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On holomorphic solutions for nonlinear singular fractional differential equations

Rabha W. Ibrahim

School of Mathematical Sciences, Faculty of Sciences and Technology, UKM, Malaysia

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ABSTRACT

The paper concerns with a nonlinear singular fractional differential equation:

$$t^{\alpha}\frac{\partial^{\alpha}u(t,z)}{\partial t^{\alpha}} = F\left(t, z, u, \frac{\partial u}{\partial z}\right), \quad 0 < \alpha < 1,$$

where $t \in J := [0, 1]$ and $z \in U := \{z \in \mathbb{C} : |z| < 1\}$. The existence and the uniqueness of holomorphic solution are established.

1. Introduction

Fractional calculus is a rapidly growing subject of interest for physicists and mathematicians. The reason for this is that problems may be discussed in a much more stringent and elegant way than using traditional methods. Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations [1–7].

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann–Liouville operators, Erdlyi–Kober operators, Weyl–Riesz operators, Caputo operators and Grünwald–Letnikov operators, have appeared during the past three decades with its applications in other fields [8–15]. Moreover, the existence and the uniqueness of holomorphic solutions for nonlinear fractional differential equations such as Cauchy problems and diffusion problems in complex domain are established and posed [16–20].

The present paper deals with a nonlinear singular fractional differential equation, in sense of the Riemann–Liouville operators, in the analytic category. The Riemann–Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann–Liouville operators. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms (see [21–23]).

Definition 1.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^{\alpha}f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\mathrm{d}\tau.$$

E-mail address: rabhaibrahim@yahoo.com.

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When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product,

$$\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0$$

and $\phi_\alpha(t) = 0,\; t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 1.2. The fractional (arbitrary) order derivative of the function f of order $0 < \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t} I_a^{1-\alpha} f(t).$$

Remark 1.1. From Definitions 1.1 and 1.2, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1;\; 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1;\; \alpha > 0.$$

Sufficient conditions to have a unique holomorphic solution for the equation

$$t^\alpha \frac{\partial^\alpha u(t,z)}{\partial t^\alpha} = F\left(t, z, u, \frac{\partial u}{\partial z}\right), \tag{1}$$

subject to the initial condition $u(0,0) = 0$, where $t \in J := [0,1]$, $z \in U$, $u(t,z)$ is an unknown function and $F(t,z,u,v)$ is a function with respect to the variables $(t,z,u,v) \in J \times U \times \mathbb{C}^2$ are given. The result is applied to obtain solution for well known problems.

We need the following assumptions and lemma which will be useful for the proof of the main result.

(H1) $F(t,z,u,v)$ is a holomorphic function defined in a neighborhood of the origin $(0,0,0,0) \in J \times U \times \mathbb{C}^2$.

(H2) $F(0,z,0,0) \equiv 0$ near $z = 0$.

Thus the function $F(t,z,u,v)$ may be expressed in the form:

$$F(t,z,u,v) = A(z)t + B(z)u + C(z)v + R_2(t,z,u,v), \tag{2}$$

where

$$A(z) := \frac{\partial F}{\partial t}(0,z,0,0), \qquad B(z) := \frac{\partial F}{\partial u}(0,z,0,0), \qquad C(z) := \frac{\partial F}{\partial v}(0,z,0,0),$$

and the degree of $R_2(t,z,u,v)$ with respect to (t,z,u,v) is greater than or equal to 2.

(H3) $C(z) := zc(z)$, $c(0) \neq 0$.

Lemma 1.1 ([24]). *Let $R > 0$ and $f(x)$ be a holomorphic function on $D_R = \{x \in \mathbb{C} : |x| < R\}$. If for any $r > 0$, $0 < r < R$, $f(x)$ satisfies*

$$\max_{|x|\le r} |f(x)| \le \frac{\rho}{(R-r)^\mu}$$

for some $\rho > 0$ and $\mu \ge 0$ then we have

$$\max_{|x|\le r}\left|\frac{\partial f(x)}{\partial x}\right| \le \frac{(\mu+1)e\rho}{(R-r)^{\mu+1}}.$$

2. Existence of unique solution

We have the following result.

