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J. Math. Anal. Appl. 302 (2005) 457–462

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

A method for solving an inverse biharmonic problem

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Received 29 April 2002

Available online 19 November 2004

Submitted by William F. Ames

Abstract

This paper deals with the problem of determining of an unknown coefficient in an inverse boundary value problem. Using a nonconstant overspecified data, it has been shown that the solution to this inverse problem exists and is unique.

Keywords: Nonconstancy overspecified condition; Existence; Uniqueness; Unknown coefficient; Bending; Inverse problem

1. Introduction

In this paper, we consider the problem of determining the unknown coefficient $D(\omega)$ which depends only on the function $\omega(x, y)$ in the following elliptic inverse nonlinear fourth order partial differential equation:

$$\nabla^2\left[\operatorname{div}\left(D(\omega)\operatorname{grad}\omega\right)\right] = q(x, y) \quad \text{in } \Omega, \tag{1}$$

where Ω is a bounded domain of R^2 with a sufficiently smooth boundary $\partial\Omega$ consisting of the union of the two arcs $\partial\Omega_1$ and $\partial\Omega_2$ with the common endpoints (x_0, y_0) and (x_1, y_1), $\nabla^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$ is a Laplace operator, and q is given piecewise-continuous function in Ω.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.07.043

Let s_1 and s_2 be the arclengths along $\partial\Omega_1$ and $\partial\Omega_2$ measured from the point (x_0, y_0), respectively. On Ω, we assume that $\omega(x, y)$ satisfies the condition

$$\operatorname{div}\big(D(\omega)\operatorname{grad}\omega\big) = f(x, y), \tag{2}$$

on $\partial\Omega_1$,

$$\omega(x, y) = f_3(s_1), \tag{3}$$

while on the Ω_2,

$$\omega(x, y) = f_1(s_2), \tag{4}$$

$$D\big(\omega(x, y)\big)\frac{\partial\omega}{\partial n}(x, y) = f_2(s_2), \tag{5}$$

where n denotes the unit outward normal to the boundary Ω_2, f, f_1, f_2, and f_3 are given continuous functions on their domains, and $D(\omega)$ is a Lipschitz continuous function satisfying $D(\omega) \geqslant D_0 > 0$, for some constant D_0, ω, and $D(\omega)$ are unknown functions which remain to be determined.

If $D(\omega)$ is given, then the problem (1)–(4) would be a well-posed problem for the function $\omega(x, y)$. For an unknown function $D(\omega)$, we must therefore provide additional information, namely (5) to provide a unique solution pair $(D(\omega), \omega)$ to the inverse problem (1)–(5).

If we determine a unique solution to the inverse problem (1)–(5), then we have obvious physical meaning, which asserts that a thin plastic plate lies on the plastic support under a load q, $D(\omega)$, the bending rigidity, and ω, deflection are given for any given boundary data f, f_1, f_2, f_3, and load q [9,11].

In many cases, the problem (1)–(5) may be occurs in theory of thin plate and fluid flow problems. For example, if $D(\omega)$ is a constant function, and $f = f_1 = f_3 = 0$, then ω in the problem (1)–(4) will be the bending of the simply supported thin plate under a load q [9,11,14,20,21].

In the next section, we consider the inverse problem (1)–(5), and describes some existence and uniqueness results for the solution pair $(D(\omega), \omega)$ satisfying (1)–(5). The coefficient $D(\omega)$ will be determine in terms of q, f, f_1, f_2, and f_3. Some conclusion are given in Section 3.

