On the Eigenvalue Problem for Toeplitz Band Matrices

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ABSTRACT

A formula is given for the characteristic polynomial of an nth order Toeplitz band matrix, with bandwidth $k < n$, in terms of the zeros of a kth degree polynomial with coefficients independent of n. The complexity of the formula depends on the bandwidth k, and not on the order n. Also given is a formula for eigenvectors, in terms of the same zeros and k coefficients which can be obtained by solving a $k \times k$ homogeneous system.

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1. INTRODUCTION

We consider the eigenvalue problem for Toeplitz band matrices, i.e., matrices of the form

$$T_n = (c_{j-i})_{i,j=0}^{n-1},$$

where there are integers r and s such that

$$r, s \geqslant 0, \qquad r + s = k < n, \tag{1}$$

and

$$c_\nu = 0 \quad \text{if} \quad \nu > r \text{ or } \nu < -s. \tag{2}$$

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Thus, the elements of T_n are constant along any stripe parallel to the main diagonal, and those more than r positions to the right or s positions to the left of the main diagonal are zero.

Because of their simplicity and importance in many applications, much has been written about Toeplitz band matrices, and efficient methods have been given for inverting them (e.g., [1], [10], [11], [12], [13], [15], [19]) and for solving linear systems with such matrices (e.g., [2], [4], [5], [9], [14], [17], [18]). However, very little has been published on methods for finding their eigenvalues and eigenvectors which take advantage of their peculiar simplicity. The tridiagonal case is well understood [6, 9], but, to the author's knowledge, only the recent paper of Bini and Capovani [3] on the eigenvalues of Hermitian Toeplitz band matrices contains any sort of general results along these lines. Grunbaum [7, 8] has considered the eigenvalue problem for Toeplitz matrices which are not necessarily banded.

Since the eigenvalue problem for triangular Toeplitz matrices is essentially trivial, we incur no significant loss of generality by assuming that

$$rsc_r c_{-s} \neq 0. \tag{3}$$

It is to be understood that (1), (2), and (3) apply throughout the paper. We take the underlying field to be the complex numbers.

Our main result reduces the evaluation of the characteristic polynomial of T_n to finding the zeros of the polynomial

$$P(z;\lambda) = \sum_{\mu=-s}^{r} c_\mu z^{\mu+s} - \lambda z^s, \tag{4}$$

and evaluating a kth order determinant whose entries are powers of these zeros, or simple related functions of the zeros if the equation

$$P(z;\lambda) = 0 \tag{5}$$

has repeated roots. We also give an explicit formula for the value of this determinant which is valid when (5) has k distinct roots. Thus, the complexity of this representation depends only on k, the bandwidth of T_n, and is independent of its order. Moreover, we give an explicit formula for the eigenvectors of T_n corresponding to a given eigenvalue, which contains k coefficients that can be obtained by solving a kth order homogeneous system with complexity independent of n.

2. THE MAIN RESULTS

We observe that (3) implies that $z = 0$ is not a root of (5). For notational reasons, we introduce the k-dimensional column vector function

$$C_n(z) = \operatorname{col}[1, z, \dots, z^{s-1}, z^{n+s}, z^{n+s+1}, \dots, z^{n+k-1}],$$

and denote its lth derivative by $C_n^{(l)}(z)$.

Definition 1. For a fixed λ, let $z_1, \dots, z_q$ be the distinct roots of (5), with multiplicities $m_1, \dots, m_q$; thus,

$$q \leqslant k; \qquad m_j \geqslant 1 \quad (1 \leqslant j \leqslant q); \qquad m_1 + \cdots + m_q = k.$$

Define the $k \times k$ matrix A_n as follows: the first m_1 columns of A_n are $C_n^{(l)}(z_1)$ $(0 \leqslant l \leqslant m_1 - 1)$, the next m_2 columns are $C_n^{(l)}(z_2)$ $(0 \leqslant l \leqslant m_2 - 1)$, and so forth. In particular, if $q = k$, so that (5) has distinct roots $z_1, \dots, z_k$, then

