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Generalized Variational Inequalities and Generalized Quasi-Variational Inequalities*

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A very general minimax inequality is first established. Three generalized variational inequalities are then derived, which improve those obtained by Tan and Browder. By applying a fixed point theorem of Himmelberg, two generalized quasi-variational inequalities are also proved, one of which generalizes those of Shih–Tan to the non-compact case with much weaker hypotheses and in a more general setting. © 1990 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For a nonempty set Y, 2^Y will denote the family of all nonempty subsets of Y. Let E and F be vector spaces over Φ, $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional, and X be a nonempty subset of E. If $S: X \to 2^X$ and $T: X \to 2^F$, the generalized quasi-variational inequality problem for the pair (S, T) is to find $\hat{y} \in X$ satisfying the following properties:

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$.

In this paper, we fist establish a very general minimax inequality, then we improve some generalized variational inequalities of Tan [15,

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Theorems 3 and 5] and of Browder [4, Theorem 6]. Next, by applying a fixed point theorem of Himmelberg [8, Theorem 2], we prove two existence theorems on solutions of the generalized quasi-variational inequality problem, one of which improves those of Shih–Tan [11, Theorems 1 and 2] to the non-compact case with much weaker hypotheses.

2. PRELIMINARIES

Let E and F be two vector spaces over Φ, and $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional. For each $x_0\in E$, each nonempty subset A of E and $\varepsilon>0$, let

$$W(x_0;\varepsilon):=\{y\in F\colon |\langle y, x_0\rangle|<\varepsilon\},$$
$$U(A;\varepsilon):=\{y\in F\colon \sup_{x\in A}|\langle y, x\rangle|<\varepsilon\}.$$

Let $\sigma\langle F, E\rangle$ be the topology on F generated by the family $\{W(x;\varepsilon)\colon x\in E$ and $\varepsilon>0\}$ as a subbase for the neighborhood system at 0. If E is a topological vector space, let $\delta\langle F, E\rangle$ be the topology on F generated by the family $\{U(A;\varepsilon)\colon A$ is a nonempty compact subset of E and $\varepsilon>0\}$ as a base for the neighborhood system at 0. We note then, F when equipped with the topology $\sigma\langle F, E\rangle$ or the topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space but not necessarily Hausdorff. If X is a nonempty subset of E, then a map $T\colon X\to 2^F$ is *monotone* (with respect to the bilinear functional $\langle\ ,\ \rangle$) if for any $x, y\in X$, $u\in T(x)$, and $w\in T(y)$, $\operatorname{Re}\langle w-u, y-x\rangle\geqslant 0$. A subset C of E is said to be $\sigma\langle E, F\rangle$-compact if C is compact in the $\sigma\langle E, F\rangle$-topology. If X is a subset of E, the function $h\colon X\to\mathbb{R}$ is said to be $\sigma\langle E, F\rangle$-lower semi-continuous if h is lower semi-continuous when X is equipped with the relative $\sigma\langle E, F\rangle$-topology. If B is a subset of a vector space, $\operatorname{co}(B)$ will denote the convex hull of B. If X and Y are two nonempty sets and $F\colon X\to 2^Y$, then the graph of F is the set $\{(x, y)\in X\times Y\colon y\in F(x)\}$.

The following result is essentially Lemma 2 of Shih–Tan in [14] and its proof is thus omitted.

LEMMA 1. *Let E and F be two topological vector spaces over Φ and $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional. Let X be a nonempty subset of E and $T\colon X\to 2^F$ be an upper semi-continuous map such that for each $x\in X$, $T(x)$ is compact. Let $\psi\colon X\times X\to\mathbb{R}$ be defined by*

$$\psi(x, y):=\inf_{w\in T(y)}\operatorname{Re}\langle w, y-x\rangle.$$

If $\langle\ ,\ \rangle$ *is continuous on compact subsets of* $F \times X$, *then for each fixed* $x \in X$, *the map* $y \to \psi(x, y)$ *is a lower semi-continuous function of* y *on* A *for each nonempty compact subset* A *of* X.

