On optimal two-level nonregular factorial split-plot designs

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\textbf{ABSTRACT}

This article studies two-level nonregular factorial split-plot designs. The concepts of indicator function and aliasing are introduced to study such designs. The minimum G-aberration criterion proposed by Deng and Tang (1999)\textsuperscript{[4]} for two-level nonregular factorial designs is extended to the split-plot case. A method to construct the whole-plot and sub-plot parts is proposed for nonregular designs. Furthermore, the optimal split-plot schemes for 12-, 16-, 20- and 24-run two-level nonregular factorial designs are searched, and many such schemes are tabulated for practical use.

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\textbf{1. Introduction}

A split-plot design is often used when it is not practical to perform all the experimental runs of a multifactorial experiment in a completely random order. Recently, many authors have focused on fractional factorial split-plot (FFSP) designs, see e.g., [8,1,19,15,17,16,3] and the references therein. To perform an FFSP design with \textit{m} factors, we often first randomly choose one of the factorial level-settings of these, say \textit{m}_1, hard-to-change factors and then run all of the level-combinations of the remaining \textit{m}_2(= \textit{m} – \textit{m}_1) factors in a random order, while holding the \textit{m}_1 factors fixed. This is repeated for each level-combination of these \textit{m}_1 factors. If the design matrix for this experimental setup is identical to a $2^{m-k}$ fractional factorial (FF) design, where \textit{m} = \textit{m}_1 + \textit{m}_2 and \textit{k} = \textit{k}_1 + \textit{k}_2, then it is said to be a $2^{(m_1+m_2)-(k_1+k_2)}$ FFSP design. The \textit{m}_1 and \textit{m}_2 factors are called whole-plot (WP) and sub-plot (SP) factors, respectively. There are \textit{k}_1 WP and \textit{k}_2 SP fractional generators. The group formed by the \textit{k} = \textit{k}_1 + \textit{k}_2 generators is called the defining contrast subgroup. Let $A_i$ denote the number of words of length \textit{i} in the defining contrast subgroup of a $2^{(m_1+m_2)-(k_1+k_2)}$ design, then the vector $W = (A_3, \ldots, A_m)$ is called the word-length pattern of the design. The maximum resolution and minimum aberration (MA) criteria can then be defined [8,1].

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All the papers mentioned above discussed only split-plot schemes for regular FF designs. However, nonregular factorial designs have some advantages over regular ones in terms of run size flexibility and estimation capacity. Therefore, they are becoming popular choices in practice, and in many situations they need to possess a split-plot structure. In this article, we extend the minimum G-aberration (MGA) criterion proposed by Deng and Tang [4] for two-level nonregular factorial designs to the split-plot case, and provide a method for constructing two-level nonregular factorial split-plot designs from Hadamard matrices.

2. Indicator function and aliasing

In this section, a polynomial representation for general two-level factorial designs is presented. It applies to any two-level factorial design, with or without replicates, regular or nonregular, and can set up the mathematical framework for studying nonregular factorial split-plot designs.

2.1. Indicator function

Let $F$ be a $2^m$ full factorial design. Without loss of generality, the levels of each factor in $F$ are denoted by 1 and $-1$. We use an $n \times m$ matrix of 1 and $-1$ to represent a factorial design $D$, where each row of the matrix corresponds to a run and each column to a factor. According to Refs. [18,2], we have

**Definition 1.** A factorial design $D$ corresponds to a unique polynomial function defined on $F$ with the form

$$F_D(x_1, \ldots, x_m) = h_0 + \sum_{k=1}^{m} \sum_{1 \leq i_1 < \cdots < i_k \leq m} h_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k},$$  \hspace{1cm} (1)

where $h_0 = n/2^m$, $h_{i_1 \cdots i_k}$ = $\frac{1}{2^m} \sum_{X \in F} x_{i_1} \cdots x_{i_k}$, and $X = (x_1, \ldots, x_m)$ represents a design point in $F$. The summation $\sum_{X \in F} x_{i_1} \cdots x_{i_k}$ can be viewed as a general inner product of $k$ columns of $D$. The polynomial function (1) is called the indicator function of $D$, the polynomial terms appearing in (1) (i.e., those polynomial terms with nonzero coefficients) are called the words of $D$, and these words form the defining contrast subgroup of $D$.

The indicator function approach can be generalized to factorial split-plot designs directly. A distinction between a completely randomized design and a factorial split-plot design is that in the latter there are two types of factors, WP factors and SP factors. Now let us see two illustrative examples. The first one is modified from an example given by Montgomery [12, p. 307] for the purpose of the illustration.

