Interpolation between $H^p$ Spaces: The Complex Method

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1. INTRODUCTION

For $0 < p < \infty$ let $H^p = H^p(\mathbb{R}^n)$ denote the real variables Hardy space as in the paper of Fefferman and Stein [11]. Also let $BMO = BMO(\mathbb{R}^n)$ denote the space of functions of bounded mean oscillation. Let $(\cdot, \cdot)_\theta$ be the complex method of interpolation as defined in Calderón [4]. The following theorem is the main result of this paper.

**THEOREM 1.** $(H^p_0, L^\infty)_\theta = (H^p_0, BMO)_\theta = H^p$, $0 < p_0 < \infty$, $0 < \theta < 1$, $1/p = (1 - \theta)/p_0$.

**COROLLARY 1.** Let $X_0$ be either $H^1$ or $L^1$ and let $X_1$ be either $L^\infty$ or $BMO$. Then $(X_0, X_1)_\theta = L^p$, $0 < \theta < 1$, $1/p = 1 - \theta$.

The corollary follows immediately from Theorem 1 and Fefferman's theorem (see [11]) that $BMO$ is the dual space of $H^1$.

Before proceeding to further results we make some comments on the history of Theorem 1. Unfortunately, the "theorem" $(H^1, L^\infty)_\theta = L^p$, $1/p = 1 - \theta$, has been folklore for many years, and is stated, e.g., in [2, 22]. This is due to a small but inopportune typographical error in [11]. It is stated on page 157 of that paper that $(H^1, L^p)_\theta = L^q$, $1/q = 1 - \theta + \theta/p$, $1 < p \leq \infty$.

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The mistake lies in the statement $1 < p \leq \infty$; the authors meant to write $1 < p < \infty$ as their methods cannot handle the case where $p = \infty$. It is also proved in that paper that $(L^p, BMO)_\theta = L^q$, $1/q = (1 - \theta)/p$, $1 < p < \infty$. In fact, this result is by duality equivalent to the result

$$(H^1, L^p)_\theta = L^q, \quad 1/q = 1 - \theta + \theta/p, \quad 1 < p < \infty.$$ 

In a later paper Calderón and Torchinsky [6] showed $(H^{p_0}, H^{p_1})_\theta = H^p, 1/p = (1 - \theta)/p_0 + \theta/p_1$, $0 < p_0 < p_1 < \infty$, and also $(H^{p_0}, BMO)_\theta \subset H^p, 1/p = (1 - \theta)/p_0$, $0 < p_0 < \infty$. (The first result of Calderón and Torchinsky had been earlier proved in dimension one by Salem and Zygmund [23].) In order to prove Theorem 1 we need therefore only demonstrate that $(H^{p_0}, L^\infty)_\theta = H^p, 1/p = (1 - \theta)/p_0$. We also remark that Theorem 1 is known in dimension one. Let $H^p_\alpha$ denote the class of functions in $H^p(\mathbb{R})$ which admit a holomorphic extension to the upper half plan $\mathbb{H}^2_+$, and let $H^\infty_\alpha$ denote the ring of (boundary values of) functions bounded and holomorphic in $\mathbb{R}^2_+$. In [16] the second author showed $(H^{p_0}_\alpha, H^{\infty}_\alpha)_\theta = H^p_\alpha, 1/p = (1 - \theta)/p_0, 0 < p_0 < \infty$. This result is stronger than the one-dimensional version of Theorem 1 of this paper. See [16] for details. On the other hand, the methods of [16] use function theory in a very crucial manner and so cannot be directly generalized to $\mathbb{R}^n$ when $n \geq 2$. Nevertheless, some of the reasoning used in [16] can be carried over to higher dimensions; the idea behind our Lemmas 4.1 and 5.1 comes from Section 4 of that paper. Our proof of Theorem 1 depends heavily upon the theory of $H^p$ atoms [5, 8, 17, 18, 26]. It has been known for some time that there is a connection between atoms and interpolation; see, e.g., [19]. Finally, we remark that the analogue of Theorem 1 for the real method of interpolation is known. See [10, 14].

Theorem 1 has analogues for martingale $H^p$ spaces. Given an increasing sequence of $\sigma$-fields we define the martingale $H^p$ spaces, $1 \leq p \leq \infty$, and $BMO$ as in Garsia's book [13]. For $0 < p < 1$ we define $H^p$ to be the class of martingales whose square functions are in $L^p$. Other definitions of $H^p$ are also possible and this will be discussed in Section 3. In analogy with the work of Calderón and Torchinsky [6] we obtain the following result.

**Theorem 2.** For any increasing sequence of $\sigma$-fields, $(H^{p_0}, L^\infty)_\theta \subset (H^{p_0}, BMO)_\theta \subset H^p, 0 < p_0 < \infty, 0 < \theta < 1, 1/p = (1 - \theta)/p_0$.

With Theorem 2 in hand we use $H^1, BMO$ duality and an abstract argument to prove

**Theorem 3.** For any increasing sequence of $\sigma$-fields, $(H^{p_0}, BMO)_\theta = H^p, 1 \leq p_0 < \infty, 0 < \theta < 1, 1/p = (1 - \theta)/p_0$.

For any general martingales it is not always the case that
(H_{p_0}, BMO)_{\theta} = H^p, \quad 1/p = (1 - \theta)/p_0, \text{ when } p_0 < 1. \text{ We exhibit an increasing sequence of } \sigma\text{-fields such that } (H_{p_0}, L^{\infty})_{\theta} = (H_{p_0}, BMO)_{\theta} \subseteq H^p \text{ whenever } 0 < p_0 < 1. \text{ (After this paper was first written our colleague Tom Wolff showed [27] that Theorem 3 remains valid if BMO is replaced by } L^{\infty} \text{ there.) However, for } m\text{-adic martingales we obtain an analogue of Theorem 1. (We say "m-adic" instead of the more commonly used "p-adic" in order to avoid notational confusion.)}

**Theorem 4.** For m-adic martingales, \((H_{p_0}, L^{\infty})_{\theta} = (H_{p_0}, BMO)_{\theta} = H^p, 0 < p_0 < \infty, 0 < \theta < 1, 1/p = (1 - \theta)/p_0.\]

**Corollary 2.** Let \(X_0\) be either \(H^1\) or \(L^1\) and let \(X_1\) be either \(L^{\infty}\) or BMO. Then for m-adic martingales, \((X_0, X_1)_{\theta} = L^p, 0 < \theta < 1, 1/p = 1 - \theta.\]

Theorem 4 and Corollary 2 hold more generally if the \(\sigma\)-fields \(\{\mathcal{F}_n\}\) satisfy the doubling condition:

Each \(\mathcal{F}_n\) is generated by a finite partition \(\{Q_{n,i}\}\) of the underlying probability space and there is a constant \(M\) such that \(|Q_{n,i}| \leq M |Q_{n+1,j}|\) whenever \(Q_{n+1,j} \subseteq Q_{n,i}\). In Garsia’s book [13, pp. 88–90] such \(\sigma\)-fields are called “regular.”

Theorem 4 is easier to prove than Theorem 1; it is usually the case in the theory of \(H^p\) spaces that a martingale theorem is easier to prove than its \(L^p\) analogue. Moreover, Theorem 4 has the pleasant consequence that the most interesting case of Theorem 1 ((\(H^1, L^{\infty}\))_{\theta} = L^p) follows from it directly. To see this, consider the problem of showing that a function \(f \in L^p(\mathbb{R}^n), 1 < p < \infty, \text{ lies in } (H^1, L^{\infty})_{\theta}.\) An easy argument shows that it is sufficient to do this (with proper norm control) when \(\text{supp } f \subseteq [0, 1]^n\) and \(\int f \, dx = 0.\) Let \(\mathcal{F}_m\) be the \(\sigma\)-field generated by the dyadic cubes lying in \([0, 1]^n\) of edgelength \(2^{-m}.\) Martingales arising from this \(\sigma\)-field are evidently \(2^m\)-adic and so Theorem 4 applies to them. Consequently, \(f \in (H^1_d, L^{\infty})_{\theta} \quad \text{(with proper norm control), where } H^1_d \text{ is dyadic } H^1. \text{ But } H^1_d \subseteq H^1(\mathbb{R}^n) \text{ and if } g \in H^1_d, \|g\|_{H^1} \leq C \|g\|_{H^1_d}. \text{ (There are several ways to see this. One way is to observe that every function in } H^1_d \text{ can be written as a sum of dyadic } H^1 \text{ atoms. Each dyadic } H^1 \text{ atom is also an } H^1 \text{ atom. Another way is to notice that } (H^1)^* = BMO \subseteq \text{dyadic } BMO = (H^1_d)^*. \text{ Therefore } f \in (H^1, L^{\infty})_{\theta} \text{ with proper norm control. We leave as an exercise to the reader the fact that this reasoning cannot be used to prove Theorem 1 when } p_0 < 1. \text{ For a further discussion of the relation between dyadic } H^p \text{ spaces and } H^p(\mathbb{R}^n) \text{ we refer the reader to the papers of Davis [9] and Garnett and Jones [12].}

The paper is organized as follows. Section 2 is devoted to a discussion of the complex method of interpolation. The complex method of interpolation is usually defined only for Banach spaces, whereas \(H^p\) is only a quasi-Banach space when \(p < 1.\) Evidently some explaining is in order. Section 3 is devoted
to general martingales. Theorems 2 and 3 are proved there and we give an example which shows that for general martingales it is possible to have \((H^{p_0}, BMO) \subseteq H^p\) if \(p_0 < 1\). Theorem 4 is proved in Section 4 and Theorem 1 is proved in Section 5.