Theorem 2.1. *Let the assumptions* (H1)–(H3) *hold. If*

$$\beta_k(z) := \frac{B(z) - \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}}{c(z)}, \quad k \in \mathbb{N}$$

satisfies that $0 < \|\beta_k\|_r < \infty$ and $\Re(\beta_k(0)) > 0$ for all $k \in \mathbb{N}$ and $0 < \alpha < 1$, then the Eq. (1) has a unique holomorphic solution $u(t,z)$ near $(0,0) \in J \times U$ with $u(0,0) = 0$.

Proof. We realize that Eq. (1) has a formal solution

$$u(t,z)=\sum_{k=1}^{\infty}u_k(z)t^k,\quad (t\in J). \tag{3}$$

Then introduce the formal series (3) into the Eq. (1) and compare the coefficients of t^k in two sides of the equation yields

$$\begin{aligned}
&u_1(z)\frac{\Gamma(2)}{\Gamma(2-\alpha)}=A(z)+B(z)u_1(z)+C(z)\frac{\partial u_1(z)}{\partial z},\\
&u_2(z)\frac{\Gamma(3)}{\Gamma(3-\alpha)}=B(z)u_2(z)+C(z)\frac{\partial u_2(z)}{\partial z}+\phi_1\left(u_1,\frac{\partial u_1(z)}{\partial z}\right),\\
&u_3(z)\frac{\Gamma(4)}{\Gamma(4-\alpha)}=B(z)u_3(z)+C(z)\frac{\partial u_3(z)}{\partial z}+\phi_2\left(u_1,u_2,\frac{\partial u_1(z)}{\partial z},\frac{\partial u_2(z)}{\partial z}\right),\\
&\vdots
\end{aligned} \tag{4}$$

Thus we obtain the following formula

$$C(z)\frac{\partial u_k(z)}{\partial z}+\left[B(z)-\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\right]u_k(z)=-\phi_{k-1}\left(u_1,u_2,\ldots,u_{k-1},\frac{\partial u_1(z)}{\partial z},\frac{\partial u_2(z)}{\partial z},\ldots,\frac{\partial u_{k-1}(z)}{\partial z}\right) \tag{5}$$

where $C(z)=zc(z)$ and $\phi_0(z)=-A(z)$. Eq. (5) is equivalent to

$$\begin{aligned}
z\frac{\partial u_k(z)}{\partial z}+\left[\frac{B(z)-\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}}{c(z)}\right]u_k(z)&=\frac{-\phi_{k-1}\left(u_1,u_2,\ldots,u_{k-1},\frac{\partial u_1(z)}{\partial z},\frac{\partial u_2(z)}{\partial z},\ldots,\frac{\partial u_{k-1}(z)}{\partial z}\right)}{c(z)}\\
&:=\Phi\left(u_1,\ldots,u_{k-1},\frac{\partial u_1(z)}{\partial z},\ldots,\frac{\partial u_{k-1}(z)}{\partial z}\right).
\end{aligned} \tag{6}$$

Now since

$$\Re\left\{\frac{B(0)-\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}}{c(0)}\right\}>0$$

then the Eq. (6) has a unique holomorphic solution $u_k(z)$ near $z=0$. Moreover, $u_k(z)$ is bounded for all $k\in\mathbb{N}$ such that

$$\|u_k\|_r\le\frac{\|\Phi\|_r}{\|\beta_k\|_r}, \tag{7}$$

where $\|\Phi\|_r=\max_{|z|\le r}|\Phi(.)|$ and

$$\beta_k(z):=\frac{B(z)-\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}}{c(z)},\quad k\in\mathbb{N}.$$

To prove inequality (7); From Eq. (6) we have

$$\frac{d}{dz}\left[e^{\int_0^z\frac{\beta_k(s)}{s}ds}\times u_k(z)\right]=e^{\int_0^z\frac{\beta_k(s)}{s}ds}\times\frac{\Phi(z)}{z}.$$