2. Existence and uniqueness

By demonstrating the following result, we will identify the function $D(\omega)$, when $(D(\omega), \omega)$ is a solution to the inverse problem (1)–(5). For this purpose, we consider some methods introduced by Cannon [2], Matsuzawa [1], DuChateau [18], Shidfar [5,10], and Rundell [6,7]. Now, let us purpose $M(x, y) = \operatorname{div}(D(\omega)\operatorname{grad}\omega)$, then equivalently, we have to couple systems of problems

$$\nabla^2 M(x, y) = q(x, y) \quad \text{in } \Omega, \tag{6}$$

$$M(x, y) = f(x, y) \quad \text{on } \partial\Omega, \tag{7}$$

and

$$\operatorname{div}\big[D\big(\omega(x,y)\big)\operatorname{grad}\omega(x,y)\big] = M(x,y) \quad \text{in } \Omega, \tag{8}$$

$$\omega(x,y) = \begin{cases} f_1(x,y) & \text{if } (x,y) \in \partial\Omega_2, \\ f_3(x,y) & \text{if } (x,y) \in \partial\Omega_1, \end{cases} \tag{9}$$

$$D\big(\omega(x,y)\big)\frac{\partial\omega}{\partial n}(x,y) = f_2(x,y) \quad \text{on } \partial\Omega_2. \tag{10}$$

The solution of the problem (6)–(7), following the argument [12] and using Green's second formula yields

$$M(x,y) = \iint\limits_{\Omega} G(\xi,\eta;x,y)q(\xi,\eta)\,d\xi\,d\eta + \oint\limits_{\partial\Omega} f\frac{\partial G}{\partial n}\,ds, \tag{11}$$

where G is Green's function for Laplace equation in Ω subject to Dirichlet condition on $\partial\Omega$, that is

$$\nabla^2 G(x,y;\xi,\eta) = \delta(x-\xi, y-\eta) \quad \text{in } \Omega,$$

$$G(x,y;\xi,\eta) = 0 \quad \text{on } \Omega,$$

where δ is Dirac delta function.

Now, using the transformation

$$T_D(s) = \int_{s_0}^{s} D(\eta)\,d\eta, \quad s \geqslant s_0 \geqslant 0,\ s_0 \text{ is a constant number,}$$

which was used by Cannon [2], Shidfar [5], and Rundell [7].

The problem (8)–(10) reduces to one with the unknown coefficient in divergence form. Note that $T_D'(s) = D(s) \geqslant D_0 > 0$, so that $T_D(s)$ is invertible. For any solution $\omega(x,y)$ of the inverse problem (8)–(10), if $\omega(x_0,y_0)$is a given nonnegative constant, then we define

$$V(x,y) = T_D\big(\omega(x,y)\big) = \int_{\omega(x_0,y_0)}^{\omega(x,y)} D(\eta)\,d\eta. \tag{12}$$

By this transformation $V(x,y)$ satisfies [2]

$$\nabla^2 V(x,y) = M(x,y) \quad \text{in } \Omega, \tag{13}$$

$$\frac{\partial V}{\partial n}(x,y) = f_2(s_2) \quad \text{on } \partial\Omega_2, \tag{14}$$

$$V(x,y) = \begin{cases} \int_{f_3(0)}^{f_3(s_1)} D(\eta)\,d\eta & \text{on } \partial\Omega_1, \\ \int_{f_1(0)}^{f_1(s_2)} D(\eta)\,d\eta & \text{on } \partial\Omega_2. \end{cases} \tag{15}$$

Now, we will assume that the Dirichlet boundary data on $\partial\Omega$ are compatible at the points (x_0,y_0) and (x_1,y_1), that is, $f_1(x_0,y_0) = f_3(x_0,y_0)$ and $f_1(x_1,y_1) = f_3(x_1,y_1)$, f_1 and f_3 are strictly monotone functions on the boundary $\partial\Omega_2$ and $\partial\Omega_1$, respectively, $\operatorname{range}_{\partial\Omega_2} f_1 \subset \operatorname{range}_{\bar\Omega}\omega$ and $\operatorname{range}_{\partial\Omega_2} f_3 \subset \operatorname{range}_{\bar\Omega}\omega$, where the ranges are not a single point. Then it will be shown that the problem (13)–(15) leads to the existence and uniqueness of the coefficient $D(\omega)$ and function $\omega(x, y)$. These ranges conditions may be guaranteed by invoking the maximum principle and suitable restricting the functions M, f_1, f_2, and f_3. We also assume that the function f_2 is continuous on $\overline{\partial\Omega}$ and without loss of generality we may assume that the data have been normalized with $f_1(x_0, y_0) = f_3(x_0, y_0) = 0$.

