A criterion for minimality of restrictions of compact minimal Abelian flows

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Abstract

For a compact minimal Abelian flow \( X = \langle T, X \rangle \) we introduce the notion of an \( X \)-enveloped subgroup of \( T \) and use it to give a criterion for minimality of the restricted flow \( X_S = \langle S, X \rangle \), \( S \) a syndetic subgroup of \( T \), in terms of the eigenvalues of \( X \). We then deduce a criterion for total minimality of \( X \). We apply these two criteria in several situations. In case of skew-extensions we get a new proof of a classical theorem of Parry. We also introduce the notion of SK groups and use it, together with the criteria, to generalize some statements about weak mixing and improve some conditions which imply non-total-minimality. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this introduction \( X = \langle T, X, \pi \rangle \) will denote a minimal flow on a compact space \( X \) with \( T \) an Abelian topological group.

In Section 2 we give the notation and the terminology we use in this paper, as well as some relevant basic facts.

In Section 3 we introduce the notions of an \( X \)-envelope of a syndetic subgroup \( S \) of \( T \) and of an \( X \)-enveloped subgroup of \( T \). We prove some properties related to these notions and give two examples which illustrate them.

In Section 4 we formulate and prove two main results, which we name as a criterion for minimality of reduced flows and a criterion for total minimality. The first one gives a necessary and sufficient condition for a restriction \( X_S = \langle S, X \rangle \), \( S \) syndetic, of a compact minimal Abelian flow \( X = \langle T, X \rangle \) to be minimal. The second one gives a necessary and
sufficient condition for a compact minimal Abelian flow \( X = \langle T, X \rangle \) to be totally minimal.

We apply these criteria to the case \( T = \mathbb{Z} \) (and get a result whose parts are well known) and to the case \( T = \mathbb{R} \) (and get a result of Egawa).

In Section 5 we investigate skew-extensions of compact minimal \( \mathbb{Z} \)-flows. In the context of a classical theorem of Parry we apply the criterion for minimality of reduced flows to an appropriately defined flow and conclude that the condition coming from the criterion is equivalent with Parry’s condition. In that way we get a new proof of this theorem.

In Section 6 we introduce the notion of SK groups, give examples and prove some properties of this notion. These properties will then be used in Sections 7 and 8.

In Section 7 we investigate the relation between weak mixing and total minimality in the context of SK acting groups (using the criteria developed in Section 4). We generalize a result of Markley (about the equivalence of weak mixing and total minimality) and a statement of Gottschalk (about total minimality of \( X \) in terms of the structure group \( \Gamma(X) \)) by extending the class of acting groups for which their statements remain valid.

In Section 8 we use the criteria developed in Section 4 to investigate some conditions which, when imposed on compact minimal Abelian flows, necessarily imply non-total-minimality. In that way we generalize and give a shorter proof of a result of Chu, as well as several other results.

2. Notations and preliminaries

2.1. If \( X \) is a set, we denote the cardinality of \( X \) by \(|X|\). If \( X, Y \) are topological spaces, then \( \text{Homeo}(X) \) denotes the group of homeomorphisms of \( X \), and \( C(X, Y) \) denotes the set of continuous maps from \( X \) to \( Y \). The map \((x, y) \mapsto x\) from \( X \times Y \) to \( X \) is denoted by \( \text{pr}_1 \).

Similarly for \( \text{pr}_2 \). All topological spaces are assumed to be Hausdorff.

Occasionally we will use the fact that in the class of compact spaces, total disconnectedness and 0-dimensionality are equivalent.

If \( X \) is a uniform space, \( \alpha \) an entourage of \( X \) and \( x \in X \), then \( \alpha[x] \) denotes the set of all \( y \in X \) such that \((x, y) \in \alpha \).

2.2. \( \mathbb{T} \) will denote the topological group of complex numbers of module 1. If \( T \) is an Abelian group, the continuous homomorphisms \( f : T \rightarrow \mathbb{T} \), will be called continuous characters of \( T \). The set of all continuous characters of \( T \) will be denoted by \( \hat{T} \).

2.3. Let \( T_1 \) and \( T_2 \) be topological groups and let \( \chi \in \hat{T_1} \times \hat{T_2} \). Then for all \( t_1 \in T_1 \) and \( t_2 \in T_2 \), \( \chi(t_1, t_2) = \chi(t_1, 1) \cdot \chi(1, t_2) \). If we denote by \( \chi_1 \) the continuous character \( t_1 \mapsto \chi(t_1, 1) \) of \( T_1 \) and by \( \chi_2 \) the continuous character \( t_2 \mapsto \chi(1, t_2) \) of \( T_2 \), we have \( \chi(t_1, t_2) = \chi_1(t_1) \chi_2(t_2) \). Whenever no confusion can arise, we will simply write \( \chi = \chi_1 \chi_2 \).

Similarly for products with any finite number of factors.

2.4. Let \( T \) be a topological group. A subset \( A \) of \( T \) is syndetic if there exists a compact subset \( K \) of \( T \) such that \( T = KA \). If \( S \) is a syndetic subgroup of \( T \), the quotient space \( T/S \) is compact. If \( T \) is locally compact, the converse is also true [11, 2.01]. (Note however
that if we have a continuous surjective homomorphism \( f : T \to K \), where \( K \) is a compact topological group, we cannot conclude in general that \( \ker(f) \) is syndetic in \( T \).

2.5. A triple \( \mathcal{X} = (T, X, \pi) \) consisting of a topological group \( T \), a Hausdorff topological space \( X \) and a continuous action \( \pi : T \times X \to X \) of \( T \) on \( X \) will be called a \( T \)-flow or simply a flow on \( X \). We write \( t.x \) or \( tx \) for \( \pi(t, x) \). We say that \( \mathcal{X} \) is compact (respectively Abelian), if \( X \) is compact (respectively if \( T \) is Abelian). We say that \( \mathcal{X} \) is trivial if \( |X| = 1 \).

For \( x \in X \) we denote by \( \mathcal{X} \) the orbital map \( t \mapsto t.x \). For \( t \in T \) we denote by \( \tau_x \) the transition homeomorphism \( x \mapsto t.x \). For \( x \in X \) and \( U, V \subset X \), the dwelling set \( D(U, V) \) (respectively \( D(x, V) \)) is the set of all \( t \in T \) such that \( t.U \cap V \neq \emptyset \) (respectively \( t.x \in V \)).

The notions of almost periodic, proximal, distal, locally almost periodic, regularly almost periodic, proximally equicontinuous, weakly mixing flow, point or pair of points (whatever is appropriate, and in the appropriate context), as well as all other notions not mentioned in this introduction, are assumed to be defined as in [11,6,1,22]. We just note that almost periodic flows are also called uniformly almost periodic and, in the case of compact flows, they are the same as equicontinuous flows.

Every flow \( \mathcal{X} = (S, X, \pi|_{X \times S}) \), where \( S \) is a subgroup of \( T \), will be called a restriction of \( \mathcal{X} \). Usually it will be denoted shortly by \( \mathcal{X}_S = (S, X) \).

A flow \( \mathcal{X} = (T, X) \) is minimal if the orbit \( T.x \) of every point \( x \in X \) is dense in \( X \). It is totally minimal if the flow \( \mathcal{X}_S \) is minimal for every syndetic (equivalently, closed syndetic) subgroup of \( T \). If \( f : \mathcal{X} \to \mathcal{Y} \) is a surjective morphism of flows, then if \( \mathcal{X} \) is minimal (respectively totally minimal), \( \mathcal{Y} \) is minimal (respectively totally minimal).

