowledge. By using graded implications, it is possible to perform graded deductions, combining the various levels of support involved in the reasoning processes. The conclusions of such deductions may be uncertain but are derived in a monotonic way. Defeasible (and possibly uncertain) conclusions may be obtained by considering the addition of hypotheses consistent with the initial theory. We have seen that enlarging the class of possible hypotheses may contribute to obtain more intuitive results in a number of situations where the expected results may vary according to the interpreted strength of several grades. The use of relaxed form of hypotheses also proves to be very useful in the case of partially inconsistent theories, where alternative definitions of extensions may be worth considering.

We have not yet investigated the complexity of our system. Without the automatic relaxation of beliefs, it is the complexity of the underlying nonmonotonic modal logic [18]. But the process of automatic relaxation of hypotheses as described in Section 4.2 may augment further the complexity, since then we have to compare subsets of the knowledge base, namely those obtained by instantiating the variable grades by constant grades.

A possible extension to this work could be to look for alternative translations of graded defaults into hypothesis theory. For instance we might consider the translation of a graded default $(p : q/r\,\alpha)$ into a formula of the form $\langle H\rangle q \rightarrow [\alpha](p \rightarrow r)$ or $[\alpha](p\wedge \langle H\rangle q \rightarrow r)$. Such translations would lead to different notions of extensions and might prove to have (or not) other interesting properties (case analysis, contraposition, ...).

We could also try to integrate in some way the level of relaxation of some hypothesis into the grade of the conclusions derived using this hypothesis. Intuitively, one would like to be able to express that the more the strength level of some hypothesis is relaxed the less confidence we should have in the corresponding conclusion. However such a behavior is difficult to achieve in practice. The first problem is that it would probably be difficult to evaluate *how much* the grade of the relaxed hypothesis should be taken into account into the conclusion. The second point is that this would require some kind of inverse relation on the lattice of grades, which would probably be difficult to define in all the cases.

Another point with investigating is the possible links between this work and so called *argumentative inference* approaches, which purpose is to confront arguments in supporting a formula (i.e. reasons to belief in) to arguments against this formula [42]. Similarly, in our logic it is possible to have simultaneously formulas $[\alpha]p$ and $[\beta]\neg p$. Several proposals have already been made concerning argumentative approaches in the framework of possibilistic logic [1]. Given the connections between our work and possibilistic logic our approach may also benefit from the results of [1].

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Appendix

Proof of Theorem 2.1. The proof of the soundness of Σ_Γ is obvious and goes along the same lines as usual, by induction on the length of derivations. The completeness proof is based on Henkin's completeness proof for classical logic [20] and is along the same lines as [37]. Given a consistent set of formulas, we define a canonical model satisfying this set.

Definition A.1. A set S of formulas is called *complete* if

(1) S is consistent.

(2) S is maximal, i.e. $\forall A \in P(S)$, if $A \notin S$ then $S \cup \{A\}$ is inconsistent.

(3) S is saturated, i.e. for every existential formula of the form $\exists x A \in S$, there is a formula $A_x[c] \in S$ for some constant c.

Lemma A.1. *Every consistent set of formulas can be extended to a complete set of formulas.*

The following properties of a complete set of formulas are straightforward.

Lemma A.2. *Let S be a complete set of formulas. Then:*
(a) *if $A \in S$ then $\neg A \notin S$,*
(b) *if $A \in \mathscr{L}^*(S)$ and $A \notin S$ then $\neg A \in S$,*
(c) *if $A \in S$ and $B \in \mathscr{L}^*(S)$ and $\vdash A \to B$ then $B \in S$,*
(d) *if $A \vee B \in S$ then $A \in S$ or $B \in S$,*
(e) *if $A \vee B \in \mathscr{L}^*(S)$, and $A \in S$ or $B \in S$, then $A \vee B \in S$,*
(f) *for any closed term a in $\mathscr{L}^*(S)$, if $A_x[a] \in S$ then $\exists x A \in S$,*
(g) *if $\exists x A \in S$ then there is a term a in $\mathscr{L}^*(S)$ such that $A_x[a] \in S$.*

Definition A.2. Let S be a complete set of formulas and let $\alpha \in \Gamma$. We define $S^\alpha = \{A\colon [\alpha]A \in S\}$.

Property A.1. (a) $\forall \alpha, \beta \in \Gamma$ *if* $\alpha \preccurlyeq \beta$ *then* $S^\beta \subseteq S^\alpha$; (b) $\forall \alpha, \beta \in \Gamma$ $S^{\alpha \vee \beta} = S^\alpha \cap S^\beta$.