The following result can be found in [2, Theorem 1.1.1, p. 41]; as Hausdorff was not needed in its proof, it is omitted in the following statement.

LEMMA 2. *Let* X *and* Y *be topological spaces,* $F, G: X \to 2^Y$ *be such that* $F(x) \cap G(x) \neq \emptyset$ *for each* $x \in X$. *Suppose that* (i) F *is upper semi-continuous at* $x_0 \in X$, (ii) $F(x_0)$ *is compact, and* (iii) *the graph of* G *is closed. Then the map* $F \cap G: x \to F(x) \cap G(x)$ *is upper semi-continuous at* x_0.

Next we prove the following:

LEMMA 3. *Let* E *be a topological vector space over* Φ, F *be a vector space over* Φ, *and* $\langle\ ,\ \rangle: F \times E \to \Phi$ *be a bilinear functional. Let* X *be a nonempty convex subset of* E, $h: X \to \mathbb{R}$ *be a convex function, and* $T: X \to 2^F$ *be lower semi-continuous along line segments in* X *to the* $\sigma\langle F, E\rangle$*-topology on* F. *If* $\hat{y} \in X$, *then the inequality*

$$\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \qquad \textit{for all} \quad x \in X$$

implies the inequality

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}), \qquad \textit{for all} \quad x \in X.$$

Proof. Fix $x \in X$. For each $t \in [0, 1]$, let $x_t = tx + (1-t)\hat{y} = \hat{y} - t(\hat{y} - x)$, then $x_t \in X$ so that $\sup_{u \in T(x_t)} \operatorname{Re}\langle u, \hat{y} - x_t\rangle \leqslant h(x_t) - h(\hat{y})$ for all $t \in [0, 1]$; thus for all $t \in [0, 1]$, $t \cdot \sup_{u \in T(x_t)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant h(tx + (1-t)\hat{y}) - h(\hat{y}) \leqslant th(x) + (1-t)h(\hat{y}) - h(\hat{y}) = t[h(x) - h(\hat{y})]$. Hence

$$\sup_{u \in T(x_t)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \qquad \text{for all} \quad t \in (0, 1]. \qquad (*)$$

Let $w_0 \in T(\hat{y})$ be arbitrarily fixed. For each $\varepsilon > 0$, let

$$U(w_0) = \{w \in F: |\langle w - w_0, \hat{y} - x\rangle| < \varepsilon\},$$

then $U(w_0)$ is an $\sigma\langle F, E\rangle$-open neighborhood of w_0. As $U(w_0) \cap T(\hat{y}) \neq \emptyset$ and T is lower semi-continuous on $L = \{x_t: t \in [0, 1]\}$, there exists an open neighborhood N of $\hat{y}$ in L such that $U(w_0) \cap T(y) \neq \emptyset$ for all $y \in N$. Since $x_t \to \hat{y}$ as $t \to 0^+$, there exists $\delta \in (0, 1)$ such that $x_t \in N$ for all

$t \in (0, \delta)$. Choose any $t \in (0, \delta)$ and $u \in U(w_0) \cap T(x_t)$, then we have $|\langle w_0 - u, \hat{y} - x\rangle| < \varepsilon$. Thus

$$\begin{aligned} \operatorname{Re}\langle w_0, \hat{y} - x\rangle &< \operatorname{Re}\langle u, \hat{y} - x\rangle + \varepsilon \\ &\leqslant h(x) - h(\hat{y}) + \varepsilon, \qquad \text{by } (*). \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $\operatorname{Re}\langle w_0, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$. As $w_0 \in T(\hat{y})$ is also arbitrary, we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \qquad \text{for all} \quad x \in X.$$

This completes the proof. ∎

3. A Minimax Inequality

The celebrated 1961 Fan's Lemma [5, Lemma 1], which is an infinite dimensional generalization of the classical Knaster–Kuratowski–Mazurkiewicz theorem [9], asserts the following:

Theorem B. *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, ..., x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$,*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

