On stability and continuity of bounded solutions of degenerate complex Monge–Ampère equations over compact Kähler manifolds

Śławomir Dinew a, Zhou Zhang b,∗

a Jagiellonian University, Kraków, Poland
b University of Michigan, Ann Arbor, MI, United States

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Abstract

We obtain a stability estimate for the degenerate complex Monge–Ampère operator which generalizes a result of Kołodziej (2003) [12]. In particular, we obtain the optimal stability exponent and also treat the case when the right-hand side is a general Borel measure satisfying certain regularity conditions. Moreover, our result holds for functions plurisubharmonic with respect to a big form, thus generalizing the Kähler form setting in Kołodziej (2003) [12]. Independently, we also provide more detail for the proof in Zhang (2006) [18] on continuity of the solution with respect to a special big form when the right-hand side is $L^p$-measure with $p > 1$.

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1. Introduction

In this work, we generalize and strengthen Kołodziej’s stability and continuity results concerning bounded solutions for complex Monge–Ampère equations, which are proved in [12] and [11] respectively (see also [13] for a nice summary). The solutions are understood in the...
sense of pluripotential theory, i.e. we do not impose any regularity assumption other than upper semi-continuity and boundedness. It is, however, a classic fact that the image of the Monge–Amplère operator can be well defined as a Borel measure in this setting in sight of the work [1] by Bedford and Taylor.

The equation is considered over a closed Kähler manifold $X$ of complex dimension $n \geq 2$. When $n = 1$, the manifold is a Riemann surface and Monge–Amplère operator is nothing but the classic Laplace operator. Since the latter is linear, the corresponding problems can be dealt with standard techniques.

Suppose $\omega$ is a real smooth closed semi-positive $(1, 1)$-form over $X$, $\Omega$ is a positive Borel measure on $X$ and $f \in L^p(X)$ for some $p > 1$ is non-negative, where the definition of the function space $L^p(X)$ is with respect to $\Omega$. The equation under consideration is

$$(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = f \Omega.$$ 

Using $d = \partial + \bar{\partial}$ and $dc := \sqrt{-1} (\bar{\partial} - \partial)$, we have $ddc = \sqrt{-1} \partial \bar{\partial} u$ and this convention is frequently used in the literature.

As mentioned above, we require regularity of $u$ much less than what is needed to make pointwise sense for the left-hand side of the equation. More precisely, we look for solutions in the function class $\text{PSH}_\omega(X) \cap L^\infty(X)$, where $u \in \text{PSH}_\omega(X)$ means that $\omega u := \omega + \sqrt{-1} \partial \bar{\partial} u$ is non-negative in the sense of distribution.

Of course, there is an obvious condition for the existence of such a solution coming from global integration over $X$, i.e. $\int_X \omega^n = \int_X f \Omega$. This condition follows from Stokes’ theorem in the smooth case, and hence (by smooth approximation) in our case as well.

Kołodziej (cf. [11] and [12]) mainly studied the case when $\omega$ is a Kähler metric (or equivalently $[\omega]$ is a Kähler class) and $\Omega$ is a smooth volume form. The existence of bounded solution in this case is achieved. In fact, even more general function class than $L^{p>1}$ function class has been treated in [11], but for our main interest, we restrict ourselves to $L^{p>1}$-functions. Furthermore, in his case, the bounded solution is always continuous as justified in [11]. So in the discussion of stability there, continuity of the solution can be assumed without any loss of generality.

In the following we state our first main result and refer to the next section for definitions of some notions appearing in the statement.

**Theorem 1.1.** Let $X$ be a compact Kähler manifold and $\omega$ is a big form on $X$. Also assume that $\Omega$ is a positive Borel measure on $X$ dominated by capacity for $L^p$-functions with some constant $p > 1$. Let $Q$ be a positive increasing function with polynomial growth that measures the domination of $\Omega$, and the function $\kappa$ be defined by

$$\kappa(r) = C_n,p \left( \int_{r^{-\frac{1}{p}}} \int_{r^{-\frac{1}{p}}} \frac{1}{y^{-\frac{1}{p} - 1}} dy + \left( Q(r^{-\frac{1}{p}}) \right)^{-\frac{1}{p}} \right),$$

where $C_n$ is a positive constant depending merely on the complex dimension $n$, $p$ and the manifold $(X, \omega)$. Define the function $\gamma$ by $\gamma(t) = C \kappa^{-1}(t)$, with $\kappa^{-1}$ being the inverse function of $\kappa$. Consider any non-negative $L^p(\Omega)$-functions $f$ and $g$ satisfying $\int_X f \Omega = \int_X g \Omega = \int_X \omega^n$. Let $\phi$ and $\psi$ in $\text{PSH}_\omega \cap L^\infty(X)$ satisfy $\omega_{\phi}^n = f \omega^n$ and $\omega_{\psi}^n = g \omega^n$ respectively and be normalized by $\max_X \{ \phi - \psi \} = \max_X \{ \psi - \phi \}$. Then for any $\epsilon > 0$, there are a constant
\( C = C(X, \omega, \| f \|_p, \| g \|_p, \epsilon) \) and a constant \( t_0 \) depending on \( \gamma \) such that for any \( 0 \leq t < t_0 \) the inequality \( \| f - g \|_{L^1} \leq \gamma(t)t^{n+\epsilon} \) implies

\[ \| \phi - \psi \|_{L^\infty} \leq C t. \]

This result essentially says that Kołodziej’s Stability Theorem still holds even if the background form is merely big. Moreover one can relax the smoothness assumptions on the measure to “being dominated by capacity”, and the result is still true. In fact, these generalizations are consequences of results from [3] and [7]. Another major improvement is the exponent from \( n + 3 \) (cf. [12]) to the optimal \( n + \epsilon \) for any small positive \( \epsilon \).

As a natural application, we have the uniqueness of bounded solution for degenerate Monge–Ampère equation. A direct corollary is the following stability estimate, which provides the optimal exponent for the stability estimate.

**Corollary 1.2.** In the same setting as Theorem 1.1, if \( \Omega \) is smooth, then there exists a constant \( c = c(p, \epsilon, c_0) \), where \( c_0 = \max\{\| f \|_p, \| g \|_p\} \), such that

\[ \| \phi - \psi \|_{L^\infty} \leq c \| f - g \|_{L^1}^{1 - \epsilon}. \]  

(1.1)

Such an inequality was recently applied to prove Hölder continuity for solutions of Monge–Ampère equations with right-hand side in \( L^{p>1} \)-spaces (see [14]). The optimal Hölder exponent in that result is yet to be sorted out. However, the bigger the exponent in the inequality (1.1) is, the better Hölder exponent would be. Thus getting an optimal result in (1.1) is quite interesting. As Example 5.2 shows, the exponent obtained above is sharp.

In Section 6, we provide a more detailed proof of a result due to the second named author that appeared previously in [18] (or [19]). The argument given in these references is a little bit too sketchy (and more importantly, scattered in several chapters of the thesis for some other reasons), which makes it hard to follow. Now we state this result with some background information.