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ z_1^{s-1} & z_2^{s-1} & \cdots & z_k^{s-1} \\ z_1^{n+s} & z_2^{n+s} & \cdots & z_k^{n+s} \\ \vdots & \vdots & & \vdots \\ z_1^{n+k-1} & z_2^{n+k-1} & \cdots & z_k^{n+k-1} \end{bmatrix}. \tag{6}$$

There is an obvious ambiguity in this definition, since the distinct zeros of (5) can be numbered arbitrarily. However, renumbering the zeros simply permutes the columns of A_n, which does not affect our results.

In the following, $x^{(\nu)}$ is the factorial polynomial:

$$x^{(0)} = 1, \qquad x^{(\nu)} = x(x-1)\cdots(x-\nu+1).$$

Theorem 1. *Let λ and $z_1, \dots, z_q$ be as in Definition 1. Then λ is an eigenvalue of T_n if and only if* $\det A_n = 0$, *in which case the corresponding* *eigenvectors*

$$U=\operatorname{col}[u_0,u_1,\dots,u_{n-1}] \tag{7}$$

are given by

$$u_i=\sum_{j=1}^{q}\sum_{\nu=0}^{m_j-1}\alpha_{\nu j}(s+i)^{(\nu)}z_j^{s+i-\nu},\qquad 0\leqslant i\leqslant n-1,$$

where the k-vector

$$X=\operatorname{col}\left[\alpha_{01},\dots,\alpha_{m_1-1,1},\dots,\alpha_{0q},\dots,\alpha_{m_q-1,q}\right] \tag{8}$$

is a nontrivial solution of the $k\times k$ system

$$A_nX=0. \tag{9}$$

Proof. With U as in (7), the equation $T_nU=\lambda U$ is equivalent to the following three equations:

$$\sum_{\mu=-i}^{r}c_\mu u_{\mu+i}=\lambda u_i,\qquad 0\leqslant i\leqslant s-1,$$

$$\sum_{\mu=-s}^{r}c_\mu u_{\mu+i}=\lambda u_i,\qquad s\leqslant i\leqslant n-r-1,$$

$$\sum_{\mu=-s}^{n-i-1}c_\mu u_{\mu+i}=\lambda u_i,\qquad n-r\leqslant i\leqslant n-1,$$

which are together equivalent to

$$\sum_{\mu=-s}^{r}c_\mu u_{\mu+i}=\lambda u_i,\qquad 0\leqslant i\leqslant n-1, \tag{10a}$$

$$u_i=0\qquad\text{if}\quad -s\leqslant i\leqslant -1\ \text{ or }\ n\leqslant i\leqslant n+r-1. \tag{10b}$$

Therefore, λ is an eigenvalue of T_n if and only if there is a nonzero $(n+k)$-vector

$$\hat{U} = \operatorname{col}[0,\ldots,0,u_0,\ldots,u_{n-1},0,\ldots,0],$$

with s zeros at the top and r zeros at the bottom, which is a solution of the finite difference boundary value problem (10). Our hypotheses imply that for every integer i, z_j is a zero of $z^iP(z;\lambda)$ with multiplicity m_j; hence, from (4),

$$0 = \left[z^iP(z;\lambda)\right]^{(\nu)}\Big|_{z_j} = \sum_{\mu=-s}^{r} c_\mu(\mu+s+i)^{(\nu)}z_j^{\mu+s-i-\nu} - \lambda(s+i)^{(\nu)}z^{s+i-\nu},$$