Proof. (a) Let $A \in S^\beta$. Then by definition of S^β, $[\beta]A \in S$. Since $\alpha \preccurlyeq \beta$, by A_2 we know that $\vdash [\beta]A \to [\alpha]A$. Since S is complete, by Lemma A.2(c) this implies $[\alpha]A \in S$. But then $A \in S^\alpha$.

(b) Since $\alpha \preccurlyeq \alpha \vee \beta$ and $\beta \preccurlyeq \alpha \vee \beta$, by (a) we know that $S^{\alpha \vee \beta} \subseteq S^\alpha$ and $S^{\alpha \vee \beta} \subseteq S^\beta$ and thus $S^{\alpha \vee \beta} \subseteq S^\alpha \cap S^\beta$. Conversely, let $A \in S^\alpha \cap S^\beta$, by definition of S^α and S^β this implies $[\alpha]A \in S$ and $[\beta]A \in S$ and thus $[\alpha]A \wedge [\beta]A \in S$. By Property 2.2(f) we know that $\vdash_\Sigma [\alpha \vee \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$. Since S is complete, by Lemma A.2(c) this implies $[\alpha \vee \beta]A \in S$ and thus $A \in S^{\alpha \vee \beta}$. Hence $S^\alpha \cap S^\beta \subseteq S^{\alpha \vee \beta}$. $\square$

Definition A.3. A system $\mathscr{S}$ of sets of formulas is called complete if
(a) Every element of $\mathscr{S}$ is a complete set of formulas.
(b) For every $S \in \mathscr{S}$ and for every $A \in \mathscr{L}^*(S)$, if $S^\alpha \cup \{A\}$ is consistent then there is $S' \in \mathscr{S}$ such that $S^\alpha \cup \{A\} \subseteq S'$.

Lemma A.3. *For every complete set of formulas S, there is complete system of sets $\mathscr{S}$ with $S \in \mathscr{S}$.*

Now we define the relations R_α, $\alpha \in \Gamma$.

Definition A.4. Given a complete system of sets $\mathscr{S}$, we define a family $(R_\alpha)_{\alpha \in \Gamma}$ of binary relations on $\mathscr{S}$.

$$\forall \alpha \in \Gamma,\ \forall S,\ S' \in \mathscr{S}, \quad S R_\alpha S' \text{ iff } S^\alpha \subseteq S'.$$

Lemma A.4. *Let $\mathscr{S}$ be a complete system of sets and $(R_\alpha)_{\alpha\in\Gamma}$ as defined in Definition* A.4, *then*

(a) $\forall\alpha,\beta\in\Gamma$, *if* $\alpha\preccurlyeq\beta$ *then* $R_\alpha\subseteq R_\beta$,

(b) $\forall\alpha,\beta\in\Gamma$, $R_{\alpha\vee\beta}\subseteq R_\alpha\cup R_\beta$,

(c) $R_\top$ *is serial* (*i.e.*, $\forall S\in\mathscr{S}$, $\exists S'\in\mathscr{S}$ *such that* $R_\top(S,S')$).

Proof. (a) Let $\alpha,\beta\in\Gamma$ such that $\alpha\preccurlyeq\beta$ and let $S,S'\in\mathscr{S}$ such that $S\,R_\alpha\,S'$. By definition of R_α, this is equivalent to $S^\alpha\subseteq S'$. But since S is complete and $\alpha\preccurlyeq\beta$ we have $S^\beta\subseteq S^\alpha$ by Property A.1(a). Thus, we have $S^\beta\subseteq S'$, i.e. $S\,R_\beta\,S'$. Hence $R_\alpha\subseteq R_\beta$.