When we have a \( \mathbb{Z} \)-flow on \( X \), \( \mathcal{X} = (\mathbb{Z}, X, \pi) \), then the transition homeomorphism \( h := \pi_1 \) completely defines the action: \( \pi(n, x) = h^n(x) \). In that case we will write shortly \( \mathcal{X} = (X, h) \) when no confusion can arise.

2.6. Let \( \mathcal{X} = (T, X) \) be a flow. For \( x \in X \) and \( S \subset T \), the \( x \)-envelope of \( S \), denoted by \( S^x \), is the set \( \{ t \in T \mid t.x \in S.x \} \). \( S^x \) is a closed subset of \( T \), it contains \( S \), and \( S.S^x = S.x \). If \( S \) is a syndetic normal subgroup of \( T \), then \( S^x \) is a closed subgroup of \( T \) [11,2.08–2.10].

If \( S \) is a normal subgroup of \( T \) the following properties are easy to verify:

(i) \( t.S.x = S.t.x \) for all \( t \in T \), \( x \in X \).

(ii) If \( S.x \) is a minimal subset of \( X \) under \( S \), then:

(1) \( t.S.x = S.t.x \) iff \( t \in S^x \);

(2) \( (\forall y \in X) \ y \in S.x \) iff \( t.S.y = S.x \).

2.7. Let \( \mathcal{X} = (T, X) \) be a flow. The following are equivalent:

(i) the set of orbit closures under \( T \) is a partition of \( X \);

(ii) \( (\forall x \in X) (\forall y \in X) \ y \in T.x \iff T.y = T.x ; \)

(iii) every orbit closure under \( T \) is minimal under \( T \) [11,2.23].

2.8. Let \( \mathcal{X} = (T, X) \) be a flow and \( S \) a syndetic normal subgroup of \( T \). Then the set of orbit closures under \( S \) is a partition of \( X \) iff the set of orbit closures under \( T \) is a partition of \( X \). In particular, if \( \mathcal{X} \) is minimal, the set of orbit closures under \( S \) is a partition of \( X \) [11,2.24].
We denote by $O_S$ the set $\{\overline{S}x \mid x \in X\}$ and by $R(O_S)$ the relation $(x, y) \in R(O_S) \iff \overline{S}x = \overline{S}y$ on $X$.

If $\mathcal{X}$ is compact minimal, $R(O_S)$ is an equivalence relation which is open [11, 2.30] and closed [11, 2.32].

2.9. Let $\mathcal{X} = (T, X)$ be a minimal flow, $S$ a syndetic normal subgroup of $T$, and $K$ a compact subset of $T$ such that $T = KS$. The following are easy to verify:

(i) $K \overline{S}x = X$ for every $x \in X$;
(ii) in particular, for every $x, y \in X$ there is a $k \in K$ such that $k \overline{S}x = \overline{S}y$ (and consequently $k.x \in \overline{S}y$).

2.10. Let $\mathcal{X} = (T, X)$ be a flow. A continuous function $\eta : X \to \mathbb{T}$ is an eigenfunction of $\mathcal{X}$ if there is a continuous character $\chi \in \hat{T}$ such that $\eta(t.x) = \chi(t)\eta(x)$ for $(t, x) \in T \times X$. In that case $\chi$ is an eigenvalue of $\mathcal{X}$ (the eigenvalue which corresponds to $\eta$) and $\eta$ is an eigenfunction which corresponds to $\chi$. The following are equivalent:

(i) $\chi$ is trivial;
(ii) $\eta$ is constant on some $T.x$ ($x \in X$);
(iii) $\eta$ is constant on every $T.x$ ($x \in X$).

If $X$ contains a point with a dense orbit, then $\chi$ is trivial iff $\eta$ is constant.

3. The notion of an $\mathcal{X}$-enveloped subgroup

Proposition 3.1. Let $\mathcal{X} = (T, X)$ be a minimal Abelian flow, and let $S$ be a syndetic subgroup of $T$. Then $S^* = S^\circ$ for every $x, y$ from $X$.

Proof. Let $K$ be a compact subset of $T$ such that $T = K + S$. Fix any $x, y$ from $X$. There is a $k$ in $K$ such that $-k.x = y' \in \overline{S}y$. Then $x = k.y', y' \in \overline{S}y$. Let $s \in S^\circ$. We have:

\[
\begin{align*}
  s.y \in s.\overline{S}y &= s.S.y = s.\overline{S} - k.x = s. - k.\overline{S}x = -k.s.\overline{S}x \\
  &= -k.\overline{S}x = -k.S.x = S. - k.x = S.y' = \overline{S}y.
\end{align*}
\]

Hence $s \in S^\circ$. Thus $S^\circ \subset S^\circ$. By symmetry $S^\circ \subset S^\circ$. Hence $S^\circ = S^\circ$. \qed

Definition 3.2. Let $\mathcal{X} = (T, X)$ be a minimal Abelian flow, and let $S$ be a syndetic subgroup of $T$. The $\mathcal{X}$-envelope (or shortly envelope) of $S$, denoted by $S^\circ$, is the subset of $T$ which is equal to $S^\circ$, where $x$ is any element of $X$.

By the previous proposition this notion is well defined.

Definition 3.3. Let $\mathcal{X}$ be a minimal Abelian flow. A syndetic subgroup $S$ of $T$ is called an $\mathcal{X}$-enveloped (or shortly enveloped) subgroup of $T$ if $S = S^\circ$.

Note that a subgroup $S$ of $T$ may be enveloped with respect to some flow $\mathcal{X} = (T, X)$, and at the same time not enveloped with respect to some other flow $\mathcal{Y} = (T, Y)$. 
Proposition 3.4. Let $X = (T, X)$ be a minimal Abelian flow and let $S$ be a syndetic subgroup of $T$. Then $S^*$ is an enveloped subgroup of $T$ and it is the smallest enveloped subgroup of $T$ containing $S$.

Proof. $S^*$ is syndetic and $(S^*)^* = S^*$. So $S^*$ is an enveloped subgroup of $T$. Let $E$ be an enveloped subgroup of $T$ containing $S$. Then

$$E = E^* = \{ t \in T \mid t \in E \} \supset \{ t \in T \mid t \in S \} = S^*.$$ 

Thus an enveloped subgroup of $T$, containing $S$, contains $S^*$. □

Example 3.5. Let $\theta \neq 0$ be a real number. Consider the compact minimal Abelian flow $X = (\mathbb{R}, T, \pi)$, defined by $\pi(t, z) = e^{2\pi i \theta z} \in \mathbb{R}$, $t \in \mathbb{R}$, $z \in \mathbb{T}$. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Consider the action of the subgroup $\mathbb{Z} \alpha$ of $\mathbb{R}$ on $\mathbb{T}$, induced by $\pi$. The orbit in this action of an element $z \in \mathbb{T}$ has the form

$$\mathbb{Z} \alpha \cdot z = \{ n\alpha \cdot z \mid n \in \mathbb{Z} \} = \{ e^{2\pi i n \alpha \theta} \cdot z \mid n \in \mathbb{Z} \}.$$

Case 1: $\alpha \theta \in \mathbb{Q}$. Suppose $\alpha \theta = k/l$, $(k, l) = 1$. Then every $e^{2\pi i n \alpha \theta} = e^{2\pi i n k/l}$ is one of the elements $1, e^{2\pi i k/l}, \ldots, e^{2\pi i (l-1)k/l}$. Hence

$$\mathbb{Z} \alpha \cdot z = \{ e^{2\pi i k/l} \cdot z, \ldots, e^{2\pi i (l-1)k/l} \}.$$ 