We remark here that in Fan's proof of the above result, the underlying topological vector space Y *need not be Hausdorff.* We shall now establish the following minimax inequality:

Theorem 1. *Let X be a nonempty convex set in a topological vector space E. Let ϕ and ψ be two extended real-valued functions on $X \times X$ having the following properties:*

(a) *$\psi(x, x) \leqslant 0$ for all $x \in X$;*

(b) *for each fixed $x \in X$, $\phi(x, y)$ is a lower semi-continuous function of y on A for each nonempty compact subset A of X;*

(c) *for each fixed $y \in X$, the set $\{x \in X: \psi(x, y) > 0\}$ contains the convex hull of the set $\{x \in X: \phi(x, y) > 0\}$;*

(d) *there exist a nonempty compact convex subset* X_0 *of* X *and a nonempty compact subset* K *of* X *such that for each* $y \in X \backslash K$, *there exists a point* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $\phi(x, y) > 0$.

Then there exists a point $\hat{y} \in K$ *such that* $\phi(x, \hat{y}) \leqslant 0$ *for all* $x \in X$.

Proof. For each $x \in X$, let

$$K(x) = \{y \in K: \phi(x, y) \leqslant 0\}.$$

By (b), $K(x)$ is closed in K for each $x \in X$. We shall show that the family $\{K(x): x \in X\}$ has the finite intersection property. Indeed, let $x_1, ..., x_m \in X$ be given. Let

$$C \equiv \mathrm{co}(X_0 \cup \{x_1, ..., x_m\}),$$

then C is a compact convex subset of X. For each $x \in C$, let

$$\Phi(x) = \{y \in C: \phi(x, y) \leqslant 0\}.$$

Then we have the following:

(i) For each $x \in C$, $\Phi(x)$ is nonempty since $x \in \Phi(x)$ by (a) and (c) and is closed in C by (b).

(ii) $\Phi: C \to 2^C$ has the property that the convex hull of every finite subset $\{u_1, ..., u_n\}$ of C is contained in the corresponding union $\bigcup_{i=1}^n \Phi(u_i)$.

If this were false, there exist $\{u_1, ..., u_n\} \subset C$, $\alpha_1, ..., \alpha_n \geqslant 0$ with $\sum_{i=1}^n \alpha_i = 1$ such that $\sum_{j=1}^n \alpha_j u_j \notin \bigcup_{i=1}^n \Phi(u_i)$; that is, $\phi(u_i, \sum_{j=1}^n \alpha_j u_j) > 0$ for all $i = 1, ..., n$. By (c), it would imply $\psi(\sum_{i=1}^n \alpha_i u_i, \sum_{j=1}^n \alpha_j u_j) > 0$ which contradicts (a). Thus by Theorem B, we have $\bigcap_{x \in C} \Phi(x) \neq \varnothing$; that is, there exists $\bar{y} \in C$ with $\phi(x, \bar{y}) \leqslant 0$ for all $x \in C$. By (d), we must have $\bar{y} \in K$ so that $\bar{y} \in \bigcap_{i=1}^m K(x_i)$. This shows that $\{K(x): x \in X\}$ is a family of closed subsets of K which has the finite intersection property. By the compactness of K, $\bigcap_{x \in X} K(x) \neq \varnothing$. Take any $\hat{y} \in \bigcap_{x \in X} K(x)$, then $\hat{y} \in K$ and $\phi(x, \hat{y}) \leqslant 0$ for all $x \in X$. ∎

The above result slightly improves a minimax inequality of Bae–Kim–Tan [3, Theorem 1] and generalizes a minimax inequality of Shih–Tan [13, Theorem 3], both in turn generalize the minimax inequalities of Fan [7, Theorem 6; 6, Theorem 1], Tan [15, Theorem 1], and Allen [1, Theorem 1], etc. We emphasize here again that the topological vector space E in Theorem 1 is not required to be Hausdorff, as should be the case for all minimax inequalities mentioned above.

Equivalent forms (i.e., fixed point version, geometric form) of Theorem 1 similar to those in [12] can be likewise stated and proved and are omitted here.