**Theorem 1.3.** Let \( X \) be a closed Kähler manifold with \( \dim \mathbb{C} X = n \geq 2 \). Suppose we have a holomorphic map \( F : X \to \mathbb{CP}^N \) with the image \( F(X) \) of the same dimension as \( X \). Let \( \omega_M \) be any (smooth) Kähler form over some neighborhood of \( F(X) \) in \( \mathbb{CP}^N \). Consider the following equation of Monge–Ampère type:

\[ (\omega + \sqrt{-1} \partial \bar{\partial} u)^n = f \Omega, \]

where \( \omega = F^* \omega_M \), \( \Omega \) is a fixed smooth (non-degenerate) volume form over \( X \) and \( f \) is a non-negative function in \( L^p(X) \) for some \( p > 1 \) satisfying \( \int_X f \Omega = \int_X (F^* \omega_M)^n \). Then we have the following statements:

1. **(A priori estimate)** If \( u \) is a weak solution in \( \text{PSH}_\omega(X) \cap L^\infty(X) \) of this equation with the normalization \( \sup_X u = 0 \), then there is a constant \( C \) such that \( \| u \|_{L^\infty} \leq C \| f \|_{L^p}^n \) where \( C \) only depends on \( F, (X, \omega) \) and \( p \).
2. **(Existence of bounded solution)** There exists a bounded (weak) solution for this equation.
3. **(Continuity and uniqueness of bounded solution)** If \( F \) is locally birational, any bounded solution is actually the unique continuous solution.
The proof of part (1) appears in [18], Sections 2 and 3 (pp. 5–12), or in [19], Sections 4.2, 5.3, 6.1 and 6.2 (pp. 144–146, 166–169 and 173–187). The proof of (2) appears in [18], Section 4 (p. 13), or in [19], Sections 4.3, 5.3, 6.1 and 6.2 (pp. 146, 166–169 and 173–187). Here we give a more detailed proof of (3) which is discussed in [18], Section 5 (pp. 13–15), or [19], Section 7.3 (pp. 194–199).

The Monge–Ampère equation in similar setting has been studied extensively in the recent years (see [2,6] and [8]). In particular, the a priori estimate was also obtained independently in [8] (even for more general big forms), and later generalized to more singular right-hand side in [6]. As for the continuity of the solution, despite of serious efforts, the situation is still a little bit unclear in the most general setting. It is not known whether continuity holds when \( \omega \) is a general big form with continuous (even smooth) potentials. This problem has attracted a lot of interest recently, and in fact this is the main motivation to present a more detailed proof of the continuity in the situation above, which contains the case with the most interest.

We wish to point out that the methods used in the proof of the stability Theorem 1.1 are independent of the regularity of solutions. So theoretically, the solutions might be discontinuous in general, but uniformly close to each other if \( f \) and \( g \) are close in \( L^1 \)-norm. Needless to say, this could be a quite intriguing situation. Thus our results strongly support (but in no way justify) the common belief that continuity would indeed hold in general.

The applications of our results could go in several directions. The semi-positive case is particularly interesting in geometry, since it appears very naturally in the study of algebraic manifolds of general type (or big line bundles in general, see [17]). In the mean time, the degeneration of the measure on the right-hand side might be useful in complex dynamics and pluripotential theory. Complex dynamics often deals with such singular measures and it is very helpful to obtain any kind of regularity for the corresponding potentials. The same question also arises in pluripotential theory in the study of extremal functions.

### 2. Preliminaries

Throughout this note, we shall work on a closed Kähler manifold \( X \) with \( \dim \mathbb{C} X = n \geq 2 \). We equip \( X \) with a big form \( \omega \), where “big” is defined below.

**Definition 2.1.** A smooth \( d \)-closed form \( \omega \) is called big if it is pointwise semi-positive and the induced volume has a positive total integral, i.e. \( \int_X \omega^n > 0 \).

One can also define bigness for currents with bounded potentials (see [6]). For simplicity, we restrict ourselves to the smooth case.

We shall use the notions in pluripotential theory introduced by Bedford and Taylor (cf. [1]) and adjust them a little to the manifold case according to the description by Kołodziej (cf. [13]). The most important tool is the relative capacity defined as follows.

**Definition 2.2.** For a Borel subset \( K \) of \( X \), we define its relative capacity with respect to \( \omega \) by

\[
\text{Cap}_{\omega}(K) := \sup \left\{ \int_K (\omega + \partial \bar{\partial} \rho)^n \mid \rho \in \text{PSH}_{\omega}(X), \ 0 \leq \rho \leq 1 \right\}.
\]
Note that Kołodziej originally defined the relative capacity with respect to a Kähler form [12]. The obvious generalization to the setting of more general background form has appeared in [8] and [18].

We study the Monge–Ampère equation with singular measure on the right-hand side. Namely, we assume that $\Omega$ is a Borel measure instead of a smooth volume form. Then we need some restriction, since weak solutions for such an equation might not be bounded anymore (for example, if $\Omega$ is the Dirac delta measure at some point). In fact, there are measures for which the existence of solutions of any kind (bounded or not) is not clear yet. Therefore we impose some natural conditions on $\Omega$ which guarantee the existence of bounded solutions (by Theorem 1.3 for example).

**Definition 2.3.** We say that a Borel measure is dominated by capacity for $L^p$ functions if there exist constants $\alpha > 0$ and $\chi > 0$, such that for any compact $K \subset X$ and non-negative $f \in L^p(\Omega)$ with $p > 1$, one has for some constant $C$ independent of $K$ that

$$\Omega(K) \leq C \cdot \text{Cap}_w(K)^{1+\alpha}, \quad \int_K f \Omega \leq C \cdot \text{Cap}_w(K)^{1+\chi}.$$ 

A very similar notion, where only the first inequality is imposed, has been introduced in [8]. Both are variations of the so-called condition (A) introduced by Kołodziej in [11]. These conditions, which actually are stronger than condition (A), ensure the existence of bounded solutions $u$ for

$$(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = f \Omega.$$ 

See [11] for the case $\omega$ is Kähler, and [8] for the case $\omega$ is big.

Let’s say a few words about the second inequality. When $\Omega$ is a smooth volume form, it is known (again see [11] and [8]) that the first condition is satisfied for every $\alpha > 0$. Hence by an elementary application of the Hölder inequality, the second inequality also holds for every $\chi > 0$. Hölder inequality indeed implies, regardless of the smoothness of $\Omega$, that the second inequality is a consequence of the first one provided $p$ is big enough (more precisely, $(1+\alpha)(p-1) > 1$). In any case, one has to impose some condition, since a priori $f \Omega$ can be a lot more singular than $\Omega$.

Meanwhile, as in [12], Lemma 2.2 or [13], Lemma 6.5, the exponent $\chi > 0$ is used to construct the admissible function $Q$ which measures the domination by capacity. In our situation, it has growth like $t^{n \chi}$, and so the function

$$\kappa(r) = C_{n,p} \left( \int_{r^{-\frac{1}{n}}}^{\infty} y^{-1} (Q(y))^{-\frac{1}{p}} \, dy + (Q(r^{-\frac{1}{n}}))^{-\frac{1}{p}} \right) \approx r^{\frac{\chi}{n}},$$ 

and its inverse $\gamma(t) \approx t^{\frac{n}{\chi}}$. When the volume form $\Omega$ is smooth, one can take arbitrary $\chi > 0$ (of course the larger, the better). Thus one can produce a function $\gamma(t)$ with growth like $t^\epsilon$, for any small $\epsilon > 0$. When $\chi$ is bounded from above, one can take $\gamma(t) \approx t^{\frac{n}{\chi}}$. In order to avoid too

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1 We assume it is a fixed constant depending on the measure $\mu$. 

much technicalities in the proof, we shall work with the assumption that \( \chi \) can be taken to be arbitrarily large. At the end (see Remark 5.1), we shall explain how to modify the argument for some fixed \( \chi \) and obtain the stability exponent in general.

The next theorem is quoted from [12], which allows us to estimate the capacity of sub-level sets of plurisubharmonic functions.

**Theorem 2.4.** Suppose \( \phi, \psi \in PSH_\omega(X) \) and \( \phi \) satisfies \( 0 \leq \phi \leq C \), then for \( s < C + 1 \), we have

\[
\text{Cap}_\omega(\{ \psi + 2s < \phi \}) \leq \left( \frac{C + 1}{s} \right)^n \int_{\{ \psi + s < \phi \}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n.
\]

The following proposition is needed later.