$0 \leqslant \nu \leqslant m_j - 1, \quad 1 \leqslant j \leqslant q.$

This means that the k sequences

$$\left\{ (s+i)^{(\nu)}z_j^{s+i-\nu} \Big| -s \leqslant i \leqslant n+r-1 \right\}, \qquad 0 \leqslant \nu \leqslant m_j - 1, \quad 1 \leqslant j \leqslant q,$$

all satisfy (10a). Since they are linearly independent, the general solution of (10a) is of the form

$$u_i = \sum_{j=1}^{q} \sum_{\nu=0}^{m_j-1} \alpha_{\nu j}(s+i)^{(\nu)}z_j^{s+i-\nu}, \qquad -s \leqslant i \leqslant n+r-1. \tag{11}$$

On recalling the definition of A_n, we see that (11) satisfies the boundary conditions (10b) if and only if X in (8) satisfies (9). ∎

As we will see in Section 3, (5) can have repeated roots for at most k values of λ. Therefore, it is worthwhile to state the following corollary of Theorem 1.

COROLLARY 1. *Suppose* (5) *has distinct roots* $z_1,\ldots,z_k$. *Then* λ *is an eigenvalue of* T_n *if and only if there are constants* $\alpha_1,\ldots,\alpha_k$, *not all zero, which satisfy the kth-order system*

$$\sum_{j=1}^{k} \alpha_j z_j^{\nu} = 0, \qquad 0 \leqslant \nu \leqslant s-1,$$

$$\sum_{j=1}^{k} \alpha_j z_j^{n+s+\nu} = 0, \qquad 0 \leqslant \nu \leqslant r-1.$$

In this case the vector (7), *with*

$$u_i = \sum_{j=1}^{k} \alpha_j z_j^{s+i}, \qquad 0 \leqslant i \leqslant n-1,$$

is an associated eigenvector.

Theorem 1 clearly implies that there is a connection between $\det A_n$ and the characteristic polynomial

$$p_n(\lambda) = \det[\lambda I_n - T_n].$$

The next theorem states this connection precisely.

THEOREM 2. *Let* λ, $z_1, \dots, z_q$, *and* A_n *be as in Definition* 1. *Then*

$$p_n(\lambda) = (-1)^{(r-1)n} c_r^n \frac{\det A_n}{\det A_0}. \tag{12}$$

Moreover, there are at most k values of λ *for which* $q < k$.

In Section 5 we give a more explicit form for (12) for the case where $q = k$. Because of its length, we leave the proof of Theorem 2 to Section 3. We now consider the possible dimensions of the eigenspaces of T_n. To this end, let $E_n(\lambda)$ be the eigenspace of T_n corresponding to an eigenvalue λ, and

$$d_n(\lambda) = \dim E_n(\lambda). \tag{13}$$

From the proof of Theorem 1,

$$d_n(\lambda) = \text{nullity of } A_n. \tag{14}$$

LEMMA 1. *Let* λ *and* $z_1, \dots, z_q$ *be as in Definition* 1. *Then* λ *is an eigenvalue of* T_n *if and only if there are polynomials*

$$f(z) = a_0 + \cdots + a_{s-1} z^{s-1} \tag{15}$$

and

$$g(z) = b_0 + \cdots + b_{r-1} z^{r-1} \tag{16}$$

such that the polynomial

$$h(z) = f(z) + z^{n+s}g(z) \tag{17}$$

is not identically zero and has zeros at $z_1, \dots, z_q$ *with multiplicities at least* $m_1, \dots, m_q$; *i.e.,*

$$h^{(l)}(z_j) = 0, \qquad 0 \leqslant l \leqslant m_j - 1, \quad 1 \leqslant j \leqslant q. \tag{18}$$

Moreover, if $H_n(\lambda)$ *is the vector space of polynomials which satisfy* (15), (16), (17), *and* (18), *then*

$$\dim H_n(\lambda) = d_n(\lambda). \tag{19}$$

Proof. It is easy to verify that a polynomial h of the form (15), (16), and (17) satisfies (18) if and only if the vector

$$Y = \operatorname{col}[a_0, \dots, a_{s-1}, b_0, \dots, b_{r-1}]$$

satisfies the system $A_n^t Y = 0$ (^{t} = transpose). Therefore,

$$\dim H_n(\lambda) = (\text{nullity of } A_n^t) = (\text{nullity of } A_n),$$

so (13) and (14) imply (19). ■

THEOREM 3. *If* λ *is an eigenvalue of* T_n, *then* $d_n(\lambda) \leqslant \min(r, s)$.