So $\overline{\mathbb{Z} \alpha \cdot z} = \mathbb{Z} \alpha \cdot z$. To calculate $(\mathbb{Z} \alpha)^*$ it is enough to calculate $(\mathbb{Z} \alpha)^*$ for any $z \in \mathbb{T}$, for example $(\mathbb{Z} \alpha)^1$. We have:

$$\beta \in (\mathbb{Z} \alpha)^1 \iff e^{2\pi i \beta \theta} \cdot 1 = e^{2\pi i n \alpha \theta} \text{ for some } n \in \mathbb{Z} \iff \mathbb{Z} \alpha + \mathbb{Z} \frac{1}{\beta} = \mathbb{Z} \frac{1}{\alpha} \iff \beta \in \mathbb{Z} \alpha + \mathbb{Z} \frac{1}{\alpha} = \mathbb{Z} \frac{1}{\alpha}.$$ 

Thus we have $\mathbb{Z} \alpha^* = \mathbb{Z} \frac{1}{\alpha}$, where $\alpha \theta = k/l$, $(k, l) = 1$.

Case 2: $\alpha \theta \notin \mathbb{Q}$. Then $\{ e^{2\pi i n \alpha \theta} \mid n \in \mathbb{Z} \}$ is dense in $\mathbb{T}$ for every $z \in \mathbb{T}$, so $\overline{\mathbb{Z} \alpha \cdot z} = \mathbb{T}$. Then $(\mathbb{Z} \alpha)^* = (\mathbb{Z} \alpha)^1 = \{ \beta \in \mathbb{R} \mid e^{2\pi i \beta \theta} \in \mathbb{T} \} = \mathbb{R}$.

More concretely, let $\theta = 1$. Then for example: $(\mathbb{Z} 2)^* = \mathbb{Z} \frac{1}{2} = (\mathbb{Z} \frac{1}{2})^*$, $(\mathbb{Z} \sqrt{3})^* = \mathbb{R}$, etc. There are many subgroups of $\mathbb{R}$ that are enveloped, and many that are not.

Example 3.6. Let $X$ be an almost periodic compact minimal Abelian flow with $X$ non-connected. $X$ can be written as a disjoint union $X = Y \cup Z$ of two nonempty clopen sets $Y$ and $Z$. Let $\alpha = (Y \times Y) \cup (Z \times Z)$. Since $X$ is compact and $\alpha$ is an open neighborhood of the diagonal $\Delta X$, $\alpha$ is an entourage of the unique uniform structure on $X$.

Claim.

$$S := \bigcap_{x \in X} D(x, \alpha[x])$$

is a proper enveloped subgroup of $T$.

Since $X$ is almost periodic, $S$ is a syndetic subset of $T$. Note that for $y \in Y$, $\alpha[y] = Y$, and for $z \in Z$, $\alpha[z] = Z$. So $S = \{ t \in T \mid \forall y \in Y \ t \cdot y \in Y \ \text{and} \ \forall z \in Z \ t \cdot z \in Z \} = \{ t \in T \mid t \cdot Y \subset Y \ \text{and} \ t \cdot Z \subset Z \} = \{ t \in T \mid t \cdot Y = Y \ \text{and} \ t \cdot Z = Z \}$. It follows that $t_1$, $t_2 \in T$ implies $t_1 + t_2 \in S$, and $t \in S$ implies $-t \in S$. Thus $S$ is a syndetic subgroup of $T$. Now consider
any element of $X$, for example some $y \in Y$. Since $X$ is a compact minimal Abelian flow, $S^* = S' = \{ t \in T \mid t.y \in S.y \} \subseteq \{ t \in T \mid t.y \in Y \}$. (The inclusion holds because $S.y \subseteq Y$ and $Y$ is closed.) Since this is true for any $y \in Y$, we have $S^* \subseteq \bigcap_{y \in Y} \{ t \in T \mid t.y \in Y \} = \{ t \in T \mid (\forall y \in Y) t.y \in Y \}$.

To prove that $S$ is proper, consider any $y \in Y$. Since $X$ is minimal and $Z$ is open, there is a $t \in T$ such that $t.y \in Z$. This $t$ does not belong to $S$.

The claim is proved.

4. A criterion for minimality of reduced flows and a criterion for total minimality

**Proposition 4.1.** Let $X = \langle T, X \rangle$ be a flow and $S$ a normal syndetic subgroup of $T$. Let $\chi \in \hat{T}$ be an eigenvalue of $X$. The following are equivalent:

(i) $\ker(\chi) \supset S$;

(ii) $\ker(\chi) \supset S^x$ for every $x \in X$;

(iii) $\ker(\chi) \supset S^x$ for some $x \in X$.

**Proof.**

(iii) $\Rightarrow$ (i): Clear, since $S^x \supset S$.

(ii) $\Rightarrow$ (iii): Clear.

(i) $\Rightarrow$ (ii): Fix any $x \in X$. Let $\eta$ be an eigenfunction of $X$ which corresponds to $\chi$. We have:

\[ \eta(tx) = \chi(t)\eta(x), \]

for all $t \in T$, and, in particular, $\eta(sx) = \eta(x)$ for all $s \in S$. If we let $z = \eta(x)$, we have $\eta(S.x) = \{ z \}$. By continuity $\eta(S^*x) = \{ z \}$ and consequently $\eta(S^*x) = \{ z \}$. Hence from (*), $\chi(s) = 1$ for all $s \in S^x$, i.e., $\ker(\chi) \supset S^x$. $\square$

**Corollary 4.2.** If in the above proposition $X$ is a minimal Abelian flow, then $\ker(\chi) \supset S$ iff $\ker(\chi) \supset S^x$.

**Theorem 4.3** (Criterion for minimality of reduced flows). Let $X = \langle T, X, \pi \rangle$ be a compact minimal Abelian flow. Let $S$ be a syndetic subgroup of $T$ and let $X_S = \langle S, X \rangle$. The following statements are equivalent:

(i) $X_S$ is a minimal flow;

(ii) $X$ has no nontrivial eigenvalue whose kernel contains $S$;

(iii) $S^* = T$.

**Proof.** By Proposition 4.1, (ii) is equivalent with

(ii') $X'$ has no nontrivial eigenvalue whose kernel contains $S^*$.

So we will prove the above theorem with (ii) replaced by (ii'). First we make some observations.
Note that $O_S = \{S,x \mid x \in X\}$ is the same as $O_{S'}$ since $S,x = S',x$. The equivalence relation $R = R(O_{S'})$, $(x,y) \in R \iff S,x = S,y$ is open and closed by 2.8. Hence $X/R$ is compact Hausdorff. We denote by $p_X : X \to X/R = \tilde{X}$ the quotient map and by $p_T : T \to T/S'$ the canonical homomorphism. The elements of $\tilde{X}$ will be denoted by $\tilde{x} = p_X(S,x)$. The map $\pi : T \times X \to X$ is compatible with the relations (mod $S'$) $\times R$ on $T \times X$ and $R$ on $X$. Hence it induces a continuous map $\tilde{\pi} : T \times X/(\text{mod } S') \times R \to \tilde{X}$. Since mod $S'$ and $R$ are both open, we may identify $T \times X/(\text{mod } S') \times R$ with $T/S' \times X/R = T/S' \times \tilde{X}$. With this identification we have

$$\tilde{\pi} \circ (p_T \times p_X) = p_X \circ \pi.$$  

It follows that $\mathcal{Y} = \langle T/S', \tilde{X}, \tilde{\pi} \rangle$, with $((t + S'), \tilde{x} = \tilde{f}(\tilde{x})$, is a flow.