**Proposition 2.5.** Let \( \phi, \psi \in PSH_\omega(X) \) satisfy \( 0 \leq \phi \leq a, 0 \leq \psi \leq a \). Then for any constants \( m, n, t > 0 \) we have

\[
\text{Cap}_\omega(\{ \psi + (m + n)t < \phi \}) \leq \left( \frac{a + 1}{nt} \right)^n \int_{\{ \psi + mt < \phi \}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n.
\]

**Proof.** If \( nt \geq a + 1 \), then \( \{ \psi + (m + n)t < \phi \} \) is empty because of the assumption on \( \psi \). If \( nt < a + 1 \), then for any function \( \rho \in PSH_\omega(X) \), \( -1 \leq \rho \leq 0 \), we get the chain of sets

\[
\{ \psi + (m + n)t < \phi \} \subset \{ \psi + mt < \left( 1 - \frac{nt}{a + 1} \right) \phi + \frac{nt}{a + 1} \rho \} \subset \{ \psi + mt < \phi \}
\]

and the proof is the same as the one of Theorem 2.4 (cf. [12]). The argument goes through for any function \( \rho \) as above, and so one can get the conclusion for relative capacity by definition. \[\square\]

In Section 6 we shall work with (locally) birational mappings. Although these are fairly standard objects, we feel that it is worth giving the definitions as well as to show some related examples.

**Definition 2.6.** A meromorphic mapping \( F : X \rightarrow Y \) between two complex varieties \( X \) and \( Y \) is called birational if it has an inverse (in the sense of meromorphic map) such that \( F^{-1} : Y \rightarrow X \) is also meromorphic.

A typical example of such a mapping is as follows. If \( X \) carries a big line bundle \( L \), then the Iitaka Fibration Theorem (cf. [16]) states that the linear series corresponding to \( L^m \) can generate (for sufficiently large \( m \in \mathbb{N} \)) a meromorphic morphism into \( \mathbb{C}P^N \) which is birational onto the image. If moreover \( L \) is semi-ample then the mapping is holomorphic, i.e. the map is defined over \( X \).

**Definition 2.7.** A meromorphic mapping \( F : X \rightarrow Y \) is called locally birational if for every small enough neighborhood \( U \) of any point on \( F(X) \), each connected component of \( F^{-1}(U) \) is birational to \( U \).
The following example is called a double point. It shows that these two notions are indeed different.

**Example 2.8.** Consider the following map

\[ F : C \ni t \rightarrow (t^2 - 1, t(t^2 - 1)) \in C^2. \]

The image \( F(\mathbb{C}) \) sits in the variety \( \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_1^3 = z_2^2 \} \). Clearly \( F \) is a bijection to its image, except for the points \( 1 \) and \( -1 \) being mapped to \((0, 0)\). Then for any small enough neighborhood \( U \) of \((0, 0)\), the pre-image is disconnected and the connected components are not birational to \( U \). However this map is birational. For the continuity result ((3) of Theorem 1.3), this kind of situation is excluded.

Theorem 1.3 is for locally birational holomorphic mapping. In sight of the discussion on pp. 124–129 in [16] on algebraic fiber space which always have fibers being connected, the local birationality is not such a restrictive assumption when considering problems with algebraic geometry background. For further introduction regarding pluripotential theory on Kähler manifolds, we refer to [13]. A good reference for the geometric part is [16].

### 3. Stability for non-degenerate Monge–Ampère equations

We begin with stating Kołodziej’s original stability theorem (Theorem 4.1 of [12]). Note however that in the Kähler case we know that the weak solutions are actually continuous (cf. Section 2.4 in [11]).

**Theorem 3.1.** Let \( \omega \) be a Kähler form on a compact manifold \( X \) and \( A \) be a fixed positive constant. Then for any non-negative \( L^p \)-functions \( f \) and \( g \) with \( p > 1 \) satisfying \( \int_X f \omega^n = \int_X g \omega^n \) and \( \| f \|_p, \| g \|_p < A \), let \( \phi \) and \( \psi \) in \( PSH_\omega(X) \cap L^\infty(X) \) satisfy \( \omega \phi^n = f \omega^n \) and \( \omega \psi^n = g \omega^n \) respectively and be normalized by \( \max_X \{ \phi - \psi \} = \max_X \{ \psi - \phi \} \). Then there exists \( t_0 > 0 \) depending on \( \gamma \) such that for every \( 0 \leq t < t_0 \) if \( \| f - g \|_{L^1} \leq \gamma(t)t^{n+3} \), then

\[ \| \phi - \psi \|_{L^\infty} \leq C t, \]

for some \( C \) depending on \( \gamma, \omega, X, \) and \( A \).

Now one gets the following corollary (cf. [12], Corollary 4.4).

**Corollary 3.2.** For any \( \epsilon > 0 \), there exists \( c = c(\epsilon, p, c_0) \) with \( c_0 \) being the upper bound for \( L^p \)-norms of \( f \) and \( g \) such that

\[ \| \phi - \psi \|_{L^\infty} \leq c \| f - g \|_{L^1}^{\frac{1}{p+3+\epsilon}} \]

provided \( \phi \) and \( \psi \) are normalized as in the theorem above.

\( \gamma \) is defined as in the statement of Theorem 1.1.
Before proceeding further, we point out an observation leading to a small improvement on the stability exponent in the above corollary. Note that in the definition of set \( G = \{ f < (1 - t^2) g \} \) in line 2 on p. 679 in [12], one can change \( t^2 \) to \( \frac{1}{b} \) for a sufficiently large constant \( b \), and the same argument still goes through except in the last step, one has to change the set \( E_4 \) in line 5 on p. 680 in [12] to \( E_s \) for some constant \( s \) depending only on \( b \). Hence using Proposition 2.5, one can get rid of the constant in the term \( \gamma(t)t^n \) which is affected by \( b \). So \( \| f - g \|_1 \leq \gamma(t)t^{n+2} \) implies \( \| \phi - \psi \|_\infty \leq Ct \). In particular, the estimate in Corollary 3.2 holds with the exponent \( \frac{1}{n+2+\epsilon} \).

4. Adjustment to the degenerate case

Now we begin to adjust Kołodziej’s argument in [12] for the situation in Theorem 1.1. The argument (with the exponent \( n+2 \)) can be repeated line-by-line except for two issues. First, one has to justify Comparison Principle in this setting. Second, prove the inequality in line 4 on p. 679 in [12] for merely bounded \( \omega \)-plurisubharmonic functions. In the following, we treat them one by one.

4.1. Comparison Principle

In [3], the authors constructed decreasing smooth approximation for bounded functions plurisubharmonic with respect to a Kähler metric. Using this, they were able to justify Comparison Principle for any bounded functions plurisubharmonic with respect to a Kähler form.

Though the version we want would be for some background form \( \omega \geq 0 \), it still follows from their version of Comparison Principle because we can perturb \( \omega \) by \( \epsilon \omega_0 \) with \( \omega_0 > 0 \) and any constant \( \epsilon > 0 \). Functions plurisubharmonic with respect to \( \omega \) would still be plurisubharmonic with respect to \( \omega + \epsilon \omega_0 \). Letting \( \epsilon \to 0 \) in the conclusion of their version of Comparison Principle, we can conclude the following result, which deals with the first issue of running through Kołodziej’s argument.

**Theorem 4.1.** For \( \phi, \psi \in \text{PSH}_\omega(X) \cap L^\infty(X) \), where \( X \) is a closed Kähler manifold and \( \omega \geq 0 \) is a big form over \( X \), one has

\[
\int_{\{ \phi < \psi \}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq \int_{\{ \phi < \psi \}} (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n.
\]

Clearly, we only need \( \omega \geq 0 \) for this theorem in general.

4.2. Inequalities for mixed measures

The first observation is that although Kołodziej (as in [12]) considered equations of the form

\[
\omega^n_{\psi} = f \omega^n, \quad \omega^n_{\phi} = g \omega^n,
\]

the volume form \( \omega^n \) would play no significant role in the proof. The essential step is to justify the following inequality.

\[ X \] being Kähler guarantees the existence of \( \omega_0 \).
For $\phi$ and $\psi$ continuous $\omega$-plurisubharmonic functions with $f$ and $g$ being integrable functions on $X$, suppose (locally) we have

$$\omega^n_\psi \geq f \omega^n, \quad \omega^n_\phi \geq g \omega^n,$$

then the following inequalities for mixed measures hold for any $k \in \{0, 1, \ldots, n\}$,

$$\omega^k_\psi \wedge \omega^{n-k}_\phi \geq f^{k/n} g^{n-k/n} \omega^n.$$

We need to generalize the above result for more general measures and moreover for merely bounded functions $\phi$ and $\psi$. The following theorem is essentially quoted from [7].