Thus

$$\int_0^z\frac{d}{dy}\left[e^{\int_0^y\frac{\beta_k(s)}{s}ds}\times u_k(y)\right]dy=\int_0^z e^{\int_0^y\frac{\beta_k(s)}{s}ds}\times\frac{\Phi(y)}{y}dy$$

that is

$$e^{\int_0^z\frac{\beta_k(s)}{s}ds}\times u_k(z)-u_k(0)=\int_0^z e^{\int_0^y\frac{\beta_k(s)}{s}ds}\times\frac{\Phi(y)}{y}dy$$

which equivalents to

$$u_k(z)=e^{-\int_0^z\frac{\beta_k(s)}{s}ds}\times\int_0^z e^{\int_0^y\frac{\beta_k(s)}{s}ds}\times\frac{\Phi(y)}{y}dy.$$

Therefore,

$$\|u_k\|_r \le \max_{|z|\le r} \frac{\left|\mathrm{e}^{-\int_0^z \frac{\beta_k(s)}{s}\mathrm{d}s} \times \int_0^z \mathrm{e}^{\int_0^y \frac{\beta_k(s)}{s}\mathrm{d}s} \times \frac{\beta_k(y)}{y}\mathrm{d}y\right|}{|\beta_k(z)|} \times \|\Phi\|_r \\ \le \frac{\|\Phi\|_r}{\|\beta_k\|_r}.$$

Now we proceed to prove that the formal series solution (3) is convergent near $(0,0)\in(J,U)$. We expand the remainder term $R_2(t,z,u,v)$ of (2) into Taylor series with respect to (t,u,v), *i.e.*

$$R_2(t,z,u,v) = \sum_{m+n+p\ge 2} a_{m,n,p}(z)t^m u^n v^p$$

such that

(i) $\frac{a_{m,n,p}(z)}{c(z)}$ is holomorphic in U.

(ii) $|\frac{a_{m,n,p}(z)}{c(z)}| \le A_{m,n,p}, A_{m,n,p} > 0$ on U.

(iii) $\sum_{m+n+p\ge 2} A_{m,n,p}t^m V^{n+p}$ converges in (t,V) where $V>0$ satisfies $|u|\le V$ and $|v|\le V$.

From the Eq. (5), we observe that

$$\left[z\frac{\partial}{\partial z}+\beta_1\right]u_1(z) = \frac{-A(z)}{c(z)}, \\ \vdots \\ \left[z\frac{\partial}{\partial z}+\beta_k\right]u_k(z) = -\sum_{m+n+p\ge 2}\left[\sum_{m+k_1+\cdots+k_n+l_1+\cdots+l_p=k} \frac{a_{m,n,p}(z)}{c(z)} \times u_{k_1}\times\cdots\times u_{k_n}\times \frac{\partial u_{l_1}}{\partial z}\times\cdots\times\frac{\partial u_{l_p}}{\partial z}\right]. \tag{8}$$

Without loss of generality we may assume that there exists a constant $K>0$ such that

$$|u_1(z)|\le K, \quad \text{and} \quad \left|\frac{\partial u_1(z)}{\partial z}\right| \le K.$$

Denoting $C := \frac{1}{\|\beta_k\|_r}$ then we pose the following formula:

$$V(t) = Kt + \frac{C}{1-r}\sum_{m+n+p\ge 2}\frac{A_{m,n,p}}{(1-r)^{m+n+p-2}}t^m V^n V^p, \tag{9}$$

where r is a parameter with $0<r<1$. Since the Eq. (9) is an analytic functional equation in V then in view of the implicit function theorem, the Eq. (9) has a unique holomorphic solution $V(t)$ in a neighborhood of $t=0$ with $V(0)=0$. Expanding $V(t)$ into Taylor series in t we have

$$V(t) = \sum_{k\ge 1} V_k t^k \tag{10}$$

where

$$V_k = \frac{C}{1-r}\sum_{m+n+p\ge 2}\left[\sum_{m+k_1+\cdots+k_n+l_1+\cdots+l_p=k}\frac{A_{m,n,p}}{(1-r)^{m+n+p-2}} \times V_{k_1}\times\cdots\times V_{k_n}\times 0!(eV_{l_1})\times\cdots\times(p-1)!(eV_{l_p})\right] \\ := \frac{C_k}{(1-r)^{k-1}}, \quad k\in\mathbb{N} \\ > 0, \tag{11}$$

with $C_1 = K$.