4. GENERALIZED VARIATIONAL INEQUALITIES

We shall first prove the following generalized variational inequalities:

THEOREM 2. *Let E be a topological vector space over Φ, F be a vector space over Φ, X be a nonempty convex subset of E, and $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional. Suppose*

(a) *$h: X\to\mathbb{R}$ is a $\sigma\langle E, F\rangle$-lower semi-continuous and convex function.*

(b) *$T: X\to 2^F$ is lower semi-continuous along line segments in X to the $\sigma\langle F, E\rangle$-topology on F such that for each $y\in X$, the set $\{x\in X: \inf_{w\in T(y)}\operatorname{Re}\langle w, y-x\rangle + h(y)-h(x)>0\}$ contains the set $\{x\in X: \sup_{u\in T(x)}\operatorname{Re}\langle u, y-x\rangle + h(y)-h(x)>0\}$.*

(c) *There exist a nonempty $\sigma\langle E, F\rangle$-compact and convex subset X_0 of X and a nonempty $\sigma\langle E, F\rangle$-compact subset K of X such that for each $y\in X\backslash K$, there is $x\in\operatorname{co}(X_0\cup\{y\})$ with $\sup_{u\in T(x)}\operatorname{Re}\langle u, y-x\rangle + h(y)-h(x)>0$.*

Then there exists a point $\hat{y}\in K$ uch that

$$\sup_{w\in T(\hat{y})}\operatorname{Re}\langle w, \hat{y}-x\rangle\leqslant h(x)-h(\hat{y}),\qquad \textit{for all}\quad x\in X.$$

Proof. Define $\phi, \psi: X\times X\to\mathbb{R}\cup\{+\infty, -\infty\}$ by

$$\phi(x, y)=\sup_{u\in T(x)}\operatorname{Re}\langle u, y-x\rangle + h(y)-h(x),$$

$$\psi(x, y)=\inf_{w\in T(y)}\operatorname{Re}\langle w, y-x\rangle + h(y)-h(x).$$

Then we have the following properties:

(i) $\psi(x, x)=0$ for all $x\in X$.

(ii) For each fixed $x\in X$, $y\to\phi(x, y)$ is a $\sigma\langle E, F\rangle$-lower semi-continuous function on X.

(iii) For each fixed $y\in X$, it is easy to see that the set $\{x\in X: \inf_{w\in T(y)}\operatorname{Re}\langle w, y-x\rangle + h(y)-h(x)>0\}$ is convex so that by hypothesis the set $\{x\in X: \psi(x, y)>0\}$ contains the convex hull of the set $\{x\in X: \phi(x, y)>0\}$.

(iv) For each $y\in X\backslash K$, there exists $x\in\operatorname{co}(X_0\cup\{y\})$ such that $\sup_{u\in T(x)}\operatorname{Re}\langle u, y-x\rangle + h(y)-h(x)>0$ so that $\phi(x, y)>0$.

Now equip E with the $\sigma\langle E, F\rangle$-topology, we see that all hypothesis of Theorem 1 are satisfied so that there exists $\hat{y}\in K$ such that $\phi(x, \hat{y})\leqslant 0$ for all $x\in X$; that is,

$$\sup_{u\in T(x)}\operatorname{Re}\langle u, \hat{y}-x\rangle + h(\hat{y})-h(x)\leqslant 0\qquad\text{for all}\quad x\in X.$$

By Lemma 3, we have

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x) \leqslant 0 \qquad \text{for all} \quad x \in X. \blacksquare$$

We note that if $T: X \to 2^F$ is monotone with respect to the bilinear functional $\langle\ ,\ \rangle$, then for each $y \in X$, the set $\{x \in X: \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle + h(y) - h(x) > 0\}$ contains the set $\{x \in X: \sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle + h(y) - h(x) > 0\}$. Thus Theorem 2 generalizes Theorem 5 and hence also Theorem 3 (by taking $h \equiv 0$) in [15]. We shall point out here that in stating Theorem 5 in [15], the assumption that "$h: X \to \mathbb{R}$ is a *weakly* lower semi-continuous convex function" was inadvertently mis-stated as "$h: X \to \mathbb{R}$ is a lower semi-continuous, convex function."