(b) Let $\alpha,\beta\in\Gamma$ and let $S,S'\in\mathscr{S}$ such that $S\,R_{\alpha\vee\beta}\,S'$. By definition of $R_{\alpha\vee\beta}$ we know that $R_{\alpha\vee\beta}(S,S')$ iff $S^{\alpha\vee\beta}\subseteq S'$. Let us suppose that we neither have $S^\alpha\subseteq S'$ nor $S^\beta\subseteq S'$. Then there must exist some formula A such that $[\alpha]A\in S$ but $A\notin S'$ and some formula B such that $[\beta]B\in S$ but $B\notin S'$. Since S' is a complete set, this implies $\neg A\in S'$ as well as $\neg B\in S'$ (Lemma A.2(b)) Since $\vdash\neg A\wedge\neg B\rightarrow\neg(A\vee B)$ and since S' is complete, this implies (Lemma A.2(c)) $\neg(A\vee B)\in S'$. Since S is a complete set, and since $\vdash[\alpha]A\rightarrow[\alpha](A\vee B)$ as well as $\vdash[\beta]B\rightarrow[\beta](A\vee B)$ (by RK_m), we have (Lemma A.2(c)) $[\alpha](A\vee B)\in S$ and $[\beta](A\vee B)\in S$. By A_2 we have $\vdash[\alpha](A\vee B)\wedge[\beta](A\vee B)\rightarrow[\alpha\vee\beta](A\vee B)$. Again, since S is complete this implies that $[\alpha\vee\beta](A\vee B)\in S$. But by definition of $R_{\alpha\vee\beta}$, this implies that $A\vee B\in S'$. But this contradicts the consistency of S'. Hence either $S\,R_\alpha\,S'$ or $S\,R_\beta\,S'$.

(c) Let $S\in\mathscr{S}$. Since $\vdash\neg[\top]$false ($\mathrm{D}_\top$) and S is consistent, $[\top]$false cannot belong to S. Hence false $\notin S^\top$ and $S^\top$ is consistent. Let $A\in\mathscr{L}^*(S)$ such that $S^\top\cup\{A\}$ is consistent. Since $\mathscr{S}$ is a complete system, we know that there is some $S'\in\mathscr{S}$ such that $S^\top\cup\{A\}\subseteq S'$ (Definition A.3). From this we get $S^\top\subseteq S'$, hence $R_\top(S,S')$. Since this holds for any $S\in\mathscr{S}$, it follows that $R_\top$ is serial. □

Lemma A.5. *Let $\mathscr{S}$ be a complete system of sets, let $S\in\mathscr{S}$. Then $[\alpha]A\in S$ iff $A\in S'$ for every S' with $S\,R_\alpha\,S'$.*

Proof. Let $S\in\mathscr{S}$.

($\Rightarrow$) By definition of S^α, $[\alpha]A\in S$ iff $A\in S^\alpha$. Let $S'\in\mathscr{S}$, by definition of R_α, $S\,R_\alpha\,S'$ iff $S^\alpha\subseteq S'$. This implies $A\in S'$.

($\Leftarrow$) Let us suppose that $\forall S'\in\mathscr{S}$ if $S\,R_\alpha\,S'$ then $A\in S'$. We show first that $S^\alpha\cup\{\neg A\}$ is inconsistent. Assume that $S^\alpha\cup\{\neg A\}$ is consistent. Then there exists $S'\in\mathscr{S}$ such that $S^\alpha\cup\{\neg A\}\subseteq S'$ from Definition A.3(b). Hence $S^\alpha\subseteq S'$ and therefore $S\,R_\alpha\,S'$. $A\in S'$ from the precondition and therefore $\neg A\notin S'$ by Lemma A.2(a) (since S' is complete). This contradicts $S^\alpha\cup\{\neg A\}\subseteq S'$. Hence $S^\alpha\cup\{\neg A\}$ is inconsistent and there exist formulas $A_1,\ldots,A_k\in S^\alpha$ such that $\vdash\neg A_1\vee\cdots\vee\neg A_k\vee A$. Then $A_1\wedge\cdots\wedge A_k\rightarrow A$ is also a theorem from which follows $\vdash[\alpha]A_1\wedge\cdots\wedge[\alpha]A_k\rightarrow[\alpha]A$ (by RK_m). But since $A_1,\ldots,A_k\in S^\alpha$, $[\alpha]A_1,\ldots,[\alpha]A_k\in S$. Since S is complete this implies $[\alpha]A\in S$ by Lemma A.2(b). □

Now we are in a position to define an $\mathscr{H}_\Gamma$-structure for a consistent set of formulas S_0. The domain will be the set of ground terms of $\mathscr{L}^*(S_0)$.

The $\mathscr{H}_\Gamma$-structure for S_0 is defined as follows. $M = \langle W, \mathscr{A}, (R_\alpha)_{\alpha\in\Gamma}\rangle$, where

(1) W is a complete system of sets for S_0 as defined in Definition A.3.

(2) $(R_\alpha)_{\alpha\in\Gamma}$ are the accessibility relations as defined in Definition A.4.