Proof. From Lemma 1, it suffices to show that $\dim H_n(\lambda) \leqslant \min(r, s)$. Suppose

$$h_j(z) = f_j(z) + z^{n+s}g_j(z), \qquad 1 \leqslant j \leqslant s+1,$$

are any $s+1$ polynomials in $H_n(\lambda)$. Since $\deg f_j \leqslant s-1$, we can choose constants $\beta_1, \dots, \beta_{s+1}$, not all zero, such that

$$\beta_1 f_1 + \cdots + \beta_{s+1} f_{s+1} = 0. \tag{20}$$

The polynomial

$$h = \beta_1 h_1 + \cdots + \beta_{s+1} h_{s+1}$$

is also in $H_n(\lambda)$ and, from (20), is of the form $h(z) = z^{n+s}g(z)$, where

$$g = \beta_1 g_1 + \cdots + \beta_{s+1} g_{s+1}.$$

Since $z_j \neq 0$ $(1 \leqslant j \leqslant q)$, (18) implies that $g^{(l)}(z_j) = 0$, $0 \leqslant l \leqslant m_j - 1$, $1 \leqslant j \leqslant q$; i.e., g has at least k zeros, counting multiplicities. Since $\deg g \leqslant s - 1 < k$, this means that $g = 0$. Hence, $h = 0$, and so

$$\beta_1 h_1 + \cdots + \beta_{s+1} h_{s+1} = 0.$$

Therefore, any $s+1$ polynomials in $H_n(\lambda)$ are linearly dependent, and so $\dim H_n(\lambda) \leqslant s$. A similar argument shows that $\dim H_n(\lambda) \leqslant r$. ■

THEOREM 4. *Suppose λ is an eigenvalue of T_n and $d_n(\lambda) = m \geqslant 2$. Then λ is also an eigenvalue of T_{n-1} and T_{n+1},*

$$d_{n-1}(\lambda) \geqslant m - 1, \tag{21}$$

and

$$d_{n+1}(\lambda) \geqslant m - 1. \tag{22}$$

Proof. First notice that if $h \in H_n(\lambda)$ and $f(0) = 0$ [see (17)], then the polynomial

$$\tilde{h}(z) = \frac{f(z)}{z} + z^{n+s-1}g(z)$$

is in $H_{n-1}(\lambda)$. Now let the polynomials

$$h_i(z) = f_i(z) + z^{n+s}g_i(z), \qquad 1 \leqslant i \leqslant m,$$

form a basis for $H_n(\lambda)$. If $f_i(0) = 0$ $(1 \leqslant i \leqslant m)$, then the argument just given implies that $d_{n-1}(\lambda) \geqslant m$. If $f_l(0) \neq 0$ for some l, then the $m-1$ polynomials

$$\tilde{h}_i(z) = f_l(0)h_i(z) - f_i(0)h_l(z), \qquad 1 \leqslant i \leqslant m, \quad i \neq l,$$

are linearly independent members of $H_n(\lambda)$ which vanish at $z=0$. This and our earlier observation imply (21).

Now observe that if $h \in H_n(\lambda)$ and $g(0)=0$, then $h \in H_{n+1}(\lambda)$, since h can be rewritten as

$$h(z)=f(z)+\left(\frac{g(z)}{z}\right)z^{n+s+1}.$$

Because of this, an argument similar to the one just given proves (22). ■

3. PROOF OF THEOREM 2

We prove Theorem 2 by means of a series of lemmas.