**Example 1.** Suppose we wish to perform an experiment to identify factor settings that will improve the efficiency of a ball mill. Engineers have identified six potentially important factors, each at two levels: motor speed $X_1$, feed mode $X_2$, feed sizing $X_3$, material type $X_4$, gain $X_5$, and screen angle $X_6$. Suppose that it is expensive or time consuming to change the levels of $X_1$, $X_2$, $X_3$ and $X_4$, and there are only enough resources to perform 16 experimental runs. Let the defining contrast subgroup for a regular FFSP design $D_1$ be $I = X_1X_2X_3X_4 = X_2X_3X_5X_6 = X_1X_4X_5X_6$, that is, $D_1$ is a $2^{4+2-1}$ FFSP design with $X_4 = X_2X_3X_5$ as the WP part and $X_6 = X_2X_3X_5$ as the SP part. The WP part for this experiment is a $2^{4-1}$ FF design, while the SP part is a design with generator $X_2X_3X_5X_6$, selected from the interactions of WP and SP factors. For $D_1$, the word-length pattern is $(0, 3, 0, 0)$, and its indicator function is:

$$F_{D_1}(x_1, \ldots, x_6) = 1 \times (1 + x_1 x_2 x_3 x_4) (1 + x_2 x_3 x_5 x_6)$$

$$= \frac{1}{4} (1 + x_1 x_2 x_3 x_4 + x_2 x_3 x_5 x_6 + x_1 x_4 x_5 x_6).$$  \hspace{1cm} (2)

For any $X = (x_1, \ldots, x_6) \in D_1$, $F_{D_1}(X) = 1$ because $x_1 x_2 x_3 x_4 = 1$ and $x_1 x_4 x_5 x_6 = 1$, while for any $X \in F \setminus D_1$ with $F = 2^6$, we have $F_{D_1}(X) = 0$, since either $x_1 x_2 x_3 x_4 = -1$ or $x_1 x_4 x_5 x_6 = -1$. Therefore, the polynomial in (2) determines the design $D_1$. 

is partially aliased with the two-factor interaction $X$ and in the defining contrast subgroup of $D$. This implies that the two-factor interactions involved in this word like $X$ function is the first four columns as WP factors and the remaining two columns as SP factors. Its indicator function is given by the shortest word-length of the words in the defining contrast subgroup.

**Example 2.** Consider the experiment in Example 1 again. Denote the nonregular design listed in Table 1 as $D_2$. Note that the first four columns of $D_2$ can be regarded as a regular FF design, where $X_1X_2X_3X_4$ is the defining word. According to the requirements of a factorial split-plot design, we assign these first four columns as WP factors and the remaining two columns as SP factors. Its indicator function is

$$F_{D_2}(x_1, \ldots, x_6) = \frac{1}{4} \left( 1 + x_1x_2x_3x_4 + \frac{1}{2}x_1x_5x_6 + \frac{1}{2}x_2x_5x_6 + \frac{1}{2}x_3x_5x_6 - \frac{1}{2}x_4x_5x_6 \right. + \left. \frac{1}{2}x_2x_3x_4x_5x_6 + \frac{1}{2}x_1x_3x_4x_5x_6 + \frac{1}{2}x_1x_2x_4x_5x_6 - \frac{1}{2}x_1x_2x_3x_5x_6 \right).$$ \hspace{1cm} (3)

This function determines $D_2$ in the same way as (2) determines $D_1$. It can be shown that $F_{D_2}(X) = 1$ for any $X = (x_1, \ldots, x_6) \in D_2$ and $F_{D_2}(X) = 0$ for any $X \in F \setminus D_2$. In addition, the coefficients of $1/2$ in (3) provide a measure for the degree of aliasing among the corresponding columns, as will be detailed discussed in the next subsection.

### 2.2. Aliasing

In the above $2^{4+2-1+1}$ regular FF design $D_1$, $X_1X_2X_3X_4$ is a word in the defining contrast subgroup. This implies that the two-factor interactions involved in this word like $X_1X_2$ and $X_2X_4$ are fully aliased. The aliasing relationship is reflected in the indicator function through the coefficient of $X_1X_2X_3X_4$, i.e., $h_{1234}/h_0 = 1$. In fact, the polynomial terms appearing in (2) are just the words contained in the defining contrast subgroup of $D_1$, and all of their coefficients are 1.