Fix a point $a \in X$. Denote $\pi |_{T \times \{a\}}$ by $\psi$, and $\tilde{\pi} |_{T/S' \times \{\tilde{a}\}}$ by $\tilde{\psi}$. We have

$$(t + S') \tilde{a} = \tilde{a} \iff \tilde{f}(\tilde{a}) = \tilde{a} \iff \tilde{S}.t.a = \tilde{S}.a = \tilde{a} \iff tS.a = S.a \iff t \in S'.$$

Hence the stabilizer of $\tilde{a}$ in the flow $\mathcal{Y}$ is the identity subgroup of $T/S'$, hence $\tilde{\psi}$ is injective. Also $\tilde{\psi}$ is surjective by 2.9. (ii) and (4.1). Now since $T/S' \times \{a\}$ is compact, $\tilde{\psi}$ is a homeomorphism.

Now using these observations we show the equivalence of the statements (i), (ii), and (iii).

(i) $\iff$ (iii): $\lambda S$ is minimal iff $\tilde{X}$ consists of one element iff $T = S'$ (since $\tilde{\psi}$ is bijective).

(iii) $\iff$ (ii): Clear.

(ii) $\implies$ (iii): Suppose $S' \neq T$. Define a continuous map $f : X \to T/S'$ by $f = pr_1 \circ \tilde{\psi}^{-1} \circ p_X$. For $t \in T$ let $\text{transl}_{p_T(t)} : T/S' \to T/S'$ be defined by $\text{transl}_{p_T(t)}(t' + S') = p_T(t' + t + S')$. Then for every $t \in T$

$$f \circ \pi_t = \text{transl}_{p_T(t)} \circ f.$$  

(4.2)

To prove (4.2), put $\tilde{f} = pr_1 \circ \tilde{\psi}^{-1}$. For $t_1 \in T$ we have

$$f(tx) = t_1 + S' \iff \tilde{f}(\tilde{tx}) = t_1 + S' \iff \tilde{\psi}(t_1 + S', \tilde{a}) = \tilde{f}(\tilde{x}) \iff \tilde{f}(\tilde{a}) = \tilde{f}(\tilde{x})$$

$$\iff \tilde{S}.t.a = \tilde{S}.a \iff t \tilde{S}.a = \tilde{S}.a \iff tS.a = S.a \iff \tilde{a} \in (\tilde{t} + \tilde{S}).a \iff \tilde{a} = \tilde{t} \tilde{a} = \tilde{t} \tilde{x}$$

$$\iff \tilde{f}(\tilde{x}) = -(t + S' + t_1 + S') \iff f(x) = (-t + S') + (t_1 + S') \iff (t + S') + f(x) = t_1 + S'.$$

Thus (4.2) holds. Now let $\chi : T/S' \to \mathbb{T}$ be any nontrivial character of $T/S'$. For $t \in T$ let $\chi_{Z(t)} : \mathbb{T} \to \mathbb{T}$ be defined by $\text{transl}_{Z(t)}(z) = \chi(t)z$. It is easy to see that for every $t \in T$

$$\chi \circ \text{transl}_{p_T(t)} = \text{transl}_{\chi_{Z(T)}(t)} \circ \chi.$$  

(4.3)

Let $\eta = \chi \circ f : X \to \mathbb{T}$. Then $\eta$ is a continuous function which satisfies

$$\eta \circ \pi_t = \text{transl}_{\chi_{Z(T)}(t)} \circ \eta$$

for all $t \in T$. (This follows from (4.2) and (4.3).) So $\eta$ is an eigenfunction of $\chi$ whose eigenvalue $\chi \circ p_T$ is nontrivial and whose kernel contains $S'$.  \[\Box\]
**Corollary 4.4** (Criterion for total minimality). Let \( X = (T, X) \) be a compact minimal Abelian flow. The following statements are equivalent:

(i) \( X \) is a totally minimal flow;
(ii) \( X \) has no nontrivial eigenvalue whose kernel is syndetic;
(iii) \( T \) has no proper \( X \)-enveloped subgroup.

**Proof.** By Corollary 4.2, (ii) is equivalent to 

(ii') \( X \) has no nontrivial eigenvalue whose kernel contains an enveloped subgroup of \( T \).

So we will prove this corollary with (ii) replaced by (ii').

(i) \( \Rightarrow \) (ii') Clear from Theorem 4.3.

(ii') \( \Rightarrow \) (iii): Clear from Theorem 4.3.

(iii) \( \Rightarrow \) (i): Suppose (iii) holds. Let \( S \) be a syndetic subgroup of \( T \). By assumption (iii), \( S^* = T \) (since \( S^* \) is enveloped). By Theorem 4.3, \( X_S \) is minimal. ✷

As the first applications of these criteria, we investigate \( \mathbb{Z} \) and \( \mathbb{R} \)-flows.

**Corollary 4.5.** Let \( X = (\mathbb{Z}, X) \) be a compact minimal \( \mathbb{Z} \)-flow. Then the following statements are equivalent:

(i) \( X \) is totally minimal;
(ii) \( X \) has no eigenvalue \( \chi_{\lambda} = e^{2\pi i \lambda \cdot} \) such that \( \lambda \in \mathbb{Q} \setminus \{0\} \);
(iii) \( X \) has no eigenvalue \( \chi(n) = z^n \) with \( z \neq 1 \) of finite order in \( T \).

**Proof.** (ii) and (iii) are clearly equivalent. We will show (i) \( \Leftrightarrow \) (ii). Every character of \( \mathbb{Z} \) has the form \( \chi_{\lambda} = e^{2\pi i \lambda \cdot} \). We have \( \ker(\chi_{\lambda}) = \{n \in \mathbb{Z} \mid e^{2\pi i \lambda n} = 1\} = \{n \in \mathbb{Z} \mid \lambda n = k \in \mathbb{Z}\} \). Now by the criterion for total minimality, \( X \) is totally minimal iff \( X \) has no nontrivial eigenvalue \( \chi_{\lambda} \) such that \( \{n \in \mathbb{Z} \mid \lambda n = k \in \mathbb{Z}\} \) is syndetic in \( \mathbb{Z} \), iff \( X \) has no eigenvalue \( \chi_{\lambda} \) with \( \lambda \in \mathbb{Q} \setminus \{0\} \). ✷

**Remark 4.6.** The direction (i) \( \Rightarrow \) (ii), i.e., (i) \( \Rightarrow \) (iii), of the Corollary 4.5 is well known; see for example [12, p. 311]. The opposite direction is probably also known, but the author could not find a reference.

**Corollary 4.7** [4, Theorem 1]. Let \( X = (\mathbb{R}, X) \) be a compact minimal \( \mathbb{R} \)-flow. Let

\[
A(X) = \{\lambda \in \mathbb{R} \mid \chi_{\lambda} = e^{2\pi i \lambda \cdot} \text{ is an eigenvalue of } X\}
\]

and let 

\[
\widetilde{A}(X) = \left\{\frac{\lambda}{n} \mid \lambda \in A(X), \ n \in \mathbb{Z} \setminus \{0\}\right\}.
\]

Let \( S = \mathbb{Z} \alpha \), where \( \alpha > 0 \) is a real number, and let \( X_S = (S, X) \). Then \( X_S \) is minimal if and only if \( 1/\alpha \notin \widetilde{A}(X) \).