Now, by substituting expression (11) in the problem (13)–(15), and using Green's second formula, we obtain

$$V(x, y) = \iint_{\Omega} G^* M \, d\xi \, d\eta - \int_{\partial\Omega_2} G^* f_2 \, ds_2 + \int_{\partial\Omega_2} \frac{\partial G^*}{\partial n} \cdot \left(\int_0^{f_3(s_1)} D(\eta) \, d\eta \right) ds_1, \quad (16)$$

where $G^*(\xi, \eta; x, y)$ is the Green's function for Laplace equation in Ω subject to Dirichlet conditions on $\partial\Omega_1$ and Neumann on $\partial\Omega_2$ [4,12,13].

Thus, from (18) and the overspecified condition (17), we find

$$\int_0^{f_1(s_2)} D(\eta) \, d\eta = \iint_{\Omega} G^* M \, d\xi \, d\eta - \int_{\partial\Omega_2} G^* f_2 \, ds_2 + \int_{\partial\Omega_2} \frac{\partial G^*}{\partial n} \cdot \left(\int_0^{f_3(s_1)} D(\eta) \, d\eta \right) ds_1, \quad (17)$$

Putting

$$\Psi = \iint_{\Omega} G^* M \, d\xi \, d\eta - \int_{\partial\Omega_2} G^* f_2 \, ds_2, \quad (18)$$

that is known and for function $\varphi(s_1)$ defined on $\partial\Omega_1$, define the mapping $K : \partial\Omega_1 \to \partial\Omega_2$ by

$$K\big[\varphi(s_1)\big] = \int_{\partial\Omega_1} \frac{\partial G^*}{\partial n} \bigg|_{s = s_2} \varphi(s_1) \, ds_1. \quad (19)$$

We may characterize K as a linear operator of Hilbert transform operator kind with the kernel $\frac{\partial G^*}{\partial n}$ which maps the solution of Laplace equation in Ω with Dirichlet data φ on $\partial\Omega_1$ and homogeneous Neumann data on $\partial\Omega_2$ to its value on $\partial\Omega_2$. Therefore from (12), (17)–(19), we obtain

$$T_D\big(f_1(s)\big) = \Psi(s) + K\big[T_D(f_3)\big]. \quad (20)$$

Now from invertibility f_1 and f_3, we find

$$T_D(\alpha) = \Psi\big(f_1^{-1}(\alpha)\big) + \int_{\partial\Omega_1} \frac{\partial}{\partial n} G^*\big(f_1^{-1}(\alpha), \beta\big) f_3'\big(f_3^{-1}(\beta)\big) T_D(\beta) \, d\beta, \quad (21)$$

or

$$T_D = \Psi + K[T_D], \quad (22)$$

where $\alpha = f_1(s_2)$ and $\beta = f_3(s_1)$. To recover function T_D from (20), it would be necessary to make the assumption that f_1 and f_3 are strictly monotone functions on their domains. This requirement is typical of such recovery problems for partial differential equations that contain an unknown function of ω, this implies that the existence of the coefficient $D(\omega)$ and ω [3,8,15–17].

The unicity solution $(D(\omega), \omega)$ to the inverse problem (1)–(5) may be obtained from the following theorem.