LEMMA 2. *There are at most k values of λ for which* (5) *has fewer than k distinct roots.*

Proof. If (5) has a repeated root, then the resultant of $P(z;\lambda)$ and $P_z(z;\lambda)$ must vanish. From a formula given in [20, p. 84], this resultant can be written explicitly as a $(2k-1)\times(2k-1)$ determinant with ith row elements (from left to right) as follows:

(a) For $1 \leqslant i \leqslant k-1$, there are $i-1$ zeros, followed by $c_r,\dots,c_1$, $c_0-\lambda$, $c_{-1},\dots,c_{-s}$, and $k-i-1$ zeros.

(b) For $k \leqslant i \leqslant 2k-1$, there are $i-k$ zeros, then the coefficients $kc_r,\dots,(s+1)c_1$, $s(c_0-\lambda),(s-1)c_{-1},\dots,c_{-s+1}$ of $P_z(z;\lambda)$, and then $2k-i-1$ zeros.

Since $c_0-\lambda$ occurs in exactly k columns of this determinant, the resultant is a polynomial of degree $\leqslant k$. ■

Henceforth we will say that a value of λ for which (5) has repeated roots is a *critical point* of $P(z;\lambda)$. All other values of λ will be called *ordinary points* of $P(z;\lambda)$.

DEFINITION 2. For a fixed λ, let $z_1,\dots,z_q$ and $m_1,\dots,m_q$ be as in Definition 1. If $n_1,\dots,n_k$ are nonnegative integers, let

$$\Gamma(z;n_1,\dots,n_k)=\operatorname{col}[z^{n_1},z^{n_2},\dots,z^{n_k}].$$

Let $D(z_1,\dots,z_q;n_1,\dots,n_k)$ be the following $k\times k$ determinant: its first m_1 columns are $\Gamma^{(l)}(z_1;n_1,\dots,n_k)$ $(0\leqslant l\leqslant m_1-1)$; its next m_2 columns are $\Gamma^{(l)}(z_2;n_1,\dots,n_k)$ $(0\leqslant l\leqslant m_2-1)$; and so forth. Now let

$$q(\lambda;n_1,\dots,n_k)=\frac{D(z_1,\dots,z_q;n_1,\dots,n_k)}{D(z_1,\dots,z_q;0,1,\dots,k-1)}. \tag{23}$$

Observe that this definition yields

$$q(\lambda;n_1,\dots n_k)=\frac{\det\left[z_j^{n_i}\right]_{i,j=1}^{k}}{\det\left[z_j^{i-1}\right]_{i,j=1}^{k}}$$

for ordinary values of λ, and for all λ,

$$q(\lambda;0,\dots,s-1,n+s,\dots,n+k-1)=\frac{\det A_n}{\det A_0}, \tag{24}$$

where A_n is the matrix introduced in Definition 1.

The denominator on the right of (23) cannot vanish, since it can be shown that

$$D(z_1,\dots,z_q;0,1,\dots,k-1)=K\prod_{1\leqslant i<j\leqslant q}(z_j-z_i)^{r_{ij}}\;(K=\text{constant}),$$

where the r_{ij}'s are positive integers; if $q=k$, then $r_{ij}=1$ for all i, j since this is the Vandermonde determinant.

We deliberately avoid denoting the root z_i as $z_i(\lambda)$, since justifying this would require an irrelevant appeal to the theory of multiple valued algebraic analytic functions. Since the columns of both determinants in (23) are permuted in the same way if $z_1,\dots,z_q$ are renumbered, $q(\lambda;n_1,\dots,n_k)$ is well defined for all λ.

In the following we adopt the convention that the zero polynomial has degree $-\infty$.