2. Complex Methods of Interpolation

There are several possible ways of defining "the complex interpolation space" and, in particular, outside the realm of Banach spaces they may conceivably give different results. Since, as we will see at the end of this section, the consideration of more than one version is advantageous in connection with interpolation of analytic families of operators, we will here study the general question. We will only treat quasi-Banach spaces (thus including \(H^p\)).

We base the discussion on the classical construction by Calderón [4]. For a couple (of Banach spaces) \(X_0\) and \(X_1\) he first defined a space \(\mathcal{S}\) of analytic functions from the strip \(\mathcal{S} = \{z: 0 \leq \Re z < 1\}\) to \(X_0 + X_1\). He then put \((X_0, X_1)_\theta = \{F(\theta) : F \in \mathcal{S}\}\), equipped with the quotient norm. (Calderón also defined another interpolation method (\(\mathcal{F}\), but we will not employ that construction.) This construction may be used with natural definitions of the space of analytic function other than Calderón's. Since there are several possible definitions, differing, e.g., in the conditions on the continuity at the boundaries, this gives conceivably different interpolation methods. Note, however, that it is possible for two different spaces \(\mathcal{S}\) to define the same interpolation space. For example, in the definitions below we may impose the condition \(F(z) \to 0\) as \(|z| \to \infty\) without changing the interpolation spaces. (To see this, multiply by suitable scalar functions.)

We assume that \(\mathcal{S}\) is normed by

\[
\|F\|_{\mathcal{S}} = \sup_{-\infty < y < \infty} \sup_{z \in \mathcal{S}} \left( \|F(iy)\|_{X_0}, \|F(1+iy)\|_{X_1}, \|F(z)\|_{X_0+X_1} \right)
\]

(The third term is superfluous when the maximum principle applies, as in Calderón's case. However, the maximum principle fails for some quasi-Banach spaces; cf. Peetre [20].) Thus all functions in \(\mathcal{S}\) are bounded. Also

\[
\|F\|_{\mathcal{S}} \geq \|F(\theta)\|_{X_0+X_1}
\]

and thus

\[
\|x\|_{X_0+X_1} \leq \|x\|_{(X_0,X_1)_\theta}.
\]

After these preliminaries we define the strong complex interpolation space
(X_0, X_1)_\theta^s$ by taking $\mathcal{F}$ to be the closure in the norm above of the set of bounded continuous functions from $\bar{S}$ to $X_0 + X_1$ that have finite-dimensional ranges included in $X_0 \cap X_1$ and are analytic in the interior, i.e., functions $\sum_k^n F_k x_k$, where $x_k \in X_0 \cap X_1$ and $F_k$ are bounded, continuous and analytic scalar functions. This is close to the definition by Riviere [21]. For Banach spaces it is equivalent to Calderón’s definition.

We also define the weak complex interpolation space $(X_0, X_1)_\theta^w$ by taking $\mathcal{F}$ to be the set of all bounded functions $F$ such that $\langle U, F(z) \rangle$ is analytic on $S$ and continuous on $\bar{S}$ for any $U \in (X_0 + X_1)^*$. Clearly the weak interpolation spaces contain the strong ones. The inclusion may be strict in the rather trivial case when $(X_0 + X_1)^* = \{0\}$ (e.g., $X_0 = L^{p_0}$, $X_1 = L^{p_1}$, $p_0 < 1$, $p_1 < \infty$), $(X_0, X_1)_w = X_0 + X_1$. It is an open question whether the weak and the strong spaces always coincide for Banach spaces $X_0$ and $X_1$, but we can prove this equality in many cases, e.g., when one of the spaces $X_0$ or $X_1$ is separable or when one of them contains the other.

For the $H^p$-spaces studied in this paper both methods may be used except in Theorem 2, where the strong one is needed. For $\mathbb{R}^n$ the construction in Section 5 together with the result by Calderón and Torchinsky [6] show that $H^p (\mathbb{R}^n) \subset (H^{p_0}, L^\infty)_\theta \subset (H^{p_0}, L^\infty)_\theta^w = H^p$. (In fact, it suffices to test functions $U$ in the definition of $(H^{p_0}, L^\infty)_\theta^w$.) The same is true for $m$-adic martingales by Sections 3 and 4. However, for general sequences of $\sigma$-fields the dual of $H^p$ may be too small to allow the weak method in Theorem 2. (E.g., take $\mathcal{F}_n$ to be the Borel $\sigma$-field on $(0, 1)$ for all $n$. Then $H^p = L^p(0, 1)$). Both the strong and the weak methods are interpolation functors; i.e., if $T$ is a linear operator from $X_0 + X_1$ to $Y_0 + Y_1$ which is bounded from $X_0$ to $Y_0$ and from $X_1$ to $Y_1$, then $T$ is bounded from $(X_0, X_1)_\theta$ to $(Y_0, Y_1)_\theta$ [2]. The reason for introducing the weak interpolation spaces is the following abstract version of the Stein interpolation theorem [24]. Let $A(S)$ be the algebra of (complex-valued) bounded, continuous functions on $\bar{S}$ that are analytic on $S$.

**Theorem 0.** Let $T_z, z \in \bar{S}$ be a family of linear operators from $X_0 \cap X_1$ to $Y_0 + Y_1$ such that $\langle U, T_z x \rangle \in A(S)$ for any $x \in X_0 \cap X_1$ and $U \in (Y_0 + Y_1)^*$ and $\sup \|T_z\|_{x_0, y_0} \|T_{z + it}\|_{x_1, y_1} < \infty$. Assume further that either the maximum principle holds for $Y_0 + Y_1$ in the weak version $\sup \|F(x)\|_{y_0 + y_1} \leq C \sup \|F(it)\|_{y_0} \|F(1 + it)\|_{y_1}$ or for all functions $F$ from $S$ to $Y_0 + Y_1$ such that $\langle U, F(x) \rangle \in A(S)$ for $U \in (Y_0 + Y_1)^*$, or that $\sup \|T_z\|_{x_0 + x_1, y_0 + y_1} < \infty$. Then $T$ maps $(X_0, X_1)_\theta$ continuously into $(Y_0, Y_1)_\theta^w$ if whenever $x \in X_0 \cap X_1$, $x = F(\theta)$ for some $F = \sum_k^n F_k x_k$ with $F_k \in A(S)$, $x_k \in X_0 \cap X_1$, and $\|F\|_{w} \leq C \|x\|_{(L_0, L_1)}$.

**Proof.** Let $\mathcal{F}_s$ and $\mathcal{F}_w$ be the spaces of analytic functions used in the definitions of $(X_0, X_1)_\theta$ and $(Y_0, Y_1)_\theta^w$, respectively. Let $F = \sum F_k x_k$, $F_k \in A(S)$, $x_k \in X_0 \cap X_1$. Then $\langle U, T_z F(x) \rangle = \sum F_k(z) \langle U, T_z x_k \rangle \in A(S)$ for
U ∈ (Y_0 + Y_1)^\ast. Hence T_z F(z) ∈ \mathcal{F}' \text{ and } \| T_z F \|_{\mathcal{F}'} \leq C \| F \|_{\mathcal{F}}. \text{ By continuity this is true for any } F ∈ \mathcal{F}.'

**Remarks.** 1. The boundedness conditions may be relaxed to "admissible growth" [24]. (Multiply by suitable scalar functions.)