(3) $\forall w \in W$, $\mathscr{A}(w) = \langle O, F_w, P_w\rangle$ is a classical structure, where

  (3.1) The domain O is the set of ground terms of $\mathscr{L}^*(S_0)$.

  (3.2) F_w corresponds to a set of functions $f(w)$ that satisfy $f(w)(t_1,\dots,t_n) = f(t_1,\dots,t_n)$.

  (3.3) P_w corresponds to a set of relations $P(w)$ that satisfy $(t_1,\dots,t_n)\in P(w)$ iff $P(t_1,\dots,t_n)\in w$.

We now consider a Γ-interpretation I_c based on this $\mathscr{H}_\Gamma$-structure M such that at each world w of W each term t is interpreted by itself, i.e. $\forall w\in W$, $I_c(w)(t) = t$. The truth value $T(w,A)$ for a closed formula A at a given world w of W is defined inductively by the above definition of functions and predicates (3.1)–(3.3).

Lemma A.6. *For every closed formula $A\in\mathscr{L}^*(S) I_c, w \models A$ iff $A\in w$.*

Proof. The proof is straightforward by induction over the structure of the formulas using the Lemmata A.2 and A.5. □

Completeness Theorem. *If $\models f$ then $\vdash_\Sigma f$.*

Proof. Assume that there is a valid formula f which is not deducible in $\mathscr{H}_\Gamma$. Then $\{\neg f\}$ is consistent and, by Lemma A.1, it can be extended to a complete set of formulas S_0 such that $\neg f \in S$. By Lemma A.3, there is a complete system of sets $\mathscr{S}$ with $S_0 \in \mathscr{S}$. From $\mathscr{S}$, we may then construct a $\mathscr{H}_\Gamma$-structure for S_0 and a Γ-interpretation I_0 based on S_0 for which Lemma A.6 holds. In the world S_0 we have in particular $\neg f \in S_0$ and thus, by Lemma A.6, $I_0, S_0 \models \neg f$, i.e. $I_0, S_0 \not\models f$ which contradicts the validity of f. □

Theorem 4.1. *Let* HT $= (S, \mathrm{Hyp})$ *be a graded hypothesis theory and let* $E = \mathrm{Th}_\Sigma(S\cup H)$. *Then E is an ε-extension of* HT *iff* $E' = \mathrm{Th}_\Sigma(S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}\cup H)$ *is an extension of* HT$' = (S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}, \mathrm{Hyp})$.

Proof. Let $E = \mathrm{Th}_\Sigma(S\cup H)$ and $E' = \mathrm{Th}_\Sigma(S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}\cup H)$, with H subset of Hyp.

We first show that E is ε-consistent iff E' is consistent By definition of ε-consistency, E is ε-consistent iff $[\varepsilon]\mathrm{false} \notin E$, i.e., iff $S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}\cup H$ is consistent iff E' is consistent.

Now we show that E is maximally ε-consistent iff E' is maximally consistent.

Assume that E is maximally ε-consistent and suppose that $E' = \mathrm{Th}_\Sigma(S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}\cup H)$ is not maximally consistent. Then there is a Γ-instance h of some hypothesis scheme of Hyp, such that $h \notin H$ and $S\cup\{\langle H_\varepsilon\rangle\mathrm{true}\}\cup H\cup\{h\}$ is consistent. Because E is maximally ε-consistent, $S \cup H \cup \{h\}$ is ε-inconsistent. Thus $[\varepsilon]\text{false} \in \mathrm{Th}_\Sigma(S \cup H \cup \{h\})$, which is equivalent to $\neg \langle H_\varepsilon \rangle \text{true} \in \mathrm{Th}_\Sigma(S \cup H \cup \{h\})$, contradicting the consistency of $S \cup \{\langle H_\varepsilon \rangle \text{true}\} \cup H \cup \{h\}$. Hence E' is maximally consistent.

Conversely, assume that E' is maximally consistent and suppose that E is not maximally ε-consistent. Since E is ε-consistent, then H is not maximal. Therefore there is a Γ-instance h of some hypothesis scheme of Hyp, such that $h \notin H$ and $S \cup H \cup \{h\}$ is ε-consistent. Consequently, $[\varepsilon]$false is not derivable from $S \cup H \cup \{h\}$. Therefore $S \cup H \cup \{h\} \cup \{\langle H_\varepsilon \rangle \text{true}\}$ must be consistent. But this would contradict the maximality of H. Hence E is maximally ε-consistent.

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