LEMMA 3. *Suppose $n_1,\dots,n_k$ are nonnegative integers, and $m=\max\{n_1,\dots,n_k\}$. Then $q(\lambda;n_1,\dots,n_k)$ is a polynomial of degree $\leqslant m-k+1$.*

Proof. We use induction on m. Obviously $q(\lambda;n_1,\dots,n_k)$ is constant (0, 1, or -1) if $0\leqslant m\leqslant k-1$. Now suppose the conclusion is valid for some

$m \geqslant k-1$, and let $\max\{n_1,\ldots,n_k\} = m+1$. Since permuting $n_1,\ldots,n_k$ merely changes the sign of $q(\lambda; n_1,\ldots,n_k)$, we may assume that $n_k = m+1$. We may also assume that

$$n_j \leqslant m, \qquad 1 \leqslant j \leqslant k-1, \tag{25}$$

since $q(\lambda; n_1,\ldots,n_k) \equiv 0$ (two identical rows) if this is not so.

Now we claim that

$$\sum_{\mu=-s}^{r} c_\mu D(z_1,\ldots,z_q; n_1,\ldots,n_{k-1}, m+\mu-r+1)$$
$$-\lambda D(z_1,\ldots,z_q; n_1,\ldots,n_{k-1}, m-r+1) = 0. \tag{26}$$

To see this, first notice that the determinants in (26) are identical in their first $k-1$ rows; hence, if $R_{m+\mu-r+1}$ denotes the kth row of $D(z_1,\ldots,z_q; n_1,\ldots,n_{k-1}, m+\mu-r+1)$, then the left side of (26) equals a determinant with last row

$$[\gamma_1,\ldots,\gamma_k] = \sum_{\mu=-s}^{r} c_\mu R_{m+\mu-r+1} - \lambda R_{m-r+1}.$$

Here a typical γ_ν is of the form

$$\gamma_\nu = \sum_{\mu=-s}^{r} c_\mu (m+\mu-r+1)^{(l)} z_j^{m+\mu-r+1-l} - \lambda (m-r+1)^{(l)} z_j^{m-r+1-l}$$

for some j in $\{1,\ldots,q\}$ and l in $\{0,\ldots,m_j-1\}$. Since z_j is a zero of $z^{m-k+1}P(z;\lambda)$ with multiplicity m_j, this implies that $\gamma_\nu = 0$ $(1 \leqslant \nu \leqslant k)$, which in turn implies (26).

From (23) and (26),

$$q(\lambda; n_1,\ldots,n_{k-1}, m+1) = c_r^{-1}\Bigg[\lambda q(\lambda; n_1,\ldots,n_{k-1}, m-r+1)$$
$$-\sum_{\mu=-s}^{r-1} c_\mu q(\lambda; n_1,\ldots n_{k-1}, m+\mu-r+1)\Bigg]. \tag{27}$$

Because of (25),

$$\max\{n_1,\dots,n_{k-1},m+\mu-r+1\}\leqslant m,\qquad -s\leqslant\mu\leqslant r-1,$$

so our induction assumption implies that the functions

$$q(\lambda;n_1,\dots,n_{k-1},m+\mu-r+1),\qquad -s\leqslant\mu\leqslant r-1,$$

are polynomials of degree $\leqslant m-k+1$. Hence (27) implies that $q(\lambda;n_1,\dots,n_{k-1},m+1)$ is a polynomial of degree $\leqslant m-k+2$. This completes the induction. ∎

LEMMA 4. *The polynomial*

$$Q_m(\lambda)=(-1)^{(r-1)m}c_r^m q(\lambda;0,\dots,s-1,m+s,\dots,m+k-1)\qquad(28)$$

is monic and of exact degree m.