**Theorem 4.2.** Suppose the non-negative Borel measure $\Omega$ is dominated by capacity, and let $\phi$ and $\psi$ be two bounded $\omega$-plurisubharmonic functions on a Kähler manifold. If the following inequalities hold

$$\omega^n_\psi \geq f \Omega, \quad \omega^n_\phi \geq g \Omega,$$

for some $f, g \in L^{p>1}(\Omega)$, then $\forall k \in \{0, 1, \ldots, n\}$

$$\omega^k_\psi \wedge \omega^{n-k}_\phi \geq f^{k/n} g^{n-k/n} \Omega.$$

In [12] (Lemma 1.2), this result was proved under the assumption that both $\phi$ and $\psi$ are continuous and $\Omega = \omega^n$. The result is clearly local, and so can be rephrased for a ball in $\mathbb{C}^n$. Then the argument makes use of approximation, for which a solution for the Dirichlet problem with continuous boundary data is needed.

Since we deal with merely bounded functions, one can’t expect continuity on the boundary of the ball in general. Fortunately, as observed in [7], we can line-by-line follow the approximation argument from [12] whenever the measure on the right-hand side is the Lebesgue measure. Indeed, approximation at the boundary will not converge uniformly towards discontinuous boundary data, but the sequence of approximation solutions is still decreasing. This implies convergence with respect to capacity using a classic result in [1], which is enough for the argument to go through. In the case when $\omega^n$ is changed to a general measure dominated by capacity one cannot rely only on the argument from [12]. Meanwhile domination by capacity would force the measure $\Omega$ to vanish on pluripolar sets, hence one can use Theorem 1.3 in [7] to draw the same conclusion. We refer to [7] for more detail.

### 5. Improvement on the stability exponent

In this section, we improve the exponent from Kołodziej’s Stability Theorem, i.e. Theorem 3.1. The strategy is to iterate the original argument, defining at each step a new function $\rho$ (cf. line 14, p. 678 in [12]) and use the previous step to get estimates for $\|\rho - \psi\|_\infty$, which in turn can be used to choose the new set $E$ (cf. line 1, p. 679 in [12]) in a better way. To the authors’ knowledge such an iteration process is quite original and could be of some interest.

To begin with, we fix a small constant $\epsilon > 0$. The argument is divided into the following three parts as follows.
The first part is the original argument quoted before with the small improvement mentioned after Corollary 3.2, which is the starting point. In the sequel, the original argument will often be referred to as Step 1.

The second part is the iteration step. Since the first step differs slightly from all the later ones, we give a detailed description of it below and then illustrate how to proceed further. The mechanism is based on the fact that \( \| f - g \|_1 \leq \gamma(t)t^\beta \) would yield \( \int \{ \psi + kat < \phi \} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq c_0 t^n \) for some constants \( k \) and \( c_0 \). In the following, \( c_i \)'s denote constants independent of the relevant quantities.

Then applying Proposition 2.5, we have a constant \( k_1 > 0 \) depending only on \( c_0 \) such that the set \( \{ \psi + ((k + k_1)a + 2)t < \phi \} \) is empty (cf. p. 680 in [12]), and so \( \| \phi - \psi \|_\infty \leq ((k + k_1)a + 2)t \). Here \( a \) is the \( L^\infty \)-bound of the solution.

Now we try to find \( \beta < n + 2 \) as small as possible for which this implication holds with uniform control on \( c_0 \) and larger \( k \) if needed. Note that from now on, instead of \( \omega^n \), we use the measure \( \Omega \). It follows from the discussion above that Step 1 is not affected by that.

Assume \( \| f - g \|_1 \leq \gamma(t)t^\beta \) with \( t < 1 \), for some \( \beta \) to be chosen later. If \( l = t^{\frac{\beta}{n+2}} \), with \( \beta < n + 2 \), we have \( \| f - g \|_1 \leq \gamma(l)l^{n+2} \), and from Step 1 we know that

\[
\int_{E_2} g \Omega \leq \gamma(l)l^n. \tag{5.1}
\]

where, as in Kołodziej’s original argument, we denote \( E_k := \{ \psi < \phi - kat \} \). Hence

\[
\int_{E_2} g \Omega \leq c_1 t^{\frac{\beta}{n+2}}, \tag{5.2}
\]

recalling that \( \gamma(t) \) is bounded and decreases to 0 as \( t \searrow 0 \).

Now we shall find such a \( \beta < n + 2 \). Let \( \delta \) be a small positive constant to be fixed later. Consider the following “new” function, comparing with the function \( g_1 \) in Kołodziej’s proof,

\[
g_1(z) = \begin{cases} (1 + \frac{t^\delta}{2})g(z), & z \in E_2, \\ c_2 g(z), & z \in X \setminus E_2, \end{cases}
\]

where \( 0 \leq c_2 \leq 1 \) is chosen such that \( \int_X g_1 \Omega = 1 \). The constant \( \frac{1}{2} \) is taken to assure that the integral over \( E_2 \) is less than 1. Note that in spite of the fact that the case \( t \) being small is of our main interest, when \( \delta \) is also small the quantity \( t^\delta \) may not be controlled by a constant less than 1.

As in Step 1, we find a solution \( \rho \) to the equation \( \omega^n = g_1 \omega^n \), \( \max_X \rho = 0 \). Also, \( \rho \succeq -a \) for \( \| g_1 \|_\rho < 3a \) and renormalize \( \rho \) by adding a constant so that \( \max_X (\psi - \rho) = \max_X (\rho - \psi) \), which can be done in a uniformly controlled way.

\[\text{In the improved original proof as Step 1, } \beta = n + 2.\]
Now by Step 1, we have
\[
\|\rho - \psi\|_\infty \leq c_3\left(\int_{E_2} |g - g_1| \omega_n + \int_{X \setminus E_2} |g - g_1| \omega_n\right)^{\frac{1}{\eta + 2 + \epsilon}}
\]
\[
= c_3\left(\frac{t}{2} \int_{E_2} g \omega_n + (1 - c_2) \int_{X \setminus E_2} g \omega_n\right)^{\frac{1}{\eta + 2 + \epsilon}}
\]
\[
= c_3\left(\frac{t}{2} \int_{E_2} g \omega_n + \int_{X \setminus E_2} g \omega_n - \int_{X \setminus E_2} g_1 \omega_n + \left(1 + \frac{t}{2}\right) \int_{E_2} g \omega_n\right)^{\frac{1}{\eta + 2 + \epsilon}}
\]
\[
= c_3\left(\frac{t}{2} \int_{E_2} g \omega_n\right)^{\frac{1}{\eta + 2 + \epsilon}} \leq c_4 t^{\frac{\delta + \frac{\eta \delta}{n + 2 + \epsilon}}{\eta + 2 + \epsilon}}.
\]

If \(\delta\) is sufficiently small and \(\beta > n\) the last exponent is less than 1\(^5\) and we define \(\alpha = 1 - \frac{\delta + \frac{\beta n}{n + 2 + \epsilon}}{n + 2 + \epsilon}\).