By little modification in the proof of Theorem 2, we have the following:

THEOREM 3. *Let E be a topological vector space over Φ, F be a vector space over Φ, X be a nonempty convex subset of E, and $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X. Suppose*

(a) *$h: X \to \mathbb{R}$ is a lower semi-continuous and convex function.*

(b) *$T: X \to 2^F$ is lower semi-continuous along line segments in X to the $\sigma\langle F, E\rangle$-topology on F such that for each $y \in X$, the set $\{x \in X: \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle + h(y) - h(x) > 0\}$ contains the set $\{x \in X: \sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle + h(y) - h(x) > 0\}$.*

(c) *There exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, there is $x \in \operatorname{co}(X_0 \cup \{y\})$ with $\sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle + h(y) - h(x) > 0$.*

Then there exists a point $\hat{y} \in K$ such that

$$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \qquad \textit{for all} \quad x \in X.$$

While the map T in Theorems 2 and 3 is assumed to be lower semi-continuous and monotone type, we shall give another generalized variational inequality below where T is assumed to be upper semi-continuous:

THEOREM 4. *Let E be a topological vector space over Φ, X be a nonempty convex subset of E, F be a vector space over Φ, and $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the $\delta\langle F, E\rangle$-topology. Suppose*

(a) *$h: X \to \mathbb{R}$ is a lower semi-continuous convex function.*

(b) *$T: X \to 2^F$ is upper semi-continuous such that for each $x \in X$, $T(x)$ is compact convex.*

(c) *There exist a nonempty compact convex subset* X_0 *of* X *and a nonempty compact subset* K *of* X *such that for each* $y \in X \backslash K$, *there exists* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $\inf_{w \in T(y)} \mathrm{Re}\langle w, y-x\rangle + h(y) - h(x) > 0$.

Then there exist $\hat{y} \in K$ *and* $\hat{w} \in T(\hat{y})$ *such that*

$$\mathrm{Re}\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \qquad \textit{for all} \quad x \in X.$$

Proof. It is easy to see that $\langle\ ,\ \rangle$ is continuous on compact subsets of $F \times X$. Define $\psi: X \times X \to \mathbb{R}$ by

$$\psi(x, y) = \inf_{w \in T(y)} \mathrm{Re}\langle w, y-x\rangle + h(y) - h(x).$$

Then we have the following properties:

(i) Clearly $\psi(x, x) = 0$ for all $x \in X$.

(ii) By Lemma 1, for each fixed $x \in X$, $y \to \psi(x, y)$ is a lower semicontinuous function of y on A for each nonempty compact subset A of X.

(iii) It is easy to see that for each fixed $y \in X$, the set $\{x \in X: \psi(x, y) > 0\}$ is convex.

(iv) By (c), for each $y \in X \backslash K$, there is $x \in \mathrm{co}(X_0 \cup \{y\})$ such that $\psi(x, y) > 0$.

Hence all hypotheses of Theorem 1 are satisfied (with $\phi \equiv \psi$) so that there exists $\hat{y} \in K$ such that $\psi(x, \hat{y}) \leqslant 0$ for all $x \in X$, i.e.,

$$\sup_{x \in X} \inf_{w \in T(\hat{y})} (\mathrm{Re}\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)) \leqslant 0. \tag{**}$$

Define $f: X \times T(\hat{y}) \to \mathbb{R}$ by

$$f(x, w) = \mathrm{Re}\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x),$$

then for each fixed $x \in X$, $w \mapsto f(x, w)$ is an affine function which is continuous on the compact convex set $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, $x \mapsto f(x, w)$ is a concave function on the convex set X. Hence by Kneser's minimax theorem [10], we have

$$\begin{aligned} &\inf_{w \in T(\hat{y})} \sup_{x \in X} (\mathrm{Re}\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)) \\ &\quad = \sup_{x \in X} \inf_{w \in T(\hat{y})} (\mathrm{Re}\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)) \\ &\quad \leqslant 0, \qquad \text{by } (**). \end{aligned}$$