Proof. We use induction on m. From (23) (with $n_i=i-1$, $1\leqslant i\leqslant k$) and (28), $Q_0=1$. Now suppose the conclusion is valid for a given $m\geqslant 0$. From an argument like that which led to (27),

$$\begin{aligned}&q(\lambda;0,\dots,s-1,m+s+1,\dots,m+k-1,m+k)\\&\quad=c_r^{-1}\lambda q(\lambda;0,\dots,s-1,m+s+1,\dots,m+k-1,m+s)+\cdots\\&\quad=(-1)^{r-1}c_r^{-1}\lambda q(\lambda;0,\dots,s-1,m+s,m+s+1,\dots,m+k-1)+\cdots\end{aligned}\qquad(29)$$

(with obvious modifications if $r=1$), where "$\cdots$" stands for a polynomial of degree $\leqslant m$, because of Lemma 3. Multiplying (29) through by $(-1)^{(r-1)(m+1)}c_r^{m+1}$ and recalling (28) shows that

$$Q_{m+1}(\lambda)=\lambda Q_m(\lambda)+\cdots,$$

and so our induction assumption implies that Q_{m+1} is monic and of exact degree $m+1$. ∎

For $n\geqslant k$, Theorem 1, (24), and Lemma 4 imply that p_n and Q_n are both monic and of degree n, and that they have the same zeros (the eigenvalues of T_n). Therefore, certainly $Q_n=p_n$ if T_n has n distinct eigenvalues. We must still rule out the possibility that T_n has only m $(<n)$ distinct eigenvalues

$\lambda_1, \ldots, \lambda_m$, and

$$p_n(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_m)^{r_m}$$

while

$$Q_n(\lambda) = (\lambda - \lambda_1)^{s_1} \cdots (\lambda - \lambda_m)^{s_m},$$

with $r_i \neq s_i$ for some i. A continuity argument excludes this possibility. For every $\epsilon > 0$ there are constants $\hat{c}_{-s}, \ldots, \hat{c}_r$ such that $\hat{c}_{-s}\hat{c}_r \neq 0$,

$$\sum_{j=-s}^{r} |\hat{c}_j - c_j|^2 < \epsilon^2,$$

and the matrix

$$\hat{T}_n = (\hat{c}_{j-i})_{i,j=0}^{n-1}$$

has n distinct eigenvalues. Let $\hat{Q}_n$ be the polynomial defined in Lemma 4 in connection with $\hat{c}_{-s}, \ldots, \hat{c}_r$, and let $\hat{p}_n$ be the characteristic polynomial of $\hat{T}_n$. By the proof just given, $\hat{Q}_n = \hat{p}_n$. But since the coefficients of $\hat{Q}_n$ and $\hat{p}_n$ are continuous functions of $\hat{c}_{-s}, \ldots, \hat{c}_r$ we can now let $\epsilon \to 0$ and infer that $Q_n = p_n$ in this case also.

4. AN EXAMPLE: THE TRIDIAGONAL CASE

Now suppose that $r = s = 1$, so that T_n is tridiagonal. Then (4) and (5) become

$$P(z; \lambda) = c_{-1} + (c_0 - \lambda)z + c_1 z^2, \tag{30}$$

which has a repeated root if and only if $\lambda = c_0 \pm 2\sqrt{c_1 c_{-1}}$; however, we will see that neither these numbers is an eigenvalue of T_n for any $n > 2$.

If λ is an ordinary point of (30), (6) becomes

$$A_n = \begin{bmatrix} 1 & 1 \\ z_1^{n+1} & z_2^{n+1} \end{bmatrix}.$$

From Theorem 2, λ is an eigenvalue of T_n if and only if $\det A_n = 0$, which is equivalent to $(z_2/z_1)^{n+1} = 1$. Since $z_1 \neq z_2$, we can rewrite this in the convenient equivalent form

$$z_1 = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right), \qquad z_2 = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right), \tag{31}$$

where $q = 1,\dots,n$ and γ_q is to be determined. Let λ_q be the eigenvalue for which (30) (with $\lambda = \lambda_q$) has roots as in (31). Then

$$\begin{aligned} c_1 z^2 + (c_0 - \lambda_q)z + c_{-1} &= c_1(z - z_1)(z - z_2) \\ &= c_1\left[z^2 - 2\gamma_q \cos\left(\frac{q\pi}{n+1}\right) + \gamma_q^2\right]. \end{aligned}$$