For \(s = \frac{2c_4}{a} + 2\), we obtain the following chain of set inclusions,
\[
E_s = \{\psi + \text{sat} < \phi\}
\]
\[
\subset E := \left\{\psi < \left(1 - \frac{1}{2} t^\alpha\right) \psi + \frac{1}{2} t^\alpha \phi + \frac{1}{2} c_4 t - \text{sat}\left(1 - \frac{1}{2} t^\alpha\right)\right\}
\]
\[
\subset \left\{\psi < \left(1 - \frac{1}{2} t^\alpha\right) \phi + \frac{1}{2} t^\alpha \psi + c_4 t - \text{sat}\left(1 - \frac{1}{2} t^\alpha\right)\right\}
\]
\[
= \left\{\psi + \left(s - \frac{c_4}{a(1 - \frac{1}{2} t^\alpha)}\right) \alpha t < \phi\right\} \subset E_2, \quad (5.3)
\]

where the term \(\frac{1}{2}\), as before, is introduced in order to estimate the term \(1 - \frac{1}{2} t^\alpha\) from below. Consider the “new” set
\[
G := \left\{f < \left(1 - \frac{t^{\alpha + 3\delta}}{8n2^{\frac{n-1}{2}}}ight) g\right\}.
\]

Since \(h(t) = (1 + \frac{t}{2})^{\frac{3}{2}} - 1 - \frac{1}{4n2^{\frac{n-1}{2}}} t^{2\delta}\) increases in \([0, 1]\) and hence is non-negative there, we conclude as in Step 1 that on \(E \setminus G\),

\(^5\) If \(\beta < n\), we are already done.
\[
\left(\frac{\omega_{\frac{1}{2}t^{\alpha} + (1 - \frac{1}{2}t^{\alpha})\phi}}{8n2^{n-1}}\right)^n \geq \left(\left(1 - \frac{1}{2}t^{\alpha}\right)\left(1 - \frac{t^{\alpha+3\delta}}{8n2^{n-1}}\right) + \left(1 + \frac{t^\delta}{2}\right)\left(\frac{1}{2}t^{\alpha}\right)\right)^n g\Omega \\
\geq \left(\left(1 - \frac{1}{2}t^{\alpha}\right)\left(1 - \frac{t^{\alpha+3\delta}}{8n2^{n-1}}\right) + \left(1 + \frac{1}{4n2^{n-1}}t^{2\delta}\right)\left(\frac{1}{2}t^{\alpha}\right)\right)^n g\Omega \\
\geq \left(1 + \frac{t^{\alpha+2\delta}}{16n2^{n-1}}\right)g\Omega. \tag{5.4}
\]

Still as in Step 1, on \(G\) we have
\[
\frac{t^{\alpha+3\delta}}{8n2^{n-1}} \int_G g\Omega \leq \int_G (g - f)\Omega \leq \gamma(t)t^\beta, \tag{5.5}
\]
so using (5.4), (5.5) and Comparison Principle, we obtain
\[
\left(1 + \frac{t^{\alpha+2\delta}}{16n2^{n-1}}\right) \int_{E\setminus G} g\Omega \leq \int_E g\Omega \leq \int_{E\setminus G} g\Omega + c_5\gamma(t)t^{\beta-\alpha-3\delta}. \tag{5.6}
\]

Finally, we arrive at
\[
\int_{E\setminus G} g\Omega \leq c_6\gamma(t)t^{\beta-2\alpha-5\delta},
\]
\[
\int_{E_s} g\Omega - 8n2^{n-1}t^{\beta-\alpha-3\delta} \leq \int_{E\setminus G} g\Omega \leq \int_{E\setminus G} g\Omega.
\]

Combine them to arrive at
\[
\int_{E_s} g\Omega \leq c_7\gamma(t)t^{\beta-2\alpha-5\delta}.
\]

If \(\beta - 2\alpha - 5\delta = n\), we can proceed as in Step 1 to get \(\max(\phi - \psi) = \max(\psi - \phi) \leq ((s + s_1)a + 2)t\) for some \(s_1\) depending only on \(c_5\) and \(\|\phi - \psi\|_{\infty} \leq C(\epsilon)\|f - g\|_{L^{1/\beta+1}}^{1/\beta+1}\). Moreover,

\[
\beta\left(1 + \frac{2n}{n+2+\epsilon}\right) = n + 2 + 5\delta - \frac{2\delta}{n+2+\epsilon}.
\]

It is clear that if \(\delta\) is sufficiently small, \(\beta\) is smaller than \(n + 2\). Hence we get an improvement.

In the third and last part we iterate the argument.

Consider \(\|f - g\|_1 \leq \gamma(t)t^{\beta k+1}\). As before, for \(l = t^{\beta k+1}\), \(\int_{E_l} g\Omega \leq C\frac{n^\beta}{\beta k+1}\), comparing with (5.1), where \(r\) is chosen so that we can use the estimate on appropriate sub-level set from the
previous step. Choosing $\delta_{k+1}$ small enough and repeating the above argument, one gets

$$\beta_{k+1} = n + 2\alpha_{k+1} + 5\delta_{k+1},$$

where $\alpha_{k+1} = 1 - \frac{\delta_{k+1} + \frac{\delta_{k+1}^n}{n+2+\epsilon}}{n+2+\epsilon}$. This yields

$$\beta_{k+1} \left(1 + \frac{2n}{\beta_k (\beta_k + \epsilon)}\right) = n + 2 + 5\delta_{k+1} - \frac{2\delta_{k+1}}{\beta_k + \epsilon}. \quad (5.7)$$

Choosing $\{\delta_k\}$ to be a sequence of sufficiently small constants decreasing to 0, one can obtain that $\{\beta_k\}$ is convergent as $n \geq 2$. Suppose $A$ is the limit of the sequence $\{\beta_k\}$, one gets

$$A \left(1 + \frac{2n}{A(A + \epsilon)}\right) = n + 2$$

which implies

$$A = \frac{n + 2 - \epsilon + \sqrt{(n - 2 + \epsilon)^2 + 8\epsilon}}{2}.$$ 

Clearly, when $\epsilon \to 0^+$, $A \to n$, so $\beta_k$ can be arbitrarily close to $n$ for $k$ big enough if we take small enough $\epsilon$. Hence we have proved Corollary 1.2.

**Remark 5.1.** In the case when the measure $\Omega$ is dominated by capacity for $L^{n+1}$ functions but the constant $\chi$ is fixed, one can construct $Q(t), \kappa(t)$ and $\gamma(t)$ in such a way that $\gamma(t) \approx t^{\frac{n}{n+2}}$. Then one can use the same iteration technique as above with the exception that inequality (5.2) should be improved to

$$\int_{E_2} g \Omega \leq C t^{\frac{n\beta}{(n+2)^+} + \frac{\beta n}{n+2}},$$

where the factor $t^{\frac{n\beta}{(n+2)^+}}$ comes from the estimate of $\gamma$. The recurrence (5.7) now reads

$$\beta_{k+1} \left(1 + \frac{2n(1 + \frac{1}{\chi})}{\beta_k (\beta_k + \frac{n}{\chi})}\right) = n + 2 + 5\delta_{k+1} - \frac{2\delta_{k+1}}{\beta_k + \frac{n}{\chi}}. \quad (5.8)$$

Again this is a convergent sequence and it can be seen that $\lim_{k \to \infty} \beta_k = n$. Hence the stability estimate in this case reads

$$\|\phi - \psi\|_\infty \leq c(\epsilon, c_0, X, \mu) \|f - g\|_{L^1(d\mu)}^{\frac{1}{\frac{n}{n+2} + \epsilon}}. \quad (5.9)$$

The following example shows that the exponent obtained in our corollary is fairly sharp.
Fix appropriate positive constants $B$, $D$ such that $D < B$ and $2^{2\alpha}B < \log 2 + D$ for some fixed $\alpha \in (0, 1)$. We have the function

\[
\hat{\rho}(z) := \begin{cases} 
B\|z\|^{2\alpha}, & \|z\| < 1, \\
\max\{B\|z\|^{2\alpha}, \log\|z\| + D\}, & 1 \leq \|z\| \leq 2, \\
\log\|z\| + D, & \|z\| > 2,
\end{cases}
\]

is plurisubharmonic in $\mathbb{C}^n$ and of logarithmic growth. One can smooth out $\hat{\rho}$ so that the new function $\rho$ is again of logarithmic growth, radially symmetric, smooth away from the origin and $\rho(z) = B\|z\|^{2\alpha}$ for $\|z\| < \frac{3}{2}$.