Again, as $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$\sup_{x \in X} (\operatorname{Re}\langle \hat{w}, \hat{y}-x \rangle + h(\hat{y}) - h(x))$$
$$= \inf_{w \in T(\hat{y})} \sup_{x \in X} (\operatorname{Re}\langle w, \hat{y}-x \rangle + h(\hat{y}) - h(x)).$$

Therefore

$$\operatorname{Re}\langle \hat{w}, \hat{y}-x \rangle \leqslant h(x) - h(\hat{y}) \qquad \text{for all} \quad x \in X. \quad \blacksquare$$

The above theorem generalizes a generalized variational inequality of Browder in [4, Theorem 6] to the non-compact case in a more general setting.

5. Generalized Quasi-Variational Inequalities

In this section we shall prove two generalized quasi-variational inequalities.

THEOREM 5. *Let E be a locally convex Hausdorff topological vector space over Φ, F be a vector space over Φ, X be a nonempty convex subset of E, C be a nonempty compact subset of X, and $\langle \ , \ \rangle: F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X. Suppose*

(a) *$S: X \to 2^X$ is upper semi-continuous such that for each $x \in X$, $S(x)$ is a closed convex subset of C;*

(b) *$h: X \to \mathbb{R}$ is a lower semi-continuous and convex function;*

(c) *$T: X \to 2^F$ is lower semi-continuous from line segments in X to the $\sigma\langle F, E \rangle$-topology on F such that for each $y \in X$, the set $\{x \in X: \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x \rangle + h(y) - h(x) > 0\}$ contains the set $\{x \in X: \sup_{u \in T(x)} \operatorname{Re}\langle u, y-x \rangle + h(y) - h(x) > 0\}$;*

(d) *there exists a family $\{X(z): z \in X\}$ of nonempty compact convex subsets of X such that for each $z \in X$ and for each $y \in X \backslash S(z)$, there exists an $x \in \operatorname{co}(X(z) \cup \{y\})$ for which $\sup_{u \in T(x)} \operatorname{Re}\langle u, y-x \rangle + h(y) - h(x) > 0$.*

Then there exists an $\hat{y} \in C$ such that

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) *$\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y}-x \rangle \leqslant h(x) - h(\hat{y})$, for all $x \in X$.*

Proof. Let $z \in X$. Since $S(z)$ is nonempty compact, by (d) and Theorem 3, there exists $z^* \in S(z)$ such that

$$\sup_{w \in T(z^*)} \mathrm{R}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0 \qquad \text{for all} \quad x \in X;$$

thus the set $\{z^* \in S(z)\colon \sup_{x \in X} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}$ is nonempty for each $z \in X$.

Define $F\colon X \to 2^X$ by

$$F(z) = \{z^* \in S(z)\colon \sup_{x \in X} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}$$

for each $z \in X$. For any given $z \in X$, by Lemma 3, the set $\{z^* \in S(z)\colon \sup_{x \in X} \sup_{u \in T(x)} \mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}\}$ is contained in the set $\{z^* \in S(z)\colon \sup_{x \in X} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}$. Conversely, if $z^* \in S(z)$ is such that $\sup_{x \in X} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0$, then $\inf_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0$ for all $x \in X$ so that by (c) $\sup_{u \in T(x)} \mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x) \leqslant 0$ for all $x \in X$. This shows that the set $\{z^* \in S(z)\colon \sup_{x \in X} \sup_{u \in T(x)} \mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}$ also contains the set $\{z^* \in S(z)\colon \sup_{x \in X} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}$. Therefore

$$F(z) = \{z^* \in S(z)\colon \sup_{x \in X} \sup_{u \in T(x)} [\mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x)] \leqslant 0\}$$

which is closed convex by (b) and the assumption that for each $f \in F$, $x \to \langle f, x\rangle$ is continuous on X. Since S is upper semi-continuous and the map $z^* \to \sup_{x \in X} \sup_{u \in T(x)} \mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x)$ is lower semi-continuous on X, we conclude that the graph of F is closed in $X \times X$. Therefore by Lemma 2, F is also upper semi-continuous. Since $F(z) \subset S(z) \subset C$ for each $z \in X$, by Himmelberg's fixed point theorem [8, Theorem 2], there exists $\hat{y} \in X$ such that $\hat{y} \in F(\hat{y})$; that is,

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in X$.