Matching the coefficients of z and z^2 shows that

$$\gamma_q = \sqrt{c_{-1}/c_1} \qquad \text{(independent of } q\text{)}, \tag{32}$$

and

$$\lambda_q = c_0 + 2\sqrt{c_1 c_{-1}}\cos\left(\frac{q\pi}{n+1}\right), \qquad 1 \leqslant q \leqslant n.$$

From (31), (32), and Corollary 1, it is straightforward to show that the vector $U_q = [u_{0q},\dots,u_{n-1,q}]$ given by

$$u_{mq} = \left(\frac{c_{-1}}{c_1}\right)^{m/2} \sin\left(\frac{q(m+1)\pi}{n+1}\right), \qquad 0 \leqslant m \leqslant n-1,$$

is an eigenvector corresponding to λ_q.

These results have been obtained by other authors (e.g., [6], [9]).

5. A MORE EXPLICIT FORMULA FOR $p_n(\lambda)$

We will now write (12) in a form more convenient for computational purposes, for the case where λ is an ordinary point of (5).

DEFINITION 3. *Let $\mathscr{U}$ be the set of all r-tuples $u=(\mu_1,\ldots,\mu_r)$, where $\mu_1,\ldots,\mu_r$ are integers such that $1\leqslant\mu_1<\mu_2<\cdots<\mu_r<k$. If $z_1,\ldots,z_k$ are arbitrary complex numbers, let $\Pi_u(z_1,\ldots,z_k)$ denote the product of all factors (z_j-z_i), where $1\leqslant i<j\leqslant k$ and exactly one of the integers i, j is among $\mu_1,\ldots,\mu_r$. Let $\sum_{\mathscr{U}}$ denote the sum over all r-tuples u in $\mathscr{U}$.*

THEOREM 5. *Suppose λ is an ordinary point of* (5) *and $z_1,\ldots,z_k$ are its zeros. Then*

$$p_n(\lambda)=(-1)^{\theta_n}c_r^n\sum_{\mathscr{U}}(-1)^{\mu_1+\cdots+\mu_r}\frac{(z_{\mu_1}\cdots z_{\mu_r})^{n+s}}{\Pi_u(z_1,\ldots,z_k)},\tag{33}$$

where

$$\theta_n=(r-1)n+\frac{r(2s+r+1)}{2}.\tag{34}$$

Proof. If $u=(\mu_1,\ldots,\mu_r)$ is a given element of $\mathscr{U}$, let $(\nu_1,\ldots,\nu_s)$ be the s-tuple of integers in $\{1,\ldots,k\}$ that are not among $\mu_1,\ldots,\mu_r$, with $\nu_1<\nu_2<\cdots<\nu_s$. Expanding the determinant of (6) by Laplace's development [16, p. 123] with respect to the last r rows and removing common factors yields

$$\det A_n=\sum_{\mathscr{U}}(-1)^{\sigma(u)}(z_{\mu_1}\cdots z_{\mu_r})^{n+s}V(z_{\mu_1},\ldots,z_{\mu_r})V(z_{\nu_1},\ldots,z_{\nu_s}),\tag{35}$$

where

$$\sigma(u)=u_1+\cdots+\mu_r+\frac{r(2s+r+1)}{2}\tag{36}$$

and the V's are Vandermonde determinants. Since $\det A_0=V(z_1,\ldots,z_k)$ and

$$\frac{V(z_{\mu_1},\ldots,z_{\mu_r})V(z_{\nu_1},\ldots,z_{\nu_s})}{V(z_1,\ldots,z_k)}=(\Pi_u(z_1,\ldots,z_k))^{-1},$$

(12), (35), and (36) imply (33) and (34). ■

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