Next our aim is to show that the series $\sum_{k\ge 1} V_k t^k$ is a majorant series for the formal series solution $\sum_{k\ge 1} u_k t^k$ near $z=0$. For this purpose we will show that

$$|u_k(z)| \le V_k \quad \text{on } U_r := \{z\in\mathbb{C} : |z|\le r,\ r<1\} \tag{12}$$

and

$$\left|\frac{\partial u_k(z)}{\partial z}\right| \le (k-1)!(eV_k) \quad \text{on } U_r. \tag{13}$$

Since $(1-r) < 1$ implies

$$\frac{1}{(1-r)^{m+n+p-2}} \ge 1, \quad r < 1$$

then we have

$$\begin{aligned}
|u_k(z)| &\le C \sum_{m+n+p\ge 2}\left[\sum_{m+k_1+\cdots+k_n+l_1+\cdots+l_p=k} A_{m,n,p} \times |u_{k_1}(z)| \times \cdots \times |u_{k_n}(z)| \times \left|\frac{\partial u_{l_1}(z)}{\partial z}\right| \times \cdots \times \left|\frac{\partial u_{l_p}(z)}{\partial z}\right|\right] \\
&\le C \sum_{m+n+p\ge 2}\left[\sum_{m+k_1+\cdots+k_n+l_1+\cdots+l_p=k} A_{m,n,p} \times V_{k_1} \times \cdots \times V_{k_n} \times 0!(eV_{l_1}) \times \cdots \times (p-1)!(eV_{l_p})\right] \\
&\le C \sum_{m+n+p\ge 2}\left[\sum_{m+k_1+\cdots+k_n+l_1+\cdots+l_p=k} \frac{A_{m,n,p}}{(1-r)^{m+n+p-2}} \times V_{k_1} \times \cdots \times V_{k_n} \times 0!(eV_{l_1}) \times \cdots \times (p-1)!(eV_{l_p})\right] \\
&\le \frac{C_k}{(1-r)^{k-2}} \le \frac{C_k}{(1-r)^{k-1}} = V_k.
\end{aligned}$$

Hence we obtain the inequality (12). Next by using Lemma 1.1, we pose that

$$\left|\frac{\partial u_k(z)}{\partial z}\right| \le (k-1)\frac{eC_k}{(1-r)^{k-1}} \le (k-1)!\frac{eC_k}{(1-r)^{k-1}} = (k-1)!(eV_k).$$

This completes the proof of Theorem 2.1. □

3. Briot–Bouquet equation

As a special case of Eq. (1) is so call the Briot–Bouquet equation. The fractional Briot–Bouquet equation has wide applications in geometric function theory (see [25–30]).

Let us recall the theory of nonlinear ordinary differential equations of the form

$$t\frac{\mathrm{d}u}{\mathrm{d}t} = f(t,u), \quad f(0,0) = 0,$$

which was first studied by Briot–Bouquet and its generalization to partial differential equations takes the form

$$t\frac{\partial u(t,x)}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right),$$

where F satisfies the assumptions (H1) and (H2) with $C(z) \equiv 0$ (see [31]).

In this section we establish the existence of unique holomorphic solution for fractional Briot–Bouquet equation

$$t^{\alpha}\frac{\partial^{\alpha} u(t,z)}{\partial t^{\alpha}} = F\left(t, z, u, \frac{\partial u}{\partial z}\right), \quad z \in U. \tag{14}$$

Thus F can be expanded into the following formula:

$$F(t,z,u,v) = A(z)t + B(z)u + R_2(t,z,u,v), \tag{15}$$

where

$$A(z) := \frac{\partial F}{\partial t}(0,z,0,0), \qquad B(z) := \frac{\partial F}{\partial u}(0,z,0,0)$$

and the degree of $R_2(t,z,u,v)$ with respect to (t,z,u,v) is greater than or equal to 2. Then we have the following result.