As $F(\hat{y}) \subset C$, $\hat{y} \in C$. This completes the proof. ∎

THEOREM 6. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty convex subset of E, C be a nonempty compact subset of X, F be a vector space over Φ, and $\langle\ ,\ \rangle\colon F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, the map $x \to \langle f, x\rangle$ is continuous on X. Suppose*

(a) *$S\colon X \to 2^X$ is upper semi-continuous such that for each $x \in X$, $S(x)$ is a closed convex subset of C;*

(b) *$h: X \to \mathbb{R}$ is lower semi-continuous and convex;*

(c) *$T: X \to 2^F$ is lower semi-continuous from line segments in X to the $\sigma\langle F, E\rangle$-topology on F such that for each $y \in X$, the set $\{x \in X: \inf_{w \in T(y)} \mathrm{Re}\langle w, y - x\rangle + h(y) - h(x) > 0\}$ contains the set $\{x \in X: \sup_{u \in T(x)} \mathrm{Re}\langle u, y - x\rangle + h(y) - h(x) > 0\}$.*

Then there exists $\hat{y} \in C$ such that

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) *$\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. For each $z \in X$, define

$$F(z) = \{z^* \in S(z): \sup_{x \in S(z)} \sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0\}.$$

Since $S(z)$ is compact convex, by taking $X \equiv X_0 \equiv K \equiv S(z)$ in Theorem 3, there exists $z^* \in S(z)$ such that

$$\sup_{w \in T(z^*)} \mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(x) \leqslant 0 \qquad \text{for all} \quad x \in S(z),$$

so that $F(z)$ is nonempty. Thus $F: X \to 2^X$. Similar to the proof of Theorem 5, by (b), (c), and Lemma 3, for each $z \in X$, the set $\{z^* \in S(z): \sup_{x \in S(z)} \sup_{w \in T(z^*)} [\mathrm{Re}\langle w, z^* - x\rangle + h(z^*) - h(z)] \leqslant 0\}$ coincides with the set $\{z^* \in S(z): \sup_{x \in S(z)} \sup_{u \in T(x)} [\mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x)] \leqslant 0\}$ so that each $F(z)$ is closed convex as the map

$$z^* \to \sup_{x \in S(z)} \sup_{u \in T(x)} [\mathrm{Re}\langle u, z^* - x\rangle + h(z^*) - h(x)]$$

is lower semi-continuous and convex. Since S is upper semi-continuous, it is easy to see that the graph of F is closed in $X \times X$. Therefore, by Lemma 2, F is also upper semi-continuous. Since $F(z) \subset C$ for each $z \in X$, by Himmelberg's fixed point theorem [8, Theorem 2], there exists $\hat{y} \in C$ such that $\hat{y} \in F(\hat{y})$; that is

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$. ∎

Remark 1. We remark here that while Theorem 6 is proved in a more general setting, it also generalizes Theorems 1 and 2 of Shih–Tan in [11] in several ways: (i) the set X need not be compact; (ii) the interacting set Σ_1 in Theorem 1 of [11] is no longer required to be open; (iii) the continuity conditions on S and T are much weaker than those assumed in Theorem 2 of [11].

Remark 2. We would like to point out the difference between the conclusions of Theorem 5 and Theorem 6: In Theorem 5, the variational inequality holds for all $x \in X$ while in Theorem 6, the variational inequality holds for all $x \in S(\hat{y})$.

Remark 3. Himmelberg's fixed point theorem can be recovered from Theorem 5 or Theorem 6 by taking $h \equiv 0$ and $T \equiv 0$.

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