Via the standard inclusion $\mathbb{C}^n \ni z \to [1 : z] \in \mathbb{CP}^n$, one identifies $\rho(z)$ with

\[
\bar{\rho}([z_0 : z_1 : \cdots : z_n]) := \rho\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) = \frac{1}{2} \log\left(1 + \frac{\|z\|^2}{|z_0|^2}\right) \in \text{PSH}(\mathbb{CP}^n, \omega_{FS}),
\]

where $\omega_{FS}$ is the Fubini–Study metric on $\mathbb{CP}^n$, and the values of $\bar{\rho}$ on the hypersurface $\{z_0 = 0\}$ are understood as limits of values of $\bar{\rho}$ when $z_0$ approaches 0. It is clear that $\omega_{\bar{\rho}}^n = (dd^c \rho)^n$ in the chart $z_0 \neq 0$ and in fact one can ignore what happens on the hypersurface at infinity.

Now for a vector $h \in \mathbb{C}^n$ (with small length) one can define $\rho_h(z) := \rho(z + h)$ and similarly the corresponding $\bar{\rho}_h$. Note that when $\|h\| \to 0$, $\bar{\rho}_h \to \bar{\rho}$. One also has

\[
B\|h\|^{2\alpha} \leq \|\bar{\rho}_h - \bar{\rho}\|_\infty. \quad (5.10)
\]

The Monge–Ampère measures of $\bar{\rho}$ and $\bar{\rho}_h$ are smooth except at the origin and $-h$ respectively, and belong to $L^p(\omega_{FS}^n)$ for some $p > 1$ depending on $\alpha$. Clearly $\int_{\mathbb{CP}^n} |\omega_{\bar{\rho}}^n - \omega_{\bar{\rho}_h}^n| = \int_{\mathbb{C}^n} |(dd^c \rho)^n - (dd^c \rho_h)^n|$. To estimate the term on the right-hand side, we divide $\mathbb{C}^n$ into three pieces to estimate the total integral:

\[
\int_{\mathbb{C}^n} |(dd^c \rho)^n - (dd^c \rho_h)^n| = \int_{\{\|z\| \leq 2\|h\|\}} \cdot \|z\| + \int_{\{2\|h\| < \|z\| \leq 1\}} \cdot \|z\| + \int_{\{\|z\| > \frac{1}{2}\}} \cdot \|z\|.
\]

Using the fact that $\rho$ and $\rho_h$ are smooth functions in a neighborhood of $\{|z| > \frac{1}{2}\}$, one can easily estimate the last term by $C_0\|h\|$ for a constant $C_0$ independent of $h$. For the first two terms, we have $(dd^c \rho)^n = B^n \|z\|^{2n(\alpha - 1)}$ and $(dd^c \rho_h)^n = B^n \|z + h\|^{2n(\alpha - 1)}$, where the standard Euclidean measure is omitted. Now for the first term, we use a computation in [15],

\[
\int_{\{\|z\| \leq 2\|h\|\}} |(dd^c \rho)^n - (dd^c \rho_h)^n| \leq 2B^n \int_{\{\|z\| \leq 3\|h\|\}} \|z\|^{2n(\alpha - 1)} = C_1\|h\|^{2n\alpha}.
\]

For the second term, we have
\[ \int_{\{2\|h\| \leq \|z\| \leq \frac{1}{2}\}} \left| (dd^c \rho)^n \right| = B^n \int_{\{2\|h\| \leq \|z\| \leq \frac{1}{2}\}} \left| z^{2n(\alpha - 1)} - \|z + h\|^{2n(\alpha - 1)} \right| \]
\[ \leq B^n \int_{\{2\|h\| \leq \|z\|\}} \int_0^1 \left| \nabla z + th^{2n(\alpha - 1)} \right| \|h\| \, dt \]
\[ \leq C_2 \|h\| \int_{\{\|h\| \leq \|z\|\}} \|z^{2n(\alpha - 1) - 1} \leq C_3 \|h\|^{2n\alpha} , \]

provided \( \alpha < \frac{1}{2n} \) so that the integral is finite. Finally we conclude for small \( \|h\| \),
\[ \int \omega_n^\alpha - \omega_n^\alpha \leq C_1 \|h\|^{2n\alpha} + C_3 \|h\| \leq C_4 \|h\|^{2n\alpha} . \tag{5.11} \]

Suppose that we have a stability estimate \( \|\phi - \psi\|_{\infty} \leq C_5 \|f - g\|^{\frac{1}{m}} \). Then combining with (5.10) and (5.11), one gets
\[ \|h\|^{2\alpha} \leq C_6 (\|h\|^{2n\alpha})^{\frac{1}{m}}, \quad \alpha \in \left(0, \frac{1}{2n}\right) . \]

As \( \|h\| \to 0 \), this can hold only if \( m \geq n \). Corollary 1.2 gets us as close to \( n \) as possible. It remains interesting to see whether \( n \) itself is allowed as the exponent.

**Remark 5.3.** A stability estimate of a different type was shown in [8]. In the setting as above with \( \Omega \) being \( \omega_n \),
\[ \|\phi - \psi\|_{\infty} \leq c(\epsilon, c_0, \omega) \|\phi - \psi\|_{L^\infty(\omega^n)} \tag{5.12} \]
where \( c_0 \) is a constant that controls \( L^p \)-norms of Monge–Ampère measures of \( \phi \) and \( \psi \). Using the same reasoning as in [8], one can show more generally that
\[ \|\phi - \psi\|_{\infty} \leq c(\epsilon, c_0, \omega) \|\phi - \psi\|_{L^1(\omega^n)} , \quad \forall s > 0 . \tag{5.13} \]

Using the same example and similar estimates one can show that this exponent is also sharp, provided that \( p < 2 \) and \( s > \frac{2np}{2-p} \). It is, however, very likely that these exponents are sharp in general.

\[ ^6 \text{The reason for these restrictions is that the second integral we estimate as in the example would be divergent otherwise.} \]
6. Continuity of solutions

In this section we give more detail for the proof of the continuity statement in Theorem 1.3. Recall that in our setting there exists a holomorphic mapping \( F : X \to \mathbb{P}^N \) such that \( \omega = F^* \omega_M \) with \( \omega_M \) being a Kähler form in the projective space. Note that \( F \) by assumption is locally birational.

Consider the image \( Y = F(X) \). By the Proper Mapping Theorem \( Y \) is a (singular in general) subvariety in \( \mathbb{P}^N \). It is also clear that \( Y \) is irreducible and locally irreducible variety where the latter follows from the local birationality assumption. Recall that an upper semi-continuous function \( u \) on a singular variety \( W \) is called weakly plurisubharmonic if for every holomorphic disk \( f : \Delta \to W \), the function \( f^* u := u \circ f \) is a subharmonic function (see [9]). Theorem 5.3.1 in that paper states that for any analytic space, any such function \( u \) can be extended locally to a plurisubharmonic function in the ambient space, i.e. in our situation, for every \( x \in Y \) there exists a small Euclidean ball \( B \) in \( \mathbb{P}^N \), centered at \( x \) and a function \( v \in PSH(B) \), such that \( v|_{B \cap Y} = u \).

The continuity is then proved by a contradiction argument. Suppose \( \phi \) is a discontinuous solution of the Monge–Ampère equation under study. Since we already know that \( \phi \) is bounded, we can also assume that it is positive by adding a uniform constant. Define \( d = \sup(\phi - \phi_s) > 0 \), where \( \phi_s \) denotes the lower semi-continuous regularization of \( \phi \). Note that the supremum is attained, and in the closed set

\[
E := \{ \phi - \phi_s = d \}
\]

there exists a point \( x_0 \) such that \( \phi(x_0) = \min_E \phi \).

By assumption there exist analytic sets \( Z \subset X \) and \( W \subset Y = F(X) \) such that \( F|_{X \setminus Z} \to Y \setminus W \) is a biholomorphism and moreover \( S := \{ \omega^n = 0 \} \subset Z \).