2. For couples of $H^p$-spaces (and $BMO$) this maximum principle holds, and it suffices to take $U$ to be test functions. This follows from [6, Theorem 3.1].

3. The maximum principle holds in Banach spaces.

The same proof shows that for any space $\mathcal{F}(Y_0, Y_1)$ which is complete and an $A(S)$-module, $T_\theta \text{ maps } (X_0, X_1)_\theta \text{ boundedly into the corresponding space } (Y_0, Y_1)_\theta'$, if $\{T_z\}$ is bounded as above and $T_z x ∈ \mathcal{F}$ for any $x ∈ X_0 ∩ X_1$.

With sufficiently strong conditions on $\{T_z\}$ we thus obtain $T_\theta: (X_0, X_1)_\theta \to (Y_0, Y_1)_\theta'$, but the conditions are awkward for applications. An interpolation method between the strong and weak ones is obtained by taking $\mathcal{F}$ as $\mathcal{F}'$ with the extra condition that $F(it)$ be continuous in $Y_0$ and $F(1 + it)$ in $Y_1$.

For Banach spaces this gives the same interpolation space as the strong method. Hence we may strengthen the conclusion of the theorem above to $T_\theta: (X_0, X_1)_\theta \to (Y_0, Y_1)_\theta'$ for Banach spaces $Y_0, Y_1$, if $T_z x$ satisfies this extra continuity condition.

### 3. General Martingales

In this section we study martingales with respect to a fixed, increasing sequence $\{X_n\}_F$ of $\sigma$-fields on some probability space $(\Omega, \mathcal{F}, P)$. For a martingale $\{f_n\}_1^\infty$ we define $f^* = \sup_n |f_n|$, $A f_n = f_n - f_{n-1}$ (where $f_0 = 0$), and $S_n = (\sum_{k=1}^{n} |A f_k|^2)^{1/2}$. The spaces $H^p$, $0 < p < \infty$, and $BMO$ are defined to consist of all martingales such that the norms (quasi-norms for $p < 1$)

$$\|f\|_{H^p} = \|S_1\|_{L^p} \quad \text{and} \quad \|f\|_{BMO} = \sup_n \|E(S_{n\infty}^2 \mid \mathcal{F}_n)\|_{L_{\infty}^2}$$

are finite. For $1 < p < \infty$ $H^p = L^p(\Omega, \mathcal{F}_\infty, P)$ and for $1 ≤ p < \infty$, $H^p = \{\{f_n\} : f^* ∈ L^p\}$. See [13] for details. For $p < 1$ the last equality holds, e.g., for $m$-adic martingales but not in general [3].

Thus our definition of $H^p$ is not the only reasonable one for $p < 1$. One alternative definition would be $f^* ∈ L^p$; others have been proposed by Herz [15]. For $m$-adic martingales all of these definitions agree. Furthermore, we give at the end of this section an example which shows that Theorem 4 is false for general martingales, and this example is equally valid for all of the above mentioned definitions of $H^p$. Therefore we will use only the definition

$$\|f\|_{H^p} = \|S_{1\infty}\|_{L^p}.$$

The following proof is patterned after the proof for $\mathbb{R}^n$ in [6].

**Proof of Theorem 2.** Let $f(z) = \{f_n(z, w)\}$ be an analytic family of
martingales such that \( \|f(it)\|_{H^p} \leq 1 \) and \( \|f(1 + it)\|_{BMO} \leq 1 \). More precisely, we assume that \( f(z) \) satisfies one of two conditions. Firstly, we could assume that \( f(z) = \sum f_k(z) g_k \) for some scalar functions \( f_k \in A(S) \) and martingales \( g_k \), where the sum is finite. The argument given below plus a limiting argument then show that the strong interpolation space \( (H^p, BMO)^\varphi \subset H^p \). Secondly, we could assume that the \( \sigma \)-fields \( \mathcal{F}_n \) are generated by countable partitions of \( \Omega \). In this case we only have to assume that \( f \in \mathcal{F}^w \), and we will obtain the stronger conclusion \( (H^p, BMO)^\varphi \subset H^p \). If either of the two above cases holds, it is not hard to check that for every \( w \in \Omega \), \( \log S_{nN}(z, w) \) is a continuous subharmonic function of \( z \in S \). We leave the verification of this fact to the reader.

First assume that \( p > 1 \). Choose \( r < \min(1, p_0) \). For some positive measures \( \mu_0 \) and \( \mu_1 \) with mass one (the Poisson kernel for the strip),

\[
\begin{align*}
    r \log S_{nN}(\theta, w) &\leq \int_{-\infty}^{\infty} r \log S_{nN}(it, w)(1 - \theta) \, d\mu_0(t) \\
    &\quad + \int_{-\infty}^{\infty} r \log S_{nN}(1 + it, w) \theta \, d\mu_1(t).
\end{align*}
\]

By Jensen's inequality,

\[
S_{nN}^r(\theta, w) \leq \int_{-\infty}^{\infty} S_{nN}^{(1 - \theta)r}(it, w) \, d\mu_0(t) \int_{-\infty}^{\infty} S_{nN}^{\theta r}(1 + it, w) \, d\mu_1(t). \tag{3.1}
\]

Letting \( N \to \infty \) and invoking Hölder's inequality, (3.1) yields

\[
\begin{align*}
    E(S_{n\infty}^r(\theta) | \mathcal{F}_n) &\leq E \left( \left( \int S_{n\infty}^{(1 - \theta)r}(it, w) \, d\mu_0 \right)^{1/(1 - \theta)} \right) \left( \int S_{n\infty}^{\theta r}(1 + it, w) \, d\mu_1 \right)^{\theta} \\
    &\quad \times \left( E \left( \int S_{n\infty}^{r}(it, w) \, d\mu_0 \right)^{1 - \theta} \right) \\
    &\quad \times \left( E \left( \int S_{n\infty}^{r}(1 + it, w) \, d\mu_1 \right)^{\theta} \right) \\
    &\leq \left( E \left( \int S_{n\infty}^{r}(it, w) \, d\mu_0 \right) \right)^{1 - \theta} \\
    &\quad \times \left( E \left( \int S_{n\infty}^{r}(1 + it, w) \, d\mu_1 \right) \right)^{\theta} \\
    &= \left( E \left( \int S_{n\infty}^{r}(it, w) \, d\mu_0 \right) \right)^{1 - \theta} \\
    &\quad \times \left( E \left( \int S_{n\infty}^{r}(1 + it, w) \, d\mu_1 \right) \right)^{\theta}.
\end{align*}
\]
\begin{equation}
\left( \sup_n E \left( \int S'_{1,\infty}(t, w) \, d\mu_0 \bigg| \mathcal{F}_n \right) \right)^{1-\theta} \\
= M(w)^{1-\theta} \quad \text{a.s.},
\end{equation}

say, since

\[ E(S'_{n,\infty}(1 + it, w) \bigg| \mathcal{F}_n) \leq E(S^2_{n,\infty}(1 + it, w) \bigg| \mathcal{F}_n)^{r/2} \leq \|f(1 + it)\|_{BMO} \leq 1 \quad \text{a.s.} \]

\( M(w) \) is the martingale maximal function of \( \int S'_{1,\infty}(it, w) \, d\mu_0 \). Consequently,

\begin{align*}
E(M(w)^{p_0/r}) &\leq CE \left( \int S'_{1,\infty}(it, w) \, d\mu_0 \right)^{p_0/r} \\
&\leq CE \left( \int S^2_{1,\infty}(it, w) \, d\mu_0 \right) \\
&= C \int \|f(it)\|_{H^{p_0}}^2 \, d\mu_0 \\
&\leq C.
\end{align*}

Let \( n(w) = \inf \{n: S_{n,\infty}(\theta, w) < 2^{1/r}M(w)^{(1-\theta)/r} \} \) and let \( \psi_n = I(n(w) > n) = I(S'_{n,\infty}(\theta, w) > 2M(w)^{1-\theta}) \). By (3.2),

\[ E(S'_{n,\infty}(\theta) \bigg| \mathcal{F}_n) \geq E(2M^{1-\theta}\psi_n \bigg| \mathcal{F}_n) \]

\[ \geq E(2E(S'_{n,\infty}(\theta) \bigg| \mathcal{F}_n) \psi_n \bigg| \mathcal{F}_n) \]

\[ = 2E(S'_{n,\infty}(\theta) \bigg| \mathcal{F}_n) E(\psi_n \bigg| \mathcal{F}_n). \]