For the nonregular design $D_2$ in Table 1, it can be observed that the two-factor interactions $X_1X_2$ and $X_2X_4$ are fully aliased, and $h_{1234}/h_0 = 1$ in the indicator function (3). Also, the WP main effect $X_1$ is partially aliased with the two-factor interaction $X_2X_6$, since $h_{156}/h_0 = \frac{1}{2}$, which is the correlation between $X_1$ and $X_2X_6$ (or between $X_5$ and $X_1X_6$, or between $X_6$ and $X_1X_5$).

In the remainder of this article, we call $h_{i_1\ldots i_k}/h_0$ the **aliasing index** of $X_{i_1} \cdots X_{i_k}$, which measures the degree of aliasing associated with the word $X_{i_1} \cdots X_{i_k}$. Now we have the following definition due to [10].

**Definition 2.** For a word $X_{i_1} \cdots X_{i_k}$ of a general design $D$ with indicator function (1), its **word-length** is defined to the number of the letters in this word plus $1 - |h_{i_1\ldots i_k}/h_0|$. Furthermore, the **resolution** of $D$ is given by the shortest word-length of the words in the defining contrast subgroup.
In nonregular designs, we can allow an SP main effect to be partially aliased with any WP main effect or WP interaction. Such a strategy does not spoil the nature of the split-plot designs, and the requirement that any SP main effect cannot be fully aliased with any WP main effect or WP interaction is still maintained. On the other hand, since each WP has to be split into several SPs with the same level-combination of WP factors appearing in all these SPs, the \( p_1 = m_1 - k_1 \) columns used for generating the WP part must be eligible, that is, these columns must form a \( 2^{p_1} \) full factorial design with equal replicates. If the products of the \( p_1 \) eligible columns appear in the design matrix, they must also be included in the WP part. Otherwise, assigning any of these products as an SP factor will result in the SP main effect being fully aliased with a WP interaction.

3. Optimality criterion and assignment of WP and SP parts

For nonregular two-level designs, MGA is a commonly used criterion based on the numbers of words. When the words have the same order (i.e., the same number of letters), MGA compares the frequencies of words with distinct word-lengths. For a two-level orthogonal factorial design, the word frequencies of words with distinct word-lengths. For a two-level orthogonal factorial design, the run size \( n \) is divisible by four. Let \( t = n/4 \). By Proposition 3 of [4], it is easy to show that the fractional part of the length of any word in the design is a value in \([0/t, 1/t, \ldots, (t-1)/t]\).

**Definition 3 ([4])**. For a two-level orthogonal design \( D \), let \( f_{kj} \) be the frequency of word-length \( k + j/t, 0 \leq j \leq t - 1 \). The confounding frequency vector (CFV) is defined as the vector of length \( (m - 2)t \):
\[
F(D) = [F_3(D); \ldots; F_m(D)], \quad \text{where } F_k(D) = (f_{k0}, \ldots, f_{k(t-1)}).
\] (4)

For two designs \( D_1 \) and \( D_2 \), let \( g_i(D_1) \) and \( g_i(D_2) \) be the \( i \)th entries of \( F(D_1) \) and \( F(D_2) \), respectively, where \( i = 1, \ldots, (m - 2)t \). Let \( l \) be the smallest integer such that \( g_i(D_1) \neq g_i(D_2) \). If \( g_i(D_1) < g_i(D_2) \), then \( D_1 \) has less G-aberration than \( D_2 \). If no design has less G-aberration than \( D_1 \), then \( D_1 \) is said to have MGA.

Here, we focus on the use of the MGA criterion in the selection of factorial split-plot designs. Since great savings in computing time can be achieved by using the two or three leading terms of the CFV in (4) to classify factorial split-plot designs, we use the following two classifiers as classification and ranking criteria.

**Definition 4 ([5])**. For a two-level orthogonal design \( D \) with the CFV in (4), we call \([F_3(D); F_4(D)]\) the MA-4 classifier, and \([F_3(D); F_4(D); F_5(D)]\) the MA-5 classifier.

By Theorem 1 of [18], it can be easily shown that \( p_1 \) columns are eligible for generating the WP part if and only if no words in the indicator function \( F_3 \) are solely associated with these columns. Therefore we can assign the WP and SP parts of a split-plot design according to the following rule.

**Rule 1.** (i) Choose \( p_1 \) eligible columns and the columns generated by these \( p_1 \) columns which exist in the original design \( D \) to form the WP part; (ii) Assign the remaining columns of \( D \) to construct the SP part.

Usually we can regard an \( n \times m \) two-level design matrix \( D \) as an orthogonal array (OA) of strength \( t, 1 \leq t \leq m \), denoted by OA\((n, 2^m, t)\), in which for any \( t \) columns, all possible level-combinations appear equally often in the matrix. As we know, when the original design \( D \) is an OA of strength \( t \), any polynomial term appearing in the indicator function must contain at least \( t + 1 \) letters, so that any \( p_1 \) columns in an OA of strength \( t \) for \( p_1 \leq t \) are eligible to construct the WP part.