There are two cases. In the case of \( x_0 \notin S \), \( \omega \) is strictly positive in a small ball centered at \( x_0 \) and repeating the argument from Section 2.4 in [11], we obtain a contradiction. So from now on we assume that \( x_0 \in S \).

Consider \( F(x_0) = z \) and take a neighborhood \( U \) of \( z \) in \( Y \), such that each component of its pre-image is birational to it. Choose the one, \( \bar{U} \), containing \( x_0 \). For the rest of the argument we restrict ourselves to

\[
F : \bar{U} \to U.
\]

Consider the push-forward of \( \phi \) on \( U \) defined below

\[
(F_*\phi)(z) := \begin{cases}
\phi(w), & z \in U \setminus W, \ w \in \bar{U} \setminus Z, \ F(w) = z, \\
\limsup_{\xi \in \bar{U} \setminus Z, F(\xi) \to z} \phi(\xi), & \text{otherwise}
\end{cases}
\]

and a local potential \( \eta \) for the Kähler form \( \omega_M \) on \( U \). Eventually, we are going to choose \( \eta \) properly, but at this moment, that is not necessary. The following lemma is important.

**Lemma 6.1.** \( \eta + F_*\phi \) is weakly plurisubharmonic on \( U \).

**Proof.** Weak plurisubharmonicity is a local property, and so it is enough to check it in a small neighborhood of any point in \( U \).
For any regular point of $U$, this is evident for the (open) part which is biholomorphic to the (open) part in $\overline{U}$ because biholomorphism preserves plurisubharmonicity. One considers a Euclidean domain in this case, and so the plurisubharmonicity for the whole neighborhood would follow from the semi-continuity of the function and the classic unique extension result for plurisubharmonic functions through subvarieties (cf. [5], Chapter I, Theorem (5.24)).

However at singular points of $U$, one might a priori run into trouble as the example of a double point shows. Indeed, take the double point variety as in Example 2.8. Fix any subharmonic function $w$ on $\mathbb{C}$ satisfying $w(1) \gg w(-1)$. Now the value of the pushforward at $(0, 0)$ equals $w(-1) = \max\{w(-1), w(1)\}$. If this pushforward were weakly subharmonic then on a small disk centered at 1 the function

$$\tilde{w}(t) := \begin{cases} w(t), & t \neq 1, \\ w(-1), & t = 1 \end{cases}$$

would be subharmonic itself. But $\tilde{w}$ does not satisfy the sub-mean value inequality at 1. Hence the push-forward of a subharmonic function $w$ on $\mathbb{C}$ cannot be weakly subharmonic on the image if $w(1) \neq w(-1)$. The assumption of local birationality is used there to rule out such case.

Observe that local birationality actually forces the analytic set $Y$ to be locally irreducible. Then our lemma would follow from a classic theorem (see [4], Theorem 1.7) stating that on a locally irreducible variety $Y$, for a locally bounded plurisubharmonic function $w$ defined on $\text{Reg} Y$, i.e. the regular part of $Y$, the extension via $\limsup$ procedure $w(z) := \limsup_{\zeta \to z, \zeta \in \text{Reg} Y} w(\zeta)$ is indeed a weak plurisubharmonic function.

A more direct argument can also be found in [19], pp. 194–197, where one goes through the definition of weak plurisubharmonic function quoted before using desingularization.

\begin{remark}
Simply speaking, the key point is that the local birationality assumption guarantees that the pre-image of each point in $F(X)$ is connected (from topological consideration) which could be just a point, when restricted to the component $\overline{U}$. Then along that variety, $\eta + \varphi$ is plurisubharmonic and so has to be a constant. This is essentially why this natural push-forward construction preserves plurisubharmonicity. We would also like to point out the classic fact that the extension of a bounded plurisubharmonic function through a subvariety is unique (as in [5]), which shows up a lot in the detail.

Recall that $\omega_M$ is the Kähler metric which defines $\omega$, i.e. $\omega = F^* \omega_M$. We need to choose a good $\eta$, the local potential of $\omega_M$ near $z = F(x_0)$ in $\mathbb{CP}^N$. We proceed exactly as in [11]. In a local coordinate ball $B''$ (in $\mathbb{CP}^N$) centered at $z$, choose a local potential $\rho$ (for $\omega_M$) which is clearly strictly plurisubharmonic and smooth. It can be expanded as

$$\rho(z + h) = \rho(z) + 2\Re \left( \sum_{j=1}^n a_j h_j + \sum_{j,k=1}^n b_{jk} h_j h_k \right) + \sum_{j,k=1}^n c_{jk} h_j \bar{h}_k + o(|h|^2)$$

$$= \Re(P(h)) + H(h) + o(|h|^2),$$

where $h$ is the coordinate system, $P$ is a complex polynomial in $h$ and $H$ is the complex Hessian at $z$. Exactly as in [11], Lemma 2.3.1, $\eta := \rho - \Re P(\cdot - z)$ is also a local potential for $\omega_M$, with the additional property that $\eta$ has a strict local minimum at $z$ using that at this point that $H$ is strictly positive definite. This means that for a smaller ball, which after possible shrinking we
still denote by $B''$, $\inf_{\partial B''} \eta > \eta(z) + b''$ for some positive constant $b''$. The ball $B''$ can be any ball centered at $z$. By adding a constant if necessary one can further assume that $\eta(z) > 0$.

Now by Fornaess and Narasimhan’s extension result for weak plurisubharmonic function in [9], over an even smaller Euclidean ball $B' \subset B''$ centered at $z$, we have a function $\psi \in PSH(B')$, such that

$$\psi|_{U \cap B'} = \eta + F_\star \phi.$$ 

On a neighborhood of a slightly smaller ball $B$, $\psi$ can be approximated by a sequence of smooth plurisubharmonic functions $\psi_j$ decreasing towards it. This can be achieved using classic convolution construction (cf. [5], Chapter I, Theorem (5.5)). And one still has that $\inf_{\partial B} \eta > \eta(z) + b$ for some constant $b > 0$ from our choice of $\eta$ because the $\eta$ chosen before has its value increasing from the strict minimum taken at the center.

Now we pull back the ball and the approximation functions to $X$. Let $V := F^{-1}(B \cap U)$ and $u_j := F_\star (\psi_j)$, which are defined only in small neighborhood of $x_0$, $V$ and still continuous plurisubharmonic functions on $V$ decreasing towards $u := F_\star \eta + \phi$.\footnote{There is no need to worry about the boundary issue from convolution construction in $U$ because all that is needed is a smooth decreasing approximation for $F_\star \eta + \phi$ over a neighborhood centered at $x_0$ from pulling back a neighborhood of $z = F(x_0)$ in $\mathbb{C}P^N$. One can always shrink the domain a little. There is some related discussion in [19], pp. 182–183.}

Note that $V$ would no longer be a Euclidean domain, i.e. it cannot be contained in $\mathbb{C}^n$. Nevertheless $F_\star \eta$ is a global potential of $\omega$ on this set by the construction. This is the essential difference between this case and the Kähler case considered by Kołodziej.

Next we give the following lemma which is essentially the same as the one in [11], Section 2.4. Of course, the geometric picture is different because we are no longer considering a Euclidean domain, but Kołodziej’s argument can be carried through line by line. We give the detail below for the convenience of the readers and the sake of completeness.

**Lemma 6.3.** There exist $a_0 > 0$, $t > 1$ such that the sets

$$W(j, c) := \{ tu + d - a_0 + c < u_j \}$$

are non-empty and relatively compact in $V$ for every constant $c$ belonging to an interval which does not depend on $j > j_0$.