**Proof.** By the criterion for minimality of reduced flows, \( X_S \) is not minimal iff \( X \) has an eigenvalue \( \chi_{\lambda}, \lambda \neq 0 \), such that \( \{t \in \mathbb{R} \mid \lambda t \in \mathbb{Z}\} \supset \mathbb{Z} \alpha \), iff \( X \) has an eigenvalue \( \chi_{\lambda}, \lambda \neq 0 \).
such that $Z_{\frac{1}{n}} \supset Z_{\alpha}$, iff $X$ has an eigenvalue $\chi_\lambda$ such that $1/\alpha = \lambda/n$, $n \in \mathbb{Z} \setminus \{0\}$, iff $1/\alpha \in \tilde{A}(X)$. □

5. A new proof of a theorem of Parry

We will now investigate skew-extensions of compact minimal $\mathbb{Z}$-flows. We will show that, in the special situation described in Proposition 5.1, the criterion for minimality of reduced flows is equivalent with the well-known Parry’s condition. The proof illustrates the way in which one can end up with Parry’s condition after a sequence of natural steps, starting with the condition from the criterion.

**Proposition 5.1** ([17, p. 98–99], [22, II(8.22)], [1, p. 72]). Let $G$ be a compact Abelian topological group. Let $Y = \langle Y, \tau \rangle$ be a compact minimal $\mathbb{Z}$-flow, $\varphi : Y \to G$ a continuous map, $X = Y \times G$ and let $\sigma \in \text{Homeo}(X)$ be defined by

$$\sigma(y, g) = \left(\tau(y), \varphi(y)g\right). \quad (5.1)$$

Then the compact $\mathbb{Z}$-flow $X = \langle X, \sigma \rangle$ is minimal iff

$$f\left(\tau(y)\right) = \gamma(\varphi(y))f(y)$$

has no solution $f \in C(Y, \mathbb{T}, \gamma \in \hat{G}$, with $\gamma \neq 1$.

**Proof.** Define a (compact Abelian) flow $Z = \langle \mathbb{Z} \times G, X \rangle$ by

$$(n, g).x = \left(\text{pr}_1(\sigma^n(x)), \text{pr}_2(\sigma^n(x))g\right), \quad (5.2)$$

for $n \in \mathbb{Z}$, $g \in G$, $x \in X$. Writing $x = (y, g')$, we get from (5.1) and (5.2)

$$\begin{align*}
(n, g).(y, g') &= \left(\tau^n(y), \prod_{i=1}^{n} \varphi(\tau^{n-i}(y)) \cdot g'g\right), \\
(-n, g).(y, g') &= \left(\tau^{-n}(y), \prod_{i=1}^{n} \varphi(\tau^{-i}(y))^{-1} \cdot g'g\right).
\end{align*}$$

for $n \geq 0$. From these formulas we can see (using minimality of $Y$) that $Z$ is minimal (the orbit $(\mathbb{Z} \times G).(y, g')$ of $(y, g')$ has the form $D \times G$, where $D$ is a dense subset of $Y$).

Let $S = \mathbb{Z} \times \{1\}$. Since $(n, 1).(y, g') = \sigma^n(y, g')$, which is $n.(y, g')$ in $X$, we may identify flows $Z_S$ and $X$. So we have

$X$ is not minimal $\iff Z_S$ is not minimal. \hfill (\ast)

Now consider the following sequence of conditions:

1. **(COND1)** $\eta((n, g).(y, g')) = \chi(n, g)\eta(y, g')$, $n \in \mathbb{Z}$, $g, g' \in G$, $y \in Y$, has a solution

$\eta \in C(X, \mathbb{T})$, $\chi \in \mathbb{Z} \times \hat{G}$, with $\chi \neq 1$ and $\ker(\chi) \supset \mathbb{Z} \times \{1\}$;

2. **(COND2)** $\eta((n, g).(y, g')) = \gamma(g)\eta(y, g')$, $n \in \mathbb{Z}$, $g, g' \in G$, $y \in Y$, has a solution

$\eta \in C(X, \mathbb{T})$, $\gamma \in \hat{G}$, with $\gamma \neq 1$;
(COND3) \( \eta((1,g),(y,g')) = \gamma(g)\eta(y,g'), \ g,g' \in G, \ y \in Y \), has a solution \( \eta \in C(X,\mathbb{T}), \gamma \in \hat{G} \), with \( \gamma \neq 1 \);

(COND4) \( \eta((1,g),(y,1)) = \gamma(g)\eta(y,1), \ g \in G, \ y \in Y \), has a solution \( \eta \in C(X,\mathbb{T}), \gamma \in \hat{G} \), with \( \gamma \neq 1 \);

(COND4') \( \eta((\tau(y),\varphi(y))g) = \gamma(g)\eta(y,1), \ g \in G, \ y \in Y \), has a solution \( \eta \in C(X,\mathbb{T}), \gamma \in \hat{G} \), with \( \gamma \neq 1 \);

(COND5) \( \eta(\tau(y),1) = \gamma(\varphi(y))^{-1}\eta(y,1), \ g \in G, \ y \in Y \), has a solution \( \eta \in C(X,\mathbb{T}), \gamma \in \hat{G} \), with \( \gamma \neq 1 \);

(COND6) \( f(\tau(y)) = \gamma(\varphi(y))^{-1}f(y), \ y \in Y \), has a solution \( f \in C(Y,\mathbb{T}), \gamma \in \hat{G} \), with \( \gamma \neq 1 \).

We have (COND1) \( \Leftrightarrow \) (COND2) by 2.3. Let us see that (COND3) \( \Rightarrow \) (COND2). If we put \( g = 1 \) in (COND3), we get

\[
\eta(\sigma(y,g')) = \eta(y,g'), \ y \in Y, \ g' \in G.
\]

Hence

\[
\eta(\sigma^n(y,g')) = \eta(y,g'), \ n \in \mathbb{Z}, \ y \in Y, \ g' \in G.
\]

(5.3)

Then we replace \((y,g')\) by \(\sigma^{n-1}(y,g') = (n-1,1),(y,g')\) in (COND3) and get (COND2) using (5.3). So (COND2) \( \Leftrightarrow \) (COND3). Also (COND3) \( \Rightarrow \) (COND4) \( \Leftrightarrow \) (COND4') \( \Rightarrow \) (COND5). Now if we put

\[
f(y) = \eta(y,1)
\]

we get (COND5) \( \Rightarrow \) (COND6). Also (COND6) \( \Leftrightarrow \) the negation of the condition from the statement of the proposition.

Conversely, suppose that (COND6) holds and define

\[
\eta(y,g') = \gamma(g')f(y).
\]

(5.4)

We will show that these \( \eta, \gamma \) satisfy (COND3). We have

\[
\eta((1,g),(y,g')) = \eta(\tau(y),\varphi(y)g') = \gamma(\varphi(y))\gamma(g')\gamma(g)f(\tau(y)) = (\text{from (COND6)}) \gamma(\varphi(y'))\gamma(g')\gamma(g)f(y) = (\text{by (5.4)}) \gamma(g')\gamma(g)\gamma(g')^{-1}\eta(y,g') = \gamma(g)\eta(y,g').
\]

Thus (COND6) \( \Rightarrow \) (COND3) \( \Leftrightarrow \) (COND1).