Thus \( P(n(w) > n \big| \mathcal{F}_n) = E(\psi_n \big| \mathcal{F}_n) \leq 1/2 \) a.s. on \( \{E(S'_{n,\infty}(\theta) \big| \mathcal{F}_n) > 0 \} \). Since \( |A_{f_n}^r| = E(|A_{f_n}^r| \bigg| \mathcal{F}_n) \leq E(S'_{n,\infty}(\theta) \bigg| \mathcal{F}_n), \) \( A_{f_n}^r = 0 \) a.s. on \( \{E(S'_{n,\infty}(\theta) \big| \mathcal{F}_n) = 0 \} \). Consequently, \( |A_{f_n}^r| \leq 2 |A_{f_n}^r| P(n(w) \leq n \big| \mathcal{F}_n) \) a.s. Now let \( q \) be the exponent conjugate to \( p \). If \( \phi \in L^q, \)

\[ |E(\phi f)| = \left| \sum_{1}^{\infty} E(\Delta \phi_n A_{f_n}) \right| \]

\[ \leq \sum_{1}^{\infty} E(|\Delta \phi_n| A_{f_n}) \]

\[ \leq 2 \sum_{1}^{\infty} E(|\Delta \phi_n| A_{f_n} P(n(w) \leq n \big| \mathcal{F}_n)) \]
\[= 2 \sum_{n=1}^{\infty} E(\|A\phi_n\| \|Af_n\| I(n(w) \leq n))\]

\[= 2E \left( \sum_{n(w)} \|A\phi_n\| \|Af_n\| \right)\]

\[\leq 2E(S_{n(w)}(\phi) S_{n(w)}(f))\]

\[\leq 2E(S_{1\infty}(\phi) 2^{1/r} M(w)^{(1-\theta)/r})\]

\[\leq C \|S_{1\infty}(\phi)\|_{L^p} E(M(w)^{(1-\theta)p/r})^{1/p}\]

the last inequality following from \((3.3)\). Consequently, \(f(0) \in L^p = H^p\).

If \(p < 1\), we conclude from the first case that \(\|S_{1\infty}(1-p/2 + it)\|_{L^2} \leq C\). An argument similar to \((3.2)\) then gives \(E(S_{1\infty}(\phi)) \leq C\), i.e., \(f(\theta) \in H^p\). (This argument shows that \((H^{p_0}, H^p')_{\theta} \in H^p\), \(1/p = (1/p_0)(1-\theta) + (1/p_1)\theta\).) Theorem 2 is proved.

**Proof of Theorem 3.** We are now dealing with Banach spaces. By the reiteration theorem it suffices to prove the result for \(p_0 = 1\). By the duality theorem and Theorem 2, \(H^p = H^{*c} \subseteq (H^1, BMO)^{1-\theta} = (BMO, H^1)^{1-\theta} = (BMO^*, H^{1*})^\theta = (BMO^*, BMO)^\theta\). If \(f \in BMO\) (which is dense in \(H^1\)) it has the same norm in \((BMO^*, BMO)^\theta\), \((BMO^*, BMO)_\theta\) and \((H^1, BMO)_\theta\) (Bergh [1]). By the inclusion above and Theorem 2, this norm is equivalent to \(\|f\|_{H^p}\), and the theorem follows.

In the next section we shall extend Theorem 2 to \(p_0 < 1\) for \(m\)-adic martingales. In general this is not possible.

**Example.** Let \(A_k\) and \(B_k\), \(k = 1, 2, \ldots\), be disjoint subsets of a probability space such that \(P(A_k) = 4^{-k}\) and \(P(B_k) = 2^{-k} - 4^{-k}\). Let \(\mathcal{F}_1 = \{\phi, \Omega\}\) and let \(\mathcal{F}_2\) and \(\mathcal{F}_3 = \mathcal{F}_4 = \cdots\) be the \(\sigma\)-fields generated by \(\{A_k \cup B_k\}\) and \(\{A_k\} \cup \{B_k\}\), respectively. We may identify martingales with functions that are constant on the sets \(A_k\) and \(B_k\). Notice that \(BMO = L^\infty\) in this example. Let \(p_0 < 1 < p < \infty\), \(1 - \theta = p_0/p\), and let \(f(z)\) be as in the proof of Theorem 2. Let \(g_k(z)\) and \(h_k(z)\) be the values of \(f(z)\) on \(A_k\) and \(B_k\). These are analytic functions. Furthermore, \(E(f(z) | \mathcal{F}_2) = 2^{-k} g_k(z) + (1 - 2^{-k}) h_k(z)\) on \(A_k \cup B_k\). If \(\text{Re } z = 0\), then \(\|f(z)\|_{H^p} \leq 1\) and

\[2^{-k(1+p_0)}(1 - 2^{-k}) \|g_k(z)\|_{p_0}^p \leq \int_{B_k} |2^{-k} g_k(z)|_{p_0}^p\]

\[\leq \int_{B_k} |E(f(z) | \mathcal{F}_2)|_{p_0}^p\]

\[+ \int_{B_k} |(1 - 2^{-k}) f(z)|_{p_0}^p \leq C.\]
Therefore $|g_k(it)| \leq C2^{k(1+1/p_0)}$. Since $|g_k(1+it)| \leq C\|f(1+it)\|_{BMO} < C$, interpolation yields $|g_k(\theta)| \leq C2^{k(1-\theta)(1+1/p_0)}$ and

$$
\int_{A_k} |f(\theta)|^p = \int_{A_k} |g_k(\theta)|^p
\leq C4^{-k2^{k(1-\theta)p(1+1/p_0)}}
= C2^{-k(1-p_0)}.
$$

Hence $f(\theta)$ cannot be an arbitrary function in $H^p = L^p$ and $(H^{p_0}, BMO) \subsetneq H^p$.

### 4. Regular Martingales

We assume that $\{\mathcal{F}_n\}$ satisfies the doubling condition stated in the introduction. In this section the letter $Q$ and the word cube are used for sets $Q_{n,i}$ in the partition defining $\mathcal{F}_n$. For any cube $Q = Q_{n,i} \in \mathcal{F}_n$, the unique cube $Q_{n-1,i} \in \mathcal{F}_{n-1}$ containing $Q$ will be denoted $\bar{Q}$ and called the double of $Q$.

Let $f \in H^p$ with norm 1. In this section we construct a function $G_z$ from $S$ to a finite-dimensional subspace of $H^{p_0} \cap L^\infty$. $G_z$ is analytic and $\|G_z\|_{H^{p_0}}$, $\|G_{1+it}\|_{L^\infty}$, $\|G_z\|_{H^{p_0}+L^\infty} \leq C$ and $\|G_\theta-f\|_{H^p} < 1/2$. From this $H^p \subset (H^{p_0}, L^\infty)_\theta$ follows by iteration and, because of Theorem 2, Theorem 4 follows.

We construct the function $G_z$ using a particular atomic decomposition of $f = \{f_n\}$. Since $f_1 \in H^{p_0} \cap L^\infty$, we may assume that $f_1 = 0$. We start by approximating $f$ by $F = f_N$, where $N$ is so large that $\|f - F\|_{H^p} \leq 1/4$. Notice that $F^* \leq f^*$.

**Step 1**

We construct an atomic decomposition using an argument similar to stopping times. (However, we peek into the future.)

**Lemma 4.1.** There is a finite collection of cubes $\{Q_j\}$ and there are functions $a_j$ supported on $Q_j$ such that $F = \sum a_j$. Also, there are integers $m(j)$ such that

$$
\sum 2^{nm(j)} |Q_j| \leq C; \quad (4.1)
$$

if $Q_k \subset Q_j$ and $k \neq j$, then $m(k) > m(j)$;

$$
\sum_{Q_k = Q_j} |Q_k| < 2 |Q_j|; \quad (4.3)
$$
\[ \|a_j\|_{L^\infty} \leq C2^{m(j)}; \quad (4.4) \]
\[ \int a_j = 0. \quad (4.5) \]

**Proof.** We will construct the cubes and functions as doubly indexed families. A rearrangement then yields the stated conclusion. Let \( \Omega_m = \{w: F^* > 2^m\} \).

At stage one let \( Q_1 = \Omega \) and let \( m(1, 1) \) be the smallest integer such that \( |\Omega_{m(1,1)}| < 1/2M \), where \( M \) is the constant in the doubling property. Let \( D_1 \) be the set of all cubes included in \( \Omega_{m(1,1)} \).

At stage two, select the maximal cubes from \( D_1 \), double each one of them, and let \( W_2 = \{Q_j^2\} \) be the resulting set, i.e., \( W_2 = \{Q: Q \text{ is maximal in } D_1\} \).