**Proof.** Define $E(0) := \{ u - u_* = d \} \cap \overline{V} = E \cap \overline{V}$, and also the sets $E(a) := E := \{ u - u_* \geq d - a \} \cap \overline{V}$. They are all closed and $E(a)$ decreases towards $E(0)$ as $a \searrow 0$. Define $c(a) := \phi(x_0) - \min_{E_a} \phi$. We have that $\limsup_{a \to 0^+} c(a) \leq 0$ because otherwise we would get a contradiction from the definition of $d$. Hence we arrive at

$$c(a) < \frac{1}{3} b \quad \text{for } 0 < a < a_0 < \min \left( \frac{1}{3} b, d \right).$$

Let $A := u(x_0)$. Note that $A > d$ since the potential is greater than 0 at $x_0$, and $\phi$, a function positive everywhere, has to be greater than $d$ at $x_0$. One can choose $t > 1$, such that it satisfies

$$(t - 1)(A - d) < a_0 < (t - 1) \left( A - d + \frac{2}{3} b \right).$$

\footnote{There is no need to worry about the boundary issue from convolution construction in $U$ because all that is needed is a smooth decreasing approximation for $F_\star \eta + \phi$ over a neighborhood centered at $x_0$ from pulling back a neighborhood of $z = F(x_0)$ in $\mathbb{C}P^N$. One can always shrink the domain a little. There is some related discussion in [19], pp. 182–183.}
Now if \( y \in \partial V \cap E(a_0) \), one gets

\[
u_*(y) \geq \eta(F(x_0)) + b + F^*F_\phi(x_0) \geq A - d + \frac{2}{3}b.
\]

Hence \( u(y) \leq u_*(y) + d < tu_*(y) + d - a_0 \). Note that this inequality still holds in a neighborhood of \( \partial V \cap E(a_0) \). Taking another neighborhood relatively compact in the first and applying Hartogs’ Lemma, one obtains

\[
j < tu(y) + d - a_0, \quad \forall j > j_1.
\]

For the rest part of \( \partial V \), the same inequality holds if we take big enough \( j_1 \) and the proof is even simpler, since \( u - u_\phi \) is less than \( d - a_0 \) there. This proves the relative compactness of \( W(j,c) \) in \( V \).

Note that from the left part of (6.1), one has \((t - 1)u_*(x_0) < a_0\), and so

\[
tu_*(x_0) < u(x_0) - d - a_1 + a_0 < u_j(x_0) - d - a_1 + a_0
\]

for some constant \( a_1 > 0 \). This implies that the sets \( W(j,c) \) for \( c \in (0, a_1) \) contain some points near \( x_0 \), and so they are non-empty. The proof of the lemma is thus finished. \( \Box \)

Now we are going to apply the version of Lemma 2.3.1 from [11] to our case. There is quite something to take care of because we are no longer considering a Euclidean domain. It can be seen that the original argument in [11] can be carried through line by line in sight of the following observations (as pointed out in [19], Chapter 5).

1. The classic definition of relative capacity for Euclidean domains can be generalized in a natural way to the current situation, preserving a lot of properties. Simply speaking, the background form of smooth local potentials can be handled by numerical manipulation. There are plenty of references on this topic, for example, [13].
2. There is no need to involve relative extremal function even in Kołodziej’s original proof. When drawing conclusion on relative capacity, one can instead go through the estimation for any admissible plurisubharmonic function in Definition 2.2. This idea has appeared in the proof of Proposition 2.5.
3. Comparison Principle can still be applied as discussed in Section 4.

Running through Kołodziej’s argument, one can bound the relative capacity, \( \text{Cap}(W(j,a_1), V) \) from below by a uniform positive constant.

On the other hand, \( W(j,a_1) \subset \{ u + (d - a_0 + a_1) < u_j \} \), and so that contradicts the fact that the decreasing sequence \( \{ u_j \} \) actually converges towards \( u \) with respect to capacity. Hence we conclude that \( \phi \) is continuous.

**Remark 6.4.** The construction of push-forward of the function to the (singular) image is crucial for the argument. In general, birationality alone is not going to guarantee this. In principle, we need to avoid the situation as indicated by Example 2.8 where some component of the pre-image of a small neighborhood is not birational to the small neighborhood itself. We have this definition of local birationality, i.e. Definition 2.7 to make this idea more transparent. Using the more
standard terminology from algebraic geometry, one has the fiber being connected for the classic notion of algebraic fiber space. This is the case when the map is generated by a semi-ample line bundle (as in [16], p. 129, Theorem 2.1.27). A birational holomorphic map with connected fibers is locally birational and in fact the pre-image only has one component in this case.

7. Final remarks

Complex Monge–Ampère equations are of great interest in various aspects of mathematics. In [19], the following version of the Monge–Ampère equation

\[(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = e^u \Omega\]

is also studied. Of course the degenerate case as in the setting of Theorem 1.3 is the main focus.

Using the argument in [12], we observe that the main result in this work would also apply there. More precisely, one has the following theorem.\(^8\)

**Theorem 7.1.** Let \(\omega\) be a big form and \(u_1\) and \(u_2\) be \(\omega\)-plurisubharmonic solutions for the following Monge–Ampère equations:

\[\omega_1^n u_1^n = e^{u_1} \Omega_1, \quad \omega_2^n u_2^n = e^{u_2} \Omega_2,\]

where \(\Omega_1\) and \(\Omega_2\) are smooth volume forms. Then for any \(\epsilon > 0\), there exist positive constants \(t_0\) and \(C\) depending only on \(\epsilon, (X, \omega)\) and \(L^p > 1\)-norms of \(\Omega_1\) and \(\Omega_2\), such that if

\[\int_X |\Omega_1 - \Omega_2| \leq \gamma(t) t^{n+\epsilon},\]

then one concludes

\[\|u_1 - u_2\|_\infty \leq C t\]

for \(0 \leq t < t_0\).

**Proof.** Since Comparison Principle with respect to a big form is available and by Theorem 1.1, we have stability with exponent \(n + \epsilon\) and the proof is entirely the same as the proof of Theorem 5.2 in [12]. \(\Box\)

The following problems are related to the results in [14] and [8], stating that when \(\omega\) is a Kähler form on a compact Kähler manifold, the solutions of

\[\omega^n f = f \omega^n, \quad f \in L^p(\omega^n) \text{ for } p > 1,\]

\[^8\] This theorem can be stated in a more general form as in [12].
are Hölder continuous. In general the Hölder exponent depends on the manifold $X$, $n$ and $p$ [14]. Under the additional assumption that $X$ is homogenous, i.e. the automorphism group $\text{Aut}(X)$ acts transitively the exponent is independent of $X$ and is not less $\frac{2}{nq+2}$ for $q = \frac{P}{p-1}$ (see [8]). One can further ask the questions below of interests.

1. Is the solution continuous when $\omega$ is semi-positive and big in general? If this is the case, how about Hölder continuity?
2. Does the Hölder exponent on a general manifold really depend on the manifold? In the corresponding result for the flat case in [10], the Hölder exponent is uniform and independent of the domain. Moreover the proof in [14] strongly depends on a regularization procedure for $\omega$-plurisubharmonic functions, which consists of patching local regularizations, and this is the point where the geometry of the manifold influences the exponent. In particular, are there other regularization procedures of a more global nature which are not so affected by the local geometry?
3. Is the exponent for the homogeneous case sharp? Note that for the flat case in [10] there is also a gap between the exponent given there $\frac{2}{qn+1}$ and the exponent $\frac{2}{qn}$, for which an example is shown.
4. It is interesting to compare the stability results we have and the one in [8]. In particular, is the stability exponent in [8] sharp in general?
5. It would be very interesting to achieve Hölder continuity for potentials of more singular measures. One possible application of such a result would be a criterion for Hölder continuity of the Siciak Extremal Function of certain compact sets in $\mathbb{C}^n$ (see [13] for more discussion). Such a property is very important from pluripotential theory point of view. So one has to study the equilibrium measure of the compact sets. The problem is that such measures are singular with respect to the Lebesgue measure, while [14] and [8] rely strongly on the smoothness of $\omega^n$. However, as argument here shows, some argument can be adjusted to singular measures as well.

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