4. MGA split-plot designs

In this section, we discuss the optimal split-plot schemes of 12-, 16-, 20- and 24-run two-level factorial designs under the MGA criterion. We focus only on the factorial split-plot designs with resolution III or higher, that is, the factorial split-plot designs whose WP and SP factors form an OA.

A factorial split-plot design with \( m_1 \) WP factors and \( m_2 \) SP factors can be constructed from an OA with \( m = m_1 + m_2 \) columns according to Rule 1. Note that the number of columns generated in
the WP part must be $m_1 - p_1$. To construct the desired WP part, it is enough to calculate first all the coefficients in the indicator function of the OA we intend to use, and then select the eligible columns used for generating the WP part according to the aliasing relationship reflected by these coefficients.

To search for MGA factorial split-plot designs, we restrict our attention to designs based on Hadamard matrices. As pointed out in [13], this is not as restrictive as it might seem and almost all OAs used in practical experiments or for other purposes are indeed derived from Hadamard matrices. Deng and Tang [5] classified and ranked designs that are based on Hadamard matrices under the MGA criterion, from which we can construct optimal factorial split-plot designs.

Denote the $i$th best design with $n$ runs and $m$ factors provided in [5] by $E_i$. The algorithm for searching factorial split-plot designs can be described as follows:

**Algorithm 1.**

**Step 1.** Specify the parameters $n$, $m$ and $m_1$, which represent the numbers of runs, factors and WP factors, respectively, where $m_1 \leq m - 2$.

**Step 2.** For the case of $m_1 = 1$ WP factor, take any column of the best design $E_1$ to form the WP part, end the algorithm; for the case of $m_1 = 2$ WP factors, take any two columns of $E_1$ to form the WP part, end the algorithm; for the case of $m_1 \geq 3$ WP factors, set $i = 1$, do the following steps.

**Step 3.** Select all the sets of eligible columns from $E_i$, record them by $IC_{ij}$ for $j = 1, \ldots, n_i$ (say). Denote the number of eligible columns in set $IC_{ij}$ by $p_{ij}$. Set $j = 1$.

**Step 4.** From $E_i$, find all the columns generated by the eligible columns in $IC_{ij}$, denote the number of these generated columns by $k_{ij}$.

**Step 5.** If $p_{ij} + k_{ij} = m_1$, take these $m_1$ columns to form the WP part, and stop the searching. Otherwise, update $j = j + 1$.

**Step 6.** If $j \leq n_i$, goto Step 4. Otherwise, update $i = i + 1$.

**Step 7.** If there exits $E_i$, goto Step 3. Otherwise, the search is stopped and we fail to find the design for the specified parameters.

**Remark 1.** Our method can be regarded as a counterpart of Method I presented in [8], which constructs FFSP MA designs from the existing FF MA designs. When an existing MA design does not have the required split-plot structure, Method II in [8], using linear integer programming and some known properties on word-length patterns, can be employed. Since the CFVs of nonregular designs corresponding to the word-length patterns of FF designs are more complicated, it seems to be a challenge to search for factorial split-plot designs in a similar fashion. Of course, this deserves future research.

4.1. 12-run designs

Two designs are said to be isomorphic if one can be obtained from the other by relabeling the factors, reordering the runs, and switching the levels of factors. Consider two nonregular factorial
Table 3
16-run optimal split-plot designs.

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<tr>
<th>$n,m_1,m_2$</th>
<th>$F_3:[0,1/2]$</th>
<th>$F_4:[0,1/2]$</th>
<th>Type</th>
<th>WP factors</th>
<th>SP factors</th>
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<td>(8 10 13)</td>
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<td>(0, 0)</td>
<td>I</td>
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<td>(10 13)</td>
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</tbody>
</table>

split-plot designs. As is well known, there is only one nonisomorphic Hadamard matrix of order 12, one version of which is given by adding the all 1’s column to the 12-run Plackett and Burman design. Lin and Draper [11] and Wang and Wu [14] found that, for $m = 5, 6$, there are exactly two nonisomorphic designs, and for any other $m$, there is exactly one nonisomorphic design. In particular for $m = 4$ and $7 \leq m \leq 11$, any orthogonal $12 \times m$ design must be isomorphic to the design obtained by taking the first $m$ columns from the design in Table 2. For $m = 5$, taking columns 1–5 yields a design which is nonisomorphic to the design obtained by taking columns 1–4 and 10. We denote the two designs by PB12$_{5a}$ and PB12$_{5b}$, respectively. For $m = 6$, the two nonisomorphic designs, denoted by PB12$_{6a}$ and PB12$_{6b}$, can be obtained by taking the columns not in PB12$_{5a}$ and PB12$_{5b}$, respectively. For $m = 5$, we obtain the MA-5 classifiers of PB12$_{5a}$ and PB12$_{5b}$ as follows:

$$[F_3(\text{PB12}_{5a}); F_4(\text{PB12}_{5a}); F_5(\text{PB12}_{5a})] = [(0, 0, 10); (0, 0, 5); (0, 0, 0)],$$

and

$$[F_3(\text{PB12}_{5b}); F_4(\text{PB12}_{5b}); F_5(\text{PB12}_{5b})] = [(0, 0, 10); (0, 0, 5); (0, 1, 0)].$$

Since $g_5(\text{PB12}_{5a}) = 0$ and $g_5(\text{PB12}_{5b}) = 1$, by the definition of CFV, PB12$_{5a}$ is better than PB12$_{5b}$. For $m = 6$,

$$[F_3(\text{PB12}_{6a}); F_4(\text{PB12}_{6a}); F_5(\text{PB12}_{6a})] = [(0, 0, 20); (0, 0, 15); (0, 1, 0)],$$

and

$$[F_3(\text{PB12}_{6b}); F_4(\text{PB12}_{6b}); F_5(\text{PB12}_{6b})] = [(0, 0, 20); (0, 0, 15); (0, 0, 0)].$$

The design PB12$_{6b}$ is better according to the MGA criterion. Clearly, in either case, the MA-5 classifier is able to differentiate the two nonisomorphic designs. Because 12 is divisible by 2 and $2^2$, a 12-run two-level OA can be arranged into two or four groups, that is, split-plot designs with at most two


4.2. 16-run designs

Because 16 is divisible by 2, 2², and 2³, we can possibly arrange a 16-run two-level OA into two, four or eight groups, which implies that split-plot designs with one, two, or three eligible WP factors can be found from it. For 3 ≤ m ≤ 14, Deng and Tang [5] performed a complete search of \( \binom{15}{m} \) designs for each of the Hadamard matrices, H16-I–H16-V [6], and gave the numbers of nonisomorphic classes identified by the MA-4 classifier as well as the total number of classes for each m.

For each \( m_1 \) and \( m_2 \) with 4 ≤ \( m_1 + m_2 \) ≤ 14, the optimal split-plot designs found by the MA-4 classifier are ranked by MGA. A partial list of these designs is provided in Tables 3 and 4. In the tables, label 16.\( m_1 . m_2 \) is used to denote the split-plot design with 16 runs and m factors which consist of \( m_1 \) WP factors and \( m_2 \) SP factors with \( m = m_1 + m_2 \). The column under “Type” indicates which Hadamard matrix a particular design comes from. The tables display the columns of the optimal split-plot designs, where WP factors and SP factors are presented separately.

4.3. 20-run designs

There are exactly three nonisomorphic Hadamard matrices of order 20, which are given by Hall [7] under labels H20-Q, H20-P and H20-N. Deng and Tang [5] discovered that for designs of 20 runs,
MA-5 has almost the same differentiating power as the full CFV. For $m = 3, \ldots, 18$, the numbers of nonisomorphic classes identified by applying the MA-5 classifier to all three Hadamard matrices are $2, 3, 10, 34, 51, 80, 125, 125, 80, 51, 34, 10, 3, 2, 1$ and $1$, respectively. We provide in Table 5 the optimal split-plot designs of 20 runs for $4 \leq m_1 + m_2 \leq 16$ under the MGA criterion.

### 4.4. 24-run designs

According to [9], there are exactly 60 nonisomorphic Hadamard matrices of order 24. Deng and Tang [5] performed a complete search for $m \leq 8$ by applying MA-5 to all the 60 Hadamard matrices. They also verified that MA-5 is sufficient to differentiate the classes for 24-run designs. The numbers of classes identified by MA-5 for $m = 3, \ldots, 8$ are 4, 10, 49, 408, 3805 and 29196.
respectively. They presented the top three designs for $m = 3, \ldots, 7$. All of these best designs can be found from the 11th Hadamard matrix of order 24, which is available at the website http://www.research.att.com/~njas/oadir/ maintained by Sloane. In Table 6, we show some of the optimal split-plot designs constructed from the 11th Hadamard matrix of order 24 for $4 \leq m_1 + m_2 \leq 8$.

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