For each cube \( Q_j^2 \in W_2 \) let \( m(2,j) \) be the smallest integer such that \( |Q_j^2 \cap \Omega_{m(2,j)}| < (1/2M)|Q_j^2| \). Let

\[ D_2 = \bigcup_{Q_j^2 \in W_2} \{Q: Q \subset Q_j^2 \cap \Omega_{m(2,j)}\}. \]

At stage three we define \( W_3 = \{Q_j^3\} = \{Q: Q \text{ is maximal in } D_2\}. \) We proceed by induction and form collections \( D_3, W_4, \ldots \). We may terminate at the \( N \)th stage (\( N \) as in the definition of \( F \)). The sought collection of cubes is \( \bigcup W_i = \{Q_j^i\} \), where \( W_1 = \{Q_1^1\} \). Any cube \( Q_j^{i+1} \in W_{i+1} \) is a subcube of exactly one cube, \( Q_k^i \), say, in \( W_i \). By construction \( Q_j^{i+1} \) is the double of a maximal cube \( Q_j^i \) in \( D_i \). Thus \( |Q_j^{i+1} \cap \Omega_{m(i,k)}| \geq |Q_j^i| > (1/2M)|Q_j^{i+1}| \). By the definition of \( m, m(i+1,j) > m(i, k) \), which gives (4.2). Keeping \( Q_k^i \) fixed we see that \( Q_k^i \cap \Omega_{m(i,k)} \) is the disjoint union of these cubes \( Q_j^i \) for \( Q_j^{i+1} \subset Q_k^i \), and

\[ \sum_{Q_j^{i+1} \subset Q_k^i} |Q_j^{i+1}| \leq M \sum |Q_j^i| = M |Q_k^i \cap \Omega_{m(i,k)}| \]

\[ < 1/2 |Q_k^i|. \]

Inequality (4.3) now follows by induction. By the definition of \( m(i,j), |Q_j^i \cap \Omega_{m(i,j)-1}| \geq (1/2M)|Q_j^i|. \) Since cubes \( Q_j^i \) with the same value of \( m(i,j) \) are disjoint by (4.2),

\[ \sum 2^{pm(i,j)} |Q_j^i| \leq 2M \sum_{m} 2^{pm} \sum_{m(i,j)=m} |Q_j^i \cap \Omega_{m-1}| \]
\[ \leq 2M \sum_{m} 2^{pm} |\Omega_{m-1}| \]
\[ = 2M \int \sum_{m} 2^{pm} \chi_{\Omega_{m-1}} \]

\[ = 2M \left( \sum_{m} 2^{pm} \right) \chi_{\Omega_{m-1}} \]
Finally, we construct the atoms. Let $F(\Omega)$ denote $\frac{1}{Q} \int_{\Omega} F$. For each cube $Q_j$ we define

$$a_j(w) = F(Q_{k+1}^j) - F(Q_j) \quad w \in Q_{k+1}^j \subset Q_j$$

$$= F(w) - F(Q_j) \quad w \in Q_j \bigcup_{k} Q_{k+1}^j$$

$$= 0 \quad w \notin Q_j.$$

Then $\int a_j = 0$ and $F = \sum a_j$. To prove bound (4.4), note that each $Q_{k+1}^j$ contained in $Q_j$, as well as $Q_j$ itself, contains points not belonging to $\Omega_{m(t,j)}$. Thus $|F(Q_{k+1}^j)|, |F(Q_j)| \leq 2^m(t,j)$. Similarly, $w \notin \bigcup_k Q_{k+1}^j$ implies that $w \notin \bigcup \Omega_{m(t,j)}$ and $|F(w)| \leq 2^m(t,j)$.

**Step 2**

We now jiggle the atoms $a_j$ to prevent them from clustering at certain points. The idea of doing this comes from Carleson [7]. Let $\lambda$ be a large integer whose value will be fixed later and put $E_j = \{w \in Q_j: \sum_{Q_k \subset Q} x_{Q_k}(w) \geq \lambda\}$. Since $\sum_{Q_k \subset Q} x_{Q_k}(x)_{L^1} \leq 2 |Q_j|$ by (4.3),

$$|E_j| \leq \frac{2}{\lambda} |Q_j|.$$

**Lemma 4.2.** $\|\sum_j x_{Q_j\cap E_j}\|_{L^\infty} < \lambda$.

**Proof:** Otherwise there is a point $w$ and cubes $Q_1 \cdots Q_{\lambda}$ such that $w \in \bigcap_1^\lambda (Q_j \backslash E_j)$. We may assume that $Q_{\lambda}$ is the largest of these cubes. Then $Q_j \subset Q_{\lambda}$, $1 \leq j \leq \lambda$, and thus $\sum_{Q_k \subset Q_{\lambda}} x_{Q_k}(w) \geq \lambda$, i.e., $w \in E_\lambda$. This is a contradiction.

Let $b_j = (a_j + c_j) x_{Q_j \cap E_j}$, where $c_j = (1/|Q_j \backslash E_j|) \int_{E_j} a_j$ are constants chosen such that $\int b_j = 0$. From (4.4) and (3.6), $|c_j| \leq (C/\lambda) 2^m(t,j)$ and $\|b_j\|_{L^\infty} \leq C2^m(t,j)$.

**Lemma 4.3.** $\|f - \sum b_j\|_{H^p} < 1/2$.

**Proof:** The proof is divided into two cases.

**Case I.** $1 < p < \infty$.

$$\|f - \sum b_j\|_{H^p} \leq \|f - \sum a_j\|_{H^p} + \|\sum a_j - \sum b_j\|_{H^p}$$
At any point the non-zero terms of \( \sum 2^{m(j)} \chi_{E_j} \) form a subsequence of a geometric series. Thus the sum is less than twice the largest term and \( \| \sum 2^{m(j)} \chi_{E_j} \|_{L^p} \leq 2^{p} \sum 2^{pm(j)} \chi_{E_j} \). Thus \( \| \sum 2^{m(j)} \chi_{E_j} \|_{L^p} \leq 2 \sum 2^{pm(j)} \chi_{E_j} = 2^{p} \sum 2^{pm(j)} |E_j| \leq (C_p/\lambda) \sum 2^{pm(j)} |Q_j| \leq C_p/\lambda. \) Similarly \( \| \sum 2^{m(j)} \chi_{Q_j} \|_{L^p} \leq C_p. \) The conclusion now follows if \( \lambda \) is large enough.

Case II. \( p \leq 1. \) Since

\[
\left\| F - \sum b_j \right\|_{H^p}^p \leq \left\| F - \sum a_j \right\|_{H^p}^p + \left\| \sum a_j - \sum b_j \right\|_{H^p}^p \\
\leq 4^{-p} + \sum \| a_j - b_j \|_{H^p}^p
\]

the lemma will follow from the estimate

\[
\| a_j - b_j \|_{H^p} \leq \frac{C_p}{\lambda^{1/2}} 2^{m(j)} |Q_j|,
\]

which follows from the fact that \( a_j - b_j \) is supported on \( Q_j \),

\[
\int a_j - b_j = 0
\]

and

\[
\| a_j - b_j \|_{L^2} \leq \| a_j \chi_{E_j} \|_{L^2} + \| c_j \chi_{Q_j} \chi_{E_j} \|_{L^2} \leq C 2^{m(j)} \left( |E_j|^{1/2} + \frac{1}{\lambda} |Q_j|^{1/2} \right) \\
\leq \frac{C}{\lambda^{1/2}} 2^{m(j)} |Q_j|^{1/2}.
\]

Step 3

We now define the function \( G_z \) by

\[
G_z = \sum (2^{m(j)}(p(1-z)/p_0-1)) b_j.
\]
$G_z$ clearly is holomorphic and bounded on $S_1$ and
\[
\|f - G_\theta\|_{H^p} = \left\| f - \sum b_j \right\|_{H^p} < 1/2
\]
If $\text{Re} \ z = 1$,
\[
|G_z| \leq \sum 2^{-m(j)} |b_j| \leq C \sum \chi Q_j |f_j| \leq C
\]
by Lemma 4.2. Let us now assume that $p_0 < 1$. Then if $\text{Re} \ z = 0$,
\[
\|G_z\|_{H^{p_0}} \leq \sum 2^{m(j)(p - p_0)} \|b_j\|_{H^{p_0}} \leq C \sum 2^{(p - p_0)(m(j))2^{p_0 m(j)}} |Q_j| \leq C,
\]
using (4.1). The case $p_0 > 1$ is similar, but we omit the details since the theorem is classical in this case. This completes the proof of Theorem 4.