Now since \( S \) is a syndetic subgroup of \( \mathbb{Z} \times G \), by the criterion for total minimality of reduced flows and \((*)\), \( X \) is not minimal iff the condition (COND1) holds. But, as we have just shown, (COND1) is equivalent with the negation of Parry’s condition from the statement of the proposition. This completes the proof. \( \square \)

Remark 5.2. There is another, more general, theorem, proved by Parry [17, Theorem 1], which can also be proved [16] using the criterion for minimality of reduced flows. It illustrates even better that in the case of skew-extensions of compact minimal flows, the condition coming from the criterion for minimality of reduced flows, applied to an appropriate flow, is in a natural way equivalent with Parry’s condition. (Using a similar procedure we prove some other statements in [16], for example we get a necessary and sufficient condition (in terms of eigenvalues) for a product of two compact minimal flows, one of which is almost periodic, to be minimal.)
6. SK groups

In this section we introduce the notion of SK groups, which will be used in Sections 7 and 8. For example, Proposition 6.5 below will play a role in the proof of Proposition 8.5(ii).

**Definition 6.1.** A topological group $T$ is said to be SK, if the kernel of every continuous character $\chi \in \hat{T}$ is a syndetic subgroup of $T$.

**Example 6.2.** (i) $\mathbb{R}$. (ii) Every compact group. (iii) Every minimally almost periodic group; in particular, every extremely amenable group. (Examples: [13, 23.32].)

(An Abelian topological group $T$ is called *minimally almost periodic* if it has no nontrivial continuous characters. A topological group $T$ is called *extremely amenable* if every $T$-flow on a compact space has a fixed point. It is easy to see that every Abelian extremely amenable group is minimally almost periodic.)

Example 6.2(iii) shows that there are non-LCA SK groups. Also, not all LCA groups are SK, for example $\mathbb{Z}$, $\mathbb{R}_d$, $T_d$, $\mathbb{R} \times \mathbb{R}_d$, etc. (Here $d$ denotes the discrete topology.)

**Proposition 6.3.** A finite product of SK groups is an SK group.

**Proof.** Let $T_1, \ldots, T_n$ be SK groups and let $T = T_1 \times \cdots \times T_n$. Let $\chi \in \hat{T}$. For each $(x_1, \ldots, x_n) \in T$ we have $\chi(x_1, \ldots, x_n) = \chi(x_1, 0, \ldots, 0) \cdots \chi(0, 0, \ldots, x_n)$. For $i = 1, 2, \ldots, n$ denote by $\chi_i$ the continuous character $\chi_i : t \mapsto \chi_i(0, x_i, \ldots, 0)$ of $T_i$. So we have $\chi(x_1, \ldots, x_n) = \chi(x_1) \cdots \chi_n(x_n)$, where $\chi_i \in \hat{T}_i$ $(i = 1, 2, \ldots, n)$. Let $S_i = \ker(\chi_i)$, and $T_i = S_i + K_i$, where $K_i$ is a compact subset of $T_i$ $(i = 1, 2, \ldots, n)$. Since $\ker(\chi) \supset S_1 \times \cdots \times S_n$, and $S_1 \times \cdots \times S_n$ is syndetic ($K = K_1 \times \cdots \times K_n$ is compact and $S_1 \times \cdots \times S_n + K = T$), $\ker(\chi)$ is also syndetic. □

**Corollary 6.4.** Every connected LCA group is SK.

**Proof.** By [13, 9.14], connected LCA groups have the form $\mathbb{R}^n \times C$, where $n \geq 0$ and $C$ is a compact connected Abelian group. Since $\mathbb{R}$ and $C$ are SK, the corollary follows from Proposition 6.3. □

**Proposition 6.5.** Let $T$ be a topological group, $S$ a subgroup of $T$. Let $\chi \in \hat{T}$ be such that: (i) $\chi(S) = T$, and (ii) $\ker(\chi|_S)$ is syndetic in $S$. Then $\ker(\chi)$ is syndetic in $T$.

**Proof.** Let $S' = \ker(\chi|_S)$. We have $S = S' + K$ for some compact subset $K$ of $S$. For each $t \in T$, let $s_t$ be an element of $S$ such that $\chi(s_t) = \chi(t)^{-1}$. Then $t + s_t + S' \subset \ker(\chi)$. (Indeed, $\chi(t + s_t + S') = \chi(t) \cdot \chi(s_t) \cdot \chi(S') = \chi(t) \cdot \chi(t)^{-1} \cdot \{1\} = \{1\}$). Thus $\ker(\chi) \supset \bigcup_{t \in T} (t + s_t + S')$. Now $\ker(\chi) + K \supset \bigcup_{t \in T} (t + s_t + S') + K = \bigcup_{t \in T} (t + s_t + S' + K) = \bigcup_{t \in T} (t + S) = T$. So $\ker(\chi)$ is syndetic in $T$. □
Corollary 6.6. Let $T$ be a topological group which contains a connected SK subgroup $S$. Let $\chi \in \hat{T}$ be such that $S \not\subset \ker(\chi)$. Then $\ker(\chi)$ is syndetic in $T$.

Proof. Conditions (i) and (ii) of the previous proposition are satisfied. □

Corollary 6.7. Let $T$ be an LCA group, $T_0$ its connected component of identity. Let $\chi \in \hat{T}$ be such that $T_0 \not\subset \ker(\chi)$. Then $\ker(\chi)$ is syndetic in $T$.

Proof. $T_0$ is connected and it is SK by Corollary 6.4. So the statement follows from Corollary 6.6. □

Example 6.8. Let $T = \mathbb{R} \times \mathbb{T}$. Then $T_0 = \mathbb{R} \times \{0\}$. If $c$ is the trivial character of $\mathbb{R}$, then $\ker(c \cdot \text{id}_T) = \mathbb{R} \times \{0\}$ and this is not a syndetic subgroup of $T$. For any other character $\chi \in \hat{\mathbb{R}}$, $\ker(\chi \cdot \text{id}_T)$ is a syndetic subgroup of $T$ by Corollary 6.7. (For notation $\chi_1 \cdot \chi_2$ see 2.3.)

Remark 6.9. Abelian SK groups in a natural way generalize minimally almost periodic groups (which are never LCA unless trivial), but also contain connected LCA groups. The fact that in recent years it has become clear that extremely amenable (and minimally almost periodic) groups are not “exotic” [18], can give some importance to SK groups. In connection with this, let us mention that it is not known if minimally almost periodic groups are extremely amenable [19] even in the case of monothetic groups. It is proved in [9, Theorem 3.3] that an example of a Polish minimally almost periodic group, which is not extremely amenable, would solve in the negative the old problem from combinatorial number theory and harmonic analysis, asking if the set $S - S$, where $S$ is a syndetic subset of $\mathbb{Z}$, is big enough to be a neighborhood of 0 in the Bohr topology on $\mathbb{Z}$. (Recall that the Bohr topology on an Abelian topological group $T$ is the weakest topology on $T$ in which all originally continuous characters of $T$ remain continuous.)

Let us also mention that an Abelian topological group $T$ is extremely amenable iff every compact minimal $T$-flow is trivial, iff the universal compact minimal $T$-flow $\mathcal{M}_T = (T, M_T)$ is trivial. (Recall that for every topological group $T$, the universal compact minimal $T$-flow $\mathcal{M}_T = (T, M_T)$ such that for every compact minimal $T$-flow $\mathcal{X} = (T, X)$ there exists a morphism of flows of $\mathcal{M}_T$ onto $\mathcal{X}$. It is well-known that $\mathcal{M}_T$ exists and is unique, see [1, p. 115–117], [6, p. 61–62], [22, IV(3.27)], and also [21, Appendix]. It is shown in [21] that $\mathcal{M}_T$ is not 3-transitive.) Obviously, if a topological group $T$ admits at least one compact minimal non-totally-minimal flow, then $\mathcal{M}_T$ is not totally minimal.