Theorem. *For any given piecewise-continuous functions q, f, f_1, f_2, and f_3 such that $f_1(x_0, y_0) = f_3(x_0, y_0)$, $f_1(x_1, y_1) = f_3(x_1, y_1)$,* $\mathrm{range}_{\partial\Omega_1} f_3 \subset \mathrm{range}_{\partial\Omega_2} f_1$, *the functions f_1, f_3 are strictly monotone, and the inverse problem* (1)–(5) *has a continuous solution on $\bar{\Omega}$, the solution pair $(D(\omega), (\omega))$ of the problem* (1)–(5) *is unique.*

Proof. From (11), clearly the continuous solution $M(x, y)$ to the problem (6)–(7) is unique. Now, if (D_1, ω_1) and (D_2, ω_2) to be two pairs of solution of problem (8)–(10), then by setting $D = D_1 - D_2$ and $V = V_1 - V_2$, where $V_1 = T_{D_1}(\omega_1)$ and $V_2 = T_{D_2}(\omega_2)$, in the problem (13)–(16), we obtain

$$\nabla^2 V(x, y) = 0 \quad \text{in } \Omega, \tag{23}$$

$$\frac{\partial V}{\partial n}(x, y) = 0 \quad \text{on } \partial\Omega_2, \tag{24}$$

$$V(x, y) = \begin{cases} \int_0^{f_3(s_1)} D(\eta)\, d\eta & \text{on } \partial\Omega_1, \\ \int_0^{f_1(s_2)} D(\eta)\, d\eta & \text{on } \partial\Omega_2. \end{cases} \tag{25}$$

Using the strong maximum principle, $V(x, y)$ may not obtain its maximum in the interior of Ω or on the arc $\partial\Omega_2$, where $\frac{\partial V}{\partial n} = 0$. Therefore the maximum values of $V(x, y)$ on $\bar{\Omega}$ must lie in the range of the condition (25) for $s_1 \in \partial\Omega_1$. This assumption implies that the range of $V(x, y)$ must lie in the range of values $V(x, y)$ defined by (25) for $s_2 \in \partial\Omega_2$. The continuity of $f_1(s_2)$ then demands that $V(x, y)$ must attain its maximum on $\partial\Omega_2$, which may only happen if $V(x, y)$ is constant. Since both of $f_1(s_2)$ and $f_3(s_1)$ may not be constant functions. Thus, we conclude that $V(x, y) = 0$, and from (25) the function $D(\omega)$ must be zero for any ω in the range of f_1. This completes the proof of the theorem. □

3. Conclusion

If f_1 and f_3 are both strictly monotonic functions on their domains and continuous at the end points (x_0, y_0) and (x_1, y_1) that implies that $\mathrm{range}_{\partial\Omega_2} f_1 = \mathrm{range}_{\partial\Omega_1} f_3$, we find that there is at most one solution for the inverse problem (1)–(5). The mapping K is a bounded positive operator from the space of $C^1(\partial\Omega_1)$ to $C^1(\partial\Omega_2)$, in fact $\|K\|_\infty = 1$, where $\|\cdot\|_\infty$ denotes the supremum operator norm.

To see this, not that for any $g(s)$ continuous on $\partial\Omega_1$, $K\{g\}$ represent the value of the solution of Laplace equation on the segment of the boundary $\partial\Omega_2$, where $\frac{\partial V}{\partial n} = 0$. As in the proof of theorem, the maximum principle shows that [19]

$$\|K\|_\infty = \frac{\sup_{\partial\Omega_2} |K[g(s)]|}{\sup_{\partial\Omega_1} |g|} \leqslant 1. \tag{26}$$

Equality follows from the fact that if $g = g^{(0)}$ for some constant $g^{(0)}$, then $K[g^{(0)}] = g^{(0)}$. This shows that if constant functions are admissible then 1 is in the spectrum of K, that is, $\frac{\partial G^*}{\partial n}$, has a singularity of the order of $[(x-\xi)^2 + (y-\eta)^2]^{-1}$. Due to the difference in the arguments of the kernel of linear transformation (20), T will not in general be a symmetric operator.

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