5. PROOF OF THEOREM 1

Select $p_0$, $p$, $\theta$ satisfying, $p_0 \leq 1$, $0 < p_0 < p < \infty$ and $1/p = (1 - \theta)/p_0$ and select $f \in H^p$ with norm $\|f\|_{H^p} = 1$. In this section we construct a function $G_z$ defined on the strip $S = \{0 \leq \text{Re} \ z \leq 1\}$. $G_z$ takes values in a finite-dimensional subspace of $H^{p_0} \cap L^\infty$ and is holomorphic and bounded on $S$. $G_z$ also satisfies
\[
\sup_{\tau \in \mathbb{R}} \|G_z \|_{H^{p_0}_{\tau}}, \quad \sup_{\tau \in \mathbb{R}} \|G_{1 + \tau} \|_{L^\infty} \leq C, \quad \|G_\theta - f\|_{H^p} < 1/2.
\]
Once the function $G_z$ is constructed, Theorem 1 follows from an easy iteration argument which we leave to the reader. The construction of the function $G_z$ depends heavily on the theory of atoms [8, 17], though this could be avoided when $p_0 > 1$ by looking at dyadic martingales on $\mathbb{R}^n$ and using the results of Sections 2 and 3. A function $a$ is an $H^q$ atom if $a$ is supported on a cube $Q_i$, $\|a\|_{L^\infty} \leq |Q_i|^{-1/q}^q$, and
\[
\int x^a a \, dx = 0, \quad |a| \leq a(q).
\]
Here $x^a$ is a monomial in $x_1, \ldots, x_n$, i.e., $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 + \cdots + a_n$. The numbers $a(q)$ are decreasing in $q$ and are picked so as to guarantee that $\|a\|_{H^q} \leq C_q$ for every $H^q$ atom. Consequently, every $H^q$ atom $a$ is also an $H^r$ atom (apart from normalization) and $\|a\|_{H^r} \leq C_{a,r} |Q_i|^{1/1-q}$ as long as $r \geq q$. The following theorem is due to Coifman [8] for $\mathbb{R}$ and Latter [17] for $\mathbb{R}^n$. Other proofs can be found in [5,
18, 26]; the proof given in [18] will be found particularly digestible by those readers comfortable with the grand maximal function of Fefferman and Stein [11].

**Theorem A.** Suppose $0 < q \leq 1$ and suppose $f \in H^q$. Then there are complex numbers $\lambda_i$ and $H^q$ atoms $a_i$ such that $f = \sum \lambda_i a_i$ and

$$\sum |\lambda_i|^q \leq C_q \|f\|_{H^q}^q.$$  

The converse to Theorem A is trivial and can easily be proved by the reader. We will need a slightly stronger version of Theorem A which is in fact the version proved (but not stated) in some of the published proofs of Theorem A. Before stating this refinement we will need to introduce some notation.

The symbol $Q$ will always denote some cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and for $\lambda > 0$, $\lambda Q$ denotes the cube with the same center as $Q$ and with $l(\lambda Q) = \lambda l(Q)$, where $l(Q)$ is the sidelength of $Q$. For a tempered distribution $f$ on $\mathbb{R}^n$ let $f^*$ denote the grand maximal function of Fefferman and Stein [11]. For $f \in H^q$ let $\Omega_k = \{f^* > 2^k\}$. Since $\Omega_k$ is open and $\Omega_k \neq \mathbb{R}^n$, $\Omega_k$ admits a Whitney decomposition, $\Omega_k = \bigcup_j Q^*_j$. Each $Q^*_j \subset \Omega_k$ is a dyadic cube and $17/16 Q^*_j \subset \Omega_k$. Furthermore, $\|\sum_j \chi_{Q^*_j}\|_{L^\infty} \leq C$. See chapter VI of [25] for a construction of the Whitney decomposition. The following theorem is the refinement of Theorem A that we will use.

**Theorem B.** Suppose $0 < p_0 < p < \infty$ and suppose $f \in H^p$. Then $f = \sum_{k,j} a_j^k$, where the $a_j^k$ satisfy the following:

1. $a_j^k$ is supported on $Q_j^k \subset \Omega_k$;  
   \[ (5.1) \]
2. $\sum_j |a_j^k(x)| \leq C_{p_0} 2^k \chi_{\Omega_k}(x)$;  
   \[ (5.2) \]
and

$$\int x^\alpha a_j^k \, dx = 0, \quad |\alpha| \leq \alpha(p_0).$$  
   \[ (5.3) \]

Furthermore, the sum $\sum_{k,j} a_j^k$ converges unconditionally in $H^p$.

Theorem B is not stated as such anywhere in the literature, but it follows from the results of [5, 17, 18]. Given $f \in H^p$, those papers show how to decompose $f$ into a sum of $H^{p_0}$ atoms and the constructions used work for all values $0 < p_0 \leq p < \infty$. The reason that Theorem B is not stated in those papers is that the converse statement is false when $p > 1$. In other words, if
$p > 1$ there are sequences $\{a_j\}$ of $H^p$ atoms and $\{\lambda_j\}$ of real numbers such that $\sum |\lambda_j|^p < \infty$ but $\sum \lambda_j a_j \notin H^p$. Notice that the final assertion of Theorem B follows from (5.2) and the fact that $\sum_{k=-\infty}^{\infty} 2^k \chi_{A_k} \in L^p$.

We now proceed with the proof of Theorem 1. The idea is to use Theorem B in combination with a stopping time argument similar to that used in the martingale case. We now fix $f \in H^p$ with $\|f\|_{H^p} = 1$.

**Step 1**

Write $f = \sum a_j^k$ as in Theorem B. Our first step is to rearrange this sum. First, however, we throw a piece of this sum away. Since $\sum a_j^k$ converges unconditionally, there is a positive integer $N$, whose value we now fix, such that $\|f - \sum_{j=N}^{\infty} a_j^k\|_{H^p} < 1/4$. Thus we need only show that $\sum_{j=-N}^{\infty} a_j^k = F \in (H^p, L^\infty)_0$ with proper control of norm. We are now ready to perform a stopping time argument.

**Lemma 5.1.** Let $F$ be as above. There is a subcollection $\{Q_j\}$ of the cubes $\{Q_j^k: k \geq -N, j \in \mathbb{Z}\}$ and there are functions $a_j$ supported on $Q_j$ such that $F = \sum a_j$. For each index $j$ there is an integer $m(j)$ such that

$$\sum 2^{pm(j)} |Q_j| \leq C_p; \tag{5.4}$$

if $Q_k \subset Q_j$ and $k \neq j$ then $m(k) > m(j)$ and $Q_k \cap Q_j = \emptyset$;

$$\sum_{Q_k \subset Q_j} |Q_k| \leq 2 |Q_j|; \tag{5.5}$$

$$\|a_j\|_{L^\infty} \leq C_{p_0} 2^{m(j)}; \tag{5.6}$$

$$\int x^\alpha a_j \, dx = 0, \quad |\alpha| \leq \alpha(p_0); \tag{5.7}$$

if $A$ is any finite set of indices, then for every $x$ there exists $j(x) \in A$ such that $\sum_{j \in A} |a_j(x)| \leq C_{p_0} 2^{m(j(x))} \chi_{Q_j(x)}(x). \tag{5.8}$$