7. Total minimality of $\mathcal{X}$ in terms of the structure group $\Gamma(\mathcal{X})$

Recall that a flow $\mathcal{X} = (T, X)$ is weakly mixing if for any open subsets $U, U', V, V'$ of $X$, $D(U, V) \cap D(U', V') \neq \emptyset$ [22, p. 273].
Remark 7.1.
(i) For compact flows we have: if $\mathcal{X}$ is weakly mixing, every eigenfunction of $\mathcal{X}$ is constant [22, p. 409].
(ii) For compact minimal Abelian flows we have: if every eigenfunction of $\mathcal{X}$ is constant, $\mathcal{X}$ is weakly mixing [22, p. 409].
(iii) Combining (i), (ii) and 2.10, we have for compact minimal Abelian flows: $\mathcal{X}$ is weakly mixing iff every eigenvalue of $\mathcal{X}$ is trivial.
(iv) From (iii) and the criterion for total minimality we have for compact minimal Abelian flows: $\mathcal{X}$ is weakly mixing implies $\mathcal{X}$ is totally minimal.

 Proposition 7.2. Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal Abelian flow and suppose that $T$ is an SK group. Then $\mathcal{X}$ is totally minimal iff $\mathcal{X}$ is weakly mixing.

 Proof. The statement follows from the criterion for total minimality and Remark 7.1(iv). \Box

 Remark 7.3. According to [14, p. 480], if $\mathcal{X}$ is a compact minimal Abelian flow, “$\mathcal{X}$ is weakly mixing iff $\mathcal{X}$ is totally minimal” was first proved by Markley for $T = \mathbb{R}$. Proposition 7.2 extends this result since $\mathbb{R}$ is SK.

 Remark 7.4. If $T$ is not SK, Proposition 7.2 is not true in general. Consider for example a compact minimal (almost periodic) $\mathbb{Z}$-flow $\mathcal{X} = (\mathbb{Z}, T, \pi)$, defined by $\pi(n, z) = e^{2\pi i n \theta z}$, for $n \in \mathbb{Z}$ and $z \in T$, where $\theta \in \mathbb{R}$ is irrational. This flow is totally minimal. (Follows from (8.1) below, but it’s also easy to check directly.) However this flow is not weakly mixing. (Follows easily from the definition of weak mixing.)

 More generally, any nontrivial compact minimal $\mathbb{Z}$-flow $\mathcal{X}$ on a connected space $X$, which satisfies $S_X^\mathcal{X} \neq X \times X$, is totally minimal but not weakly mixing. (Total minimality follows from (8.1) below. Weak mixing follows from [20, p. 279], which states that a compact minimal Abelian flow $\mathcal{X}$ is weakly mixing iff $S_X^\mathcal{X} = X \times X$, but can also be deduced from [14, Theorem 3.1] or [22, V(1.19)].)

 (Here $S_X^\mathcal{X}$ is the equicontinuous structure relation of $\mathcal{X}$, i.e., the smallest closed invariant equivalence relation on $X$ such that the flow $\mathcal{X}/S_X^\mathcal{X}$ is almost periodic [22, p. 398].)

 Recall that the structure group $\Gamma(\mathcal{X})$ of the flow $\mathcal{X}$ is the Ellis semigroup $E(\mathcal{X}/S_X^\mathcal{X})$ (which is then necessarily a topological group) of the almost periodic flow $\langle T, X/S_X^\mathcal{X} \rangle$.

 Corollary 7.5. Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal Abelian flow and suppose that $T$ is SK. Then $\mathcal{X}$ is totally minimal iff $\Gamma(\mathcal{X})$ is trivial.

 Proof. For compact minimal Abelian flows, $\mathcal{X}$ is weakly mixing iff $S_X^\mathcal{X} = X \times X$ ([20, p. 279] or [22, V(1.19)]). Hence (by the previous proposition) $\mathcal{X}$ is totally minimal iff $S_X^\mathcal{X} = X \times X$. Finally, since $\mathcal{X}$ is minimal, $S_X^\mathcal{X} = X \times X$ iff $\Gamma(\mathcal{X})$ is trivial. (Indeed, if $S_X^\mathcal{X} = X \times X$, then clearly $\Gamma(\mathcal{X})$ is trivial. Conversely, if $\Gamma(\mathcal{X})$ is trivial, $E(X/S_X^\mathcal{X}) = \{id_X/S_X^\mathcal{X}\}$. So all elements of $T$ fix every element of $X/S_X^\mathcal{X}$. This means that for every
equivalence class $C \subset X$ of the relation $S_X^t$, i.e. $C \subset C$ for all $t \in T$. Thus $C$ is a (closed) invariant subset of $X$ under $T$. Since $X$ is minimal, there is only one equivalence class; i.e., $\Gamma(X)$ is trivial. 

Remark 7.6. Thus the class of Abelian topological groups for which the total minimality of every compact minimal Abelian flow $X$ is equivalent with the triviality of $\Gamma(X)$, includes Abelian SK groups. (It would be interesting to characterize this class.) In the case that $T$ is a connected LCA group, the previous corollary was stated by Gottschalk [10, p. 56]. Since, by Corollary 6.4, connected LCA groups are SK, we have a larger class of acting groups for which the statement holds.

8. Compact minimal Abelian flows that are not totally minimal

8.1. Although total minimality is a strong condition, there are many examples of totally minimal flows. For example, the following statement holds [11, 2.28]: every minimal flow $X = (T, X)$, with $T$ discrete and $X$ connected, is totally minimal.

Indeed, if $S$ is a syndetic normal subgroup of $T$, then $T = FS = SF$ for some finite set $F \subset T$. Let $x \in X$. Then (using 2.6–2.8) $X = \overline{TX} = \overline{TXS} = F.Sx = \bigcup_{t \in F_1} t.Sx$, where $F_1$ is some subset of $F$ and the union is disjoint. Since $X$ is connected, it cannot be a finite union of $>1$ disjoint closed sets. So $X = Sx$. Since this holds for any $x \in X$, $X$ is totally minimal.

8.2. In [8, p. 36] a family of examples of minimal $\mathbb{Z}$-flows on $T^n \ (n \geq 1$ any integer) is given. By 8.1, these flows are necessarily totally minimal. (See also [22, III(1.18)–III(1.20)] and [7].)

8.3. There exists a minimal continuous $\mathbb{R}$-flow on $T^2$, with no nontrivial continuous eigenvalue [15]. By 7.1(ii) and (iv), this flow is necessarily totally minimal.

8.4. Note that by 8.1 and Corollary 4.5, every nontrivial eigenvalue $\chi_\lambda = e^{2\pi i \lambda}$ of a compact minimal $\mathbb{Z}$-flow on a connected space $X$, satisfies $\lambda \notin \mathbb{Q}$. (But not every such flow has a nontrivial eigenvalue. However, it is proved in [7, Theorem 5.1] that every minimal $\mathbb{Z}$-flow on $T^2$, $X = (T^2, h)$, such that the homeomorphism $h$ is not homotopic to the identity transformation, has a nontrivial eigenvalue.)

We will now give some conditions on compact minimal Abelian flows which necessarily imply non-total-minimality.

Proposition 8.5. Let $X = (T, X, \pi)$ be a compact minimal Abelian flow. Then in each of the following situations $X$ is not totally minimal:

(i) $X$ almost periodic, $X$ non-connected;

(ii) $X$ almost periodic, $T$ contains a connected SK subgroup which acts nontrivially on $X$;
(iii) $\mathcal{X}$ proximally equicontinuous, $T$ contains a connected SK subgroup which acts nontrivially on $X$;

(iv) $\mathcal{X}$ distal, $X/S_{\mathcal{X}}^*$ non-connected;

(v) $\mathcal{X}$ distal, $X$ totally disconnected, $|X| > 1$;

(vi) $\mathcal{X}$ point-distal, $T$ SK, $X$ metric, $|X| > 1$;

(vii) $\mathcal{X}$ regularly almost periodic at at least one point, $T$ contains a connected SK subgroup which acts nontrivially on $X$, $X$ metric.

**Proof.** (i) Follows from Example 3.6 and the criterion for total minimality.

(ii) Let $S$ be a connected SK subgroup of $T$ and $a \in X$, and suppose that $|S.a| > 1$. By [22, IV(3.42)], $X$ has a compact Abelian group structure such that $a$ is the identity element and the orbital map $\pi^a : T \to X$ is a continuous group homomorphism and $\overline{\pi^a(T)} = \overline{T.a} = X$. Since $S.a$ is a nontrivial closed connected subgroup of $X$, there is a surjective continuous character of $S.a$, $f_0 : \overline{S.a} \to T$. Let $f$ be a continuous character of $X$ which extends $f_0$.

Define $\chi \in \widehat{T}$ by $\chi = f \circ \pi^a$. Since $\pi^a(S) = S.a$ is a nontrivial connected subgroup of $X$, which is dense in $\overline{S.a}$, $\chi(S) = f(\pi^a(S))$ is a nontrivial connected subgroup of $T$. Hence $\chi(S) = T$. By Corollary 6.6, $\ker(\chi)$ is a syndetic subgroup of $T$. Since $f(\ker(\chi).a) = (f \circ \pi^a)(\ker(\chi)) = [1]$, we have $f(\ker(\chi).a) = [1]$. If $t \in \ker(\chi)^a$, then $\chi(t) = f(t.a) \in \ker(\chi).a = [1]$. Hence $t \in \ker(\chi)$. Thus $\ker(\chi)^a \subset \ker(\chi)$. Hence $\ker(\chi)^a = \ker(\chi)^a = \ker(\chi)$. So $\ker(\chi)$ is a proper enveloped subgroup of $T$. Now by the criterion for total minimality $(\text{i}) \iff (\text{iii}) \mathcal{X}$ is not totally minimal.

(iii) Let $S$ be a connected SK subgroup of $T$ and $a \in X$, and suppose that $|S.a| > 1$. Since $\mathcal{X}$ is proximally equicontinuous, $P_X = S_{\mathcal{X}}^*$ [22, V(1.7)2]. Consider the almost periodic flow $(T, X/S_{\mathcal{X}}^*)$. This flow satisfies the conditions of (ii). Indeed, if $b \neq a$ is an element of $S.a$, then $a$ and $b$ are distal [11, 10.07]. Hence the images $\tilde{a}$ and $\tilde{b}$ of $a$ and $b$ in $X/S_{\mathcal{X}}^*$ are distinct points, and $\tilde{b} \neq S.a$. Now by (ii), $(T, X/S_{\mathcal{X}}^*)$ is not totally minimal. Consequently $\mathcal{X}$ is not totally minimal.

(iv) Since $\mathcal{X}$ is distal and $|X| > 1$, $S_{\mathcal{X}}^* \neq X \times X$ ([22, V(3.33)] or [1, p. 104]). The flow $(T, X/S_{\mathcal{X}}^*)$ is almost periodic with a non-connected phase space. By (i), $(T, X/S_{\mathcal{X}}^*)$ is not totally minimal. Consequently $\mathcal{X}$ is not totally minimal.

(v) Since $\mathcal{X}$ is distal and $|X| > 1$, $S_{\mathcal{X}}^* \neq X \times X$ ([22, V(3.33)] or [1, p. 104]). The canonical map $\varphi : X \to X/S_{\mathcal{X}}^*$ is not only closed, but also open ([1, p. 98], [22, V(2.3)]). Since $X$ is totally disconnected, its image $X/S_{\mathcal{X}}^*$ under a continuous clopen map is totally disconnected. In particular, $X/S_{\mathcal{X}}^*$ (having more than one element) is not connected.

By (iv), $\mathcal{X}$ is not totally minimal.

(vi) Follows from the following: Proposition 7.2, the fact that every Abelian group is amenable and [22, V(1.18)].

(vii) By [11, 5.24] $\mathcal{X}$ is locally almost periodic. Hence it is proximally equicontinuous. Now by (iii) $\mathcal{X}$ is not totally minimal. \(\Box\)

**Remark 8.6.** (i) The statement 8.5(ii) was first proved in the case $T = \mathbb{R}$ by Floyd (see [11, 4.55 and 4.87]). It was generalized by H. Chu to non-totally-disconnected LCA groups.
with the connected component of the identity acting nontrivially on $X$ [2]. We extend the class of acting groups for which the statement is true. Also our proof is simpler than that in [2].

(ii) We could finish the proof of 8.5(ii) in a different way, by showing that $\chi$ is a nontrivial eigenvalue of $X$ whose kernel is a syndetic subgroup of $T$. For this purpose it remains to show that $f(t.x) = \chi(t)f(x)$ for all $(t, x) \in T \times X$. Denote the operation in $X$ by $\ast$. We have $(t_1 + t_2).a = \pi^n(t_1 + t_2) = \pi^n(t_1) \ast \pi^n(t_2) = t_1.a \ast t_2.a$ for any $t_1, t_2 \in T$. For any $x \in X$ there is a net $t_\lambda.a \to x$. Hence for any $t \in T$, $t.a \ast x = t.a \ast (\lim t_\lambda.a) = \lim (t.a \ast t_\lambda.a) = \lim (t + t_\lambda).a = \lim t.(t_\lambda.a) = t.(\lim t_\lambda.a) = t.x$. So $f(t.x) = f(t.a \ast x) = f(t.a)f(x) = \chi(t)f(x)$ for any $(t, x) \in T \times X$. Now we use the criterion for total minimality ((i) $\iff$ (ii)).

(iii) The statement 8.5(iii) for $X$ locally almost periodic (hence proximally equicontinuous) and $T$ non-totally-disconnected LCA group with the connected component of identity acting nontrivially on $X$, was proved in ([3, p. 380]). We extend the class of acting groups and replace “locally almost periodic” by a weaker condition “proximally equicontinuous”. The part of the proof in which the statement (iii) is reduced to the statement (ii) follows [3]. The proofs of (ii) are different.

(iv) The statement 8.5(v) was proved in [14, 3.2] as an application of a criterion for weak mixing that was formulated and proved there. Although 8.5(v) implies 8.5(iv), we stated both of them since the proof of 8.5(v) reduces to the proof of 8.5(iv).

(v) A complete characterization of flows which satisfy 8.5(vii), with $T = \mathbb{R}$, in terms of their eigenvalues, is given in [5, Theorem 2].

Remark 8.7. Note that proximal compact minimal Abelian flows are trivial [22, IV(2.18)], in particular totally minimal.

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References