**Proof:** As previously, $Q_j^k$ always denotes a Whitney cube in $Q_j$. We will inductively define a sequence $W_1 = \{R_1^l\}$ of subcollections of $\{Q_j^k: k \geq -N, j \in \mathbb{Z}\}$ and corresponding functions $A_j^l$ and integers $m(l, s)$. The sought collection is obtained by rearranging $\bigcup W_i$ into a sequence. Notice that if $Q_k^i$ is a Whitney cube in $Q_k$, then there is a Whitney cube $Q_{k-1}^{i-1}$ in $Q_{k-1}$ such that $Q_k^i \subset Q_{k-1}^{i-1}$ and $Q_k^i \cap Q_{k-1}^{i-1} = \emptyset$. Thus if the interiors of two Whitney cubes $Q_k^i$ and $Q_{k-1}^{i-1}$ are not disjoint, either $k > l$ and $Q_k^i \subset Q_l^j$, $k > l$ and $Q_l^j \subset Q_k^i$, or $k = l$ and $i = j$. At stage one let $W_1 = \{R_1^l\} = \{Q_j^{i}: j \in \mathbb{Z}\}$. If $k > -N$ then $Q_k^i \subset R_1^l$ for some cube $R_1^l \subset W_1$. For each cube $R_1^l \subset W_1$, let $m(l, s)$ be the smallest integer such that $|R_1^l \cap \Omega_{m(l, s)}| < 1/2 |R_1^l|$. Since $f \in H^p$ such an integer exists and necessarily $m(1, s) > -N$. Then $|R_1^l \cap \Omega_{m(l, s)-1}| > 1/2 |R_1^l|$. Let $S(R_1^l) = \{Q_j^k \subset R_1^l: -N < k < m(1, s)\}$ and let $S_1 = \bigcup_s S(R_1^l)$. The collection
of cubes not put in $S_1$ is $D_1 = \{Q_j^k : k \geq -N\} \setminus S_1$. At stage two let $W_2 = \{R_j^2\}$ be the collection of all maximal cubes in $D_1$. (A cube $Q_j^k \in D_1$ is maximal if whenever $Q_f^j \in D_1$ is such that the interiors of $Q_j^k$ and $Q_f^j$ are not disjoint then $k < l$, whence $Q_j^k \subseteq Q_f^j$.) For each cube $R_j^2 \in W_2$ let $m(2, s)$ be the smallest integer such that $|R_j^2 \cap \Omega_{m(2, s)}| < 1/2 |R_j^2|$. Then $|R_j^2 \cap \Omega_{m(2, s) - 1}| \geq 1/2 |R_j^2|$. Let $S(R_j^2) = \{Q_j^k \subseteq R_j^2 : Q_j^k \in D_1, k < m(2, s)\}$ and let $S_2 = \bigcup S(R_j^2)$. Also put $D_2 = \{Q_j^k : k \geq -N\} \setminus (S_1 \cup S_2) = D_1 \setminus S_2$ and put $W_3 = \{R_j^3\} = \{Q_j^k \in D_2 : Q_j^k$ is maximal $\} = \bigcup S(R_j^3) \subseteq \{Q_j^k \in D_2 : Q_j^k \subseteq R_j^2\}$.

Using step two as a model we proceed by induction to form collections of cubes $W_3 = \{R_j^3\}$, $W_4 = \{R_j^4\}$, ..., $S_3 = \bigcup S(R_j^3)$, $S_4 = \bigcup S(R_j^4)$, ..., and $D_3 = D_2 \setminus S_3$, $D_4 = D_3 \setminus S_4$, .... These collections have the property that $\bigcup S_i = \{Q_j^k : k \geq -N\}$ and every cube $Q_j^k$ is contained in exactly one $S(R_j^i)$ if $k \geq -N$. For each cube $R_j^t \in W_t$ there is an integer $m(t, s)$ such that $S(R_j^t) = \{Q_j^k \subseteq R_j^t : Q_j^k \subseteq D_{t-1}, k < m(t, s)\}$ and

$$|R_j^t \cap \Omega_{m(t, s)}| < 1/2 |R_j^t|, \quad (5.10)$$

$$|R_j^t \cap \Omega_{m(t, s) - 1}| \geq 1/2 |R_j^t|. \quad (5.11)$$

An immediate consequence of (5.10) is

$$\sum_{R_j^{t+1} \subseteq R_j^t} |R_j^{t+1}| < 1/2 |R_j^t|, \quad R_j^t \in W_t, \quad t \geq 1. \quad (5.12)$$

For each $R_j^t \in W_t$ put $A_j^t = \sum_{Q_j^k \in S(R_j^t)} a_j^k$. Then $F = \sum A_j^t$ and since each function $a_j^k$ is supported on $\hat{Q}_j^k$, $A_j^t$ is supported on $\hat{R}_j^t$. By condition (5.2),

$$\|A_j^t\|_L^2 \leq C_{p_0} \sum_{k = -N}^{m(t, s)} 2^k \leq C_{p_0} 2^{m(t, s)} \quad (5.13)$$

and by condition (5.3),

$$\int x^\alpha A_j^t \, dx = 0, \quad |\alpha| \leq \alpha(p_0). \quad (5.14)$$

Suppose $B$ is a finite subset of $\mathbb{Z}^2$ and let $m(x) = \max\{m(t, s) : (t, s) \in B, x \in \hat{R}_j^t\}$. Then by condition (5.2),

$$\sum_{(t, s) \in B} |A_j^t(x)| \leq \sum_{k \leq m(x)} \sum_j |a_j^k(x)| \leq \sum_{k = -\infty}^{m(x)} C_{p_0} 2^k = C_{p_0} 2^{m(x)}. \quad (5.15)$$

We now rewrite the sum $\sum A_j^t$ as $\sum a_j$. Then (5.5) follows from the construction of $W_t$, (5.6) follows from (5.12) and an iteration argument,
(5.7) is (5.13), (5.8) is (5.14), and (5.9) is (5.15). Finally, since cubes \( R'_s \) with the same value of \( m(t, s) \) have disjoint interiors, (5.11) yields

\[
\sum_{t, s} 2^{pm(t, s)} \| R'_s \| \leq 2 \sum_m 2^{pm} \sum_{m(t, s) = m} | R'_s \cap \Omega_{m-1} | \\
\leq 2 \sum_m 2^{pm} | \Omega_{m-1} | \\
= 2 \int \sum_m 2^{pm} x_{\Omega_{m-1}} \, dx \\
\leq C_p \int | f^* |^p \, dx = C_p.
\]

This is (5.4) and the lemma is completely proved.

**Step 2**

We now take \( F = \sum a_j \) as in the statement of Lemma 4.1 and jiggle the functions \( a_j \). For each cube \( Q_j \) let \( J(Q_j) = \{ Q_{k} \subset 5Q_j : | Q_k | \leq | Q_j | \} \). Then by condition (5.6),

\[
\left\| \sum_{Q_k \in J(Q_j)} x_{\tilde{Q}_k} \right\|_{L^1} \leq 2 \cdot 5^{2n} | Q_j |. \tag{5.16}
\]

Let \( \lambda > 0 \) be a large integer whose value will be fixed later and put

\[
E_j = \left\{ x \in \tilde{Q}_j : \sum_{Q_k \in J(Q_j)} x_{\tilde{Q}_k(x)} \geq \lambda \right\}.
\]

By condition (5.16),

\[
| E_j | \leq \frac{2 \cdot 5^{2n}}{\lambda} | Q_j |. \tag{5.17}
\]

**Lemma 5.2.** \( \| \sum_{j} x_{\tilde{Q}_j \cap E_j} \|_{L^\infty} \leq 3^n \lambda. \)

**Proof:** For each integer \( m \) there are at most \( 3^n \) cubes \( Q_j \) such that \( \tilde{Q}_j \) contains a given point \( x \) and such that \( l(Q_j) = 2^m \). Consequently, if the lemma is false there are cubes \( Q_1, Q_2, \ldots, Q_{\lambda} \) such that \( l(Q_1) < l(Q_2) < \cdots < l(Q_{\lambda}) \) and such that there is a point \( x \in \bigcap_{j=1}^{\lambda} (\tilde{Q}_j \setminus E_j) \). But then \( Q_j \subset 5Q_{\lambda} \), \( 1 \leq j \leq \lambda \), and consequently

\[
\sum_{Q_k \subset 3Q_{\lambda}} x_{\tilde{Q}_k(x)} \geq \lambda, \quad | Q_k | \leq | Q_{\lambda} |,
\]

i.e., \( x \in E_{\lambda} \). This is a contradiction.
We now jiggle the functions \( a_j \) to make them have support on \( \tilde{Q}_j \setminus E_j \). The idea of doing this comes from Carleson [7]. Let \( \tilde{a}_j = a_j \chi_{\tilde{Q}_j \setminus E_j} \). Then by (5.7), (5.8), and (5.17),

\[
\left| \int (x - x_j)^{\alpha} \tilde{a}_j \, dx \right| \leq C_{p_0} 2^{m(j)} \frac{2 \cdot 5^n}{\lambda} |Q_j| l(Q_j)^{\alpha}, \quad \alpha \leq \alpha(p_0). \tag{5.18}
\]

Here \( x_j \) denotes the center of \( Q_j \).

**Lemma 5.3.** There is a polynomial \( P_j \) such that

\[
\| P_j \|_{L^\infty(Q)} \leq C_{p_0} \frac{2^{m(j)}}{\lambda}
\]

and

\[
\int x^{\alpha} (\tilde{a}_j - P_j \chi_{Q_j \setminus E_j}) \, dx = 0, \quad |\alpha| \leq \alpha(p_0).
\]

**Proof.** Inequality (5.18) and a scaling argument show that it is sufficient to prove the lemma for the case where \( Q_j = [0, 1]^n \equiv Q \) and \( m(j) = 0 \). An elementary finite-dimensional Hilbert space argument shows that if to every multi-index \( \alpha, \, |\alpha| \leq \alpha(p_0) \), there is an associated complex number \( \beta_\alpha \) satisfying \( |\beta_\alpha| \leq 1 \), then there is a polynomial \( P \) of degree \( \leq \alpha(p_0) \) such that

\[
\| P \|_{L^\infty(Q)} \leq C_{p_0} \text{ and }
\]

\[
\int_Q x^{\alpha} P \, dx = \beta_\alpha, \quad |\alpha| \leq \alpha(p_0). \tag{5.19}
\]

Let \( \beta_\alpha = \int_Q x^{\alpha} \tilde{a}_j \, dx, \quad |\alpha| \leq \alpha(p_0) \) and let \( P_0 \) be the polynomial satisfying (5.19). Then by (5.18), \( \| P_0 \|_{L^\infty(Q)} \leq C_{p_0}/\lambda \). Let \( T_0 = P_0 \chi_{Q \setminus E} \), where \( E = E_j \) is the set deleted from \( Q_j = Q \). Then by (5.17),

\[
\left| \int x^{\alpha} (\tilde{a}_j - T_0) \, dx \right| = \left| \int_E x^{\alpha} P_0 \, dx \right| \leq (C_{p_0}/\lambda)^2, \quad |\alpha| \leq \alpha(p_0).
\]

An iteration argument now shows that if \( \lambda \) is large enough there is a polynomial \( P \) such that \( \| P \|_{L^\infty(Q)} \leq C_{p_0}/\lambda \) and \( x^{\alpha} (\tilde{a}_j - P \chi_{Q \setminus E}) \, dx = 0, \quad |\alpha| \leq \alpha(p_0) \). The lemma is proved.

Now put \( b_j = \tilde{a}_j - P_j \chi_{Q_j \setminus E_j} \). Then \( b_j \) is supported on \( \tilde{Q}_j \setminus E_j \),

\[
\| b_j \|_{L^\infty} \leq C_{p_0} 2^{m(j)}
\]
and

$$\int x^\alpha b_j \, dx = 0, \quad |\alpha| \leq a(p_0).$$

**Lemma 5.4.** $\|f - \sum b_j\|_{H^p} < 1/2$.

**Proof.** The proof is divided into two cases.

**Case I.** $1 < p < \infty$. Since $F = \sum a_j$,

$$\|f - \sum h_j\|_{H^p} \leq \|f - F\|_{H^p} + \left\| \sum a_j - \sum h_j \right\|_{L^p},$$

$$\leq 1/4 + C_p \left\| \sum a_j - \sum b_j \right\|_{L^p},$$

$$\leq 1/4 + C_p \left\| \sum a_j \chi_{E_j} \right\|_{L^p} + C_p \left\| \sum P_j \chi_{Q_j} \chi_{E_j} \right\|_{L^p}.$$

Almost every $x$ belongs to only finitely many $E_j$, and thus by (5.9),

$$|\sum a_j(x) \chi_{E_j}(x)| \leq C_{p_0} 2^{m(j(x))}, \text{ where } x \in E_{j(x)}.$$

Consequently,

$$\left| \sum a_j(x) \chi_{E_j}(x) \right|^p \leq C_{p_0}^p 2^{pm(j(x))} \chi_{E_{j(w)}}(x)$$

and

$$\left\| \sum a_j \chi_{E_j} \right\|_{L^p}^p \leq C_{p_0}^p \int \sum 2^{pm(j)} \chi_{E_j} \, dx$$

$$= C_{p_0}^p \sum 2^{pm(j)} |E_j|$$

$$\leq C_{p_0}^p \sum 2^{pm(j)} |Q_j|$$

$$\leq C_{p_0}^p \lambda^{-1}$$

by (5.17) and (5.4). Similarly

$$\sum 2^{m(j)} \chi_{Q_j}(x) \leq 2 \cdot 2^{m(j(x))} \chi_{Q_{j(0)}}(x)$$

for almost all $x$ and

$$\left| \sum 2^{m(j)} \chi_{Q_j} \right|^p \leq 2^p \sum 2^{pm(j)} \chi_{Q_j}.$$
Consequently,
\[
\left\| \sum P_j \chi_{Q_j \cap E_j} \right\|_{L^p}^p \leq \left( \frac{C_{p_0}}{\lambda} \right)^p \left\| \sum 2^{m(j)} \chi_{Q_j} \right\|_{L^p}^p \\
\leq 2^p \left( \frac{C_{p_0}}{\lambda} \right)^p \int \sum 2^{pm(j)} \chi_{Q_j} \, dx \leq C_{p, p_0} \lambda^{-p}.
\]
Thus if \( \lambda \) is large enough, the proof for Case I is established.

**Case II.** \( p \leq 1 \). Since
\[
\left\| f - b_j \right\|_{H^p}^p \leq \left\| f - F \right\|_{H^p}^p + \left\| F - \sum b_j \right\|_{H^p}^p
\]
and since \( \left\| f - F \right\|_{H^p}^p \leq (1/4)^p \), we need only show that \( \left\| F - \sum b_j \right\|_{H^p}^p < (1/2)^p - (1/4)^p \) if \( \lambda \) is large enough. We first establish the estimate
\[
\left\| a_j - b_j \right\|_{H^p} \leq C_{p_0} \lambda^{-p/2} 2^{pm(j)} |Q_j|.
\]
(5.20)
This estimate follows easily from the fact that \( a_j - b_j \) is supported on \( \tilde{Q}_j \),
\[
\int x^\alpha(a_j - b_j) \, dx = 0, \quad |\alpha| \leq \alpha(p_0),
\]
and
\[
\left\| a_j - b_j \right\|_{L^2} \leq \left\| a_j \chi_{E_j} \right\|_{L^2} + \left\| P_j \chi_{Q_j} \right\|_{L^2} \\
\leq C_{p_0} 2^{m(j)} |Q_j|^{1/2} + C_{p_0} 2^{m(j)} \lambda^{-1} |Q_j|^{1/2}.
\]
The last inequality above follows from (5.17) and Lemma 5.3. Now by (5.4) and (5.20),
\[
\left\| F - \sum b_j \right\|_{H^p}^p \leq \sum \left\| a_j - b_j \right\|_{H^p}^p \\
\leq C_{p_0} \lambda^{-p/2} \sum 2^{pm(j)} |Q_j| \\
\leq C_{p, p_0} \lambda^{-p/2} \\
\leq (1/2)^p - (1/4)^p,
\]
if \( \lambda \) is large enough. The proof for Case II is established.

**Step 3**
To establish Theorem 1, we need only show that \( \sum b_j \in (H^{p_0}, L^\infty)_b \). Since
the sum $\sum b_j$ converges in $H^p$ norm, we may now assume that this sum is finite. Let $\alpha(z) = p((1 - z)/p_0) - 1$ and let

$$G_z = \sum (2^m)^{\alpha(z)} b_j.$$

Then since there are only a finite number of functions $b_j$, $\|G_z\|_{HP_0} + \|G_z\|_{L^\infty} \leq M$ for all $z$ in the strip $S$, where $M$ is some large number. When $\Re z = 1$ we have

$$\|G_z\|_{L^\infty} \leq \left\| \sum 2^{-m(j)} |b_j| \right\|_{L^\infty} \leq C_{p_0} \left\| \sum \zeta \varphi_j \right\|_{L^\infty} \leq C_{p_0}$$

by Lemma 4.2. Let us now assume that $p_0 \leq 1$ since the theorem is known for $p_0 > 1$ (and follows from our proof for $p_0 = 1$ by interpolation). Then if $\Re z = 0$,

$$\|G_z\|_{H^{p_0}} \leq \sum 2^{(p-p_0)m(j)} \|b_j\|_{H^{p_0}} \leq C_{p_0} \sum 2^{(p-p_0)m(j)} 2^{p_0 m(j)} |Q_j| \leq C_{p,p_0},$$

the last inequality following from (5.4). Since $G_0 = \sum b_j$, the proof of Theorem 1 is complete.

**References**

14. R. Hanks, Interpolation by the real method between BMO, $L^a$ ($0 < a < \infty$) and $H^p$ ($0 < a < \infty$), *Indiana Math. J.* 26 (1977), 679–690.