Theorem 3.1. *Let the assumptions* (H1), *and* (H2) *hold. If*

$$\Re(B(z)) \ne \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}, \quad \forall k \in \mathbb{N} \text{ and } z \in U,$$

then the Eq. (14) *has a unique holomorphic solution* $u(t,z)$ *near* $(0,0) \in J \times U$ *with* $u(0,0) = 0$.

Proof. Consider that Eq. (14) has a formal solution

$$u(t,z)=\sum_{k=1}^{\infty}u_k(z)t^k, \quad (t\in J). \tag{16}$$

Then Eq. (16) can decomposed into two equations

$$u_1(z)\left[\frac{\Gamma(2)}{\Gamma(2-\alpha)}-B(z)\right]=A(z), \qquad k=1$$

and

$$u_k(z)\left[\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}-B(z)\right]=\sum_{2\le i+j+p\le k}a_{ijp}(z)\left[\sum_{i+|m|+|n|=k}u_{m1}\dots u_{mj}\times\frac{\partial u_{n1}}{\partial z}\cdots\frac{\partial u_{np}}{\partial z}\right], \quad k\ge 2,$$

where $|m|=m_1+\cdots+m_j$ and $|n|=n_1+\cdots+n_p$. Therefor, if $\Re(B(z))\neq\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}$, $\forall k\in\mathbb{N}$ and $z\in U$ then (16) has a unique holomorphic solution $u(t,z)$ near $(0,0)\in(J,U)$. $\square$

4. Applications

In this section, we illustrate two examples of singularity in t and u to apply Theorem 2.1.

Example 4.1. Consider the following equation

$$\begin{cases}\dfrac{t^{0.5}}{1.128}\dfrac{\partial^{0.5}u(t,z)}{\partial t^{0.5}}+4z\dfrac{\partial u(t,z)}{\partial z}=(10+z)t+zt^2, \quad t\in J=[0,1]\\ u(0,z)=0, \quad \text{in a neighborhood of } z=0\end{cases} \tag{17}$$

where $u(t,z)$ is the unknown function. By putting

$$u(t,z)=\mu(z)t+v(t,z) \quad (v(t,z)=O(t^2))$$

as a formal solution. Computations give

$$t^{0.5}\frac{\partial^{0.5}u(t,z)}{\partial t^{0.5}}=1.128\mu(z)t+t^{0.5}v_\alpha(t,z)$$

and

$$z\frac{\partial u(t,z)}{\partial z}=z\mu'(z)+zv_z(t,z).$$

Therefore, $\mu(z)$ satisfies

$$\mu(z)+4z\mu'(z)-10-z=0.$$

Now by letting

$$\mu(z):=p+\phi(z),$$

where p is a constant and $\phi(z)=O(z)$ we obtain that $p=10$. Hence we have the following equation:

$$\begin{cases}4z\phi'(z)=z-\phi(z), \quad p=10\\ \phi(0)=0,\end{cases} \tag{18}$$

such that the holomorphic solution $\phi(z)$ exists uniquely and converges in a neighborhood of the origin.

Example 4.2. Assume the following equation

$$\begin{cases}\dfrac{u(t,z)}{1.128}t^{0.5}\dfrac{\partial^{0.5}u(t,z)}{\partial t^{0.5}}+16z\dfrac{\partial u(t,z)}{\partial z}=zt+(1+z)t^2, \quad t\in J=[0,1]\\ u(0,z)=0, \quad \text{in a neighborhood of } z=0\end{cases} \tag{19}$$

where $u(t,z)$ is the unknown function. By putting

$$u(t,z)=\mu(z)t+v(t,z) \quad (v(t,z)=O(t^2))$$

as a formal solution. Therefore, $\mu(z)$ satisfies

$$\mu(z)^2 + 16z\mu'(z) - 1 - z = 0.$$

Now by assuming

$$\mu(z) := q + \psi(z),$$

where q is a constant and $\psi(z) = O(z)$ we obtain that $q = \pm 1$. Hence we impose the following equations:

$$\begin{cases} 16z\psi'(z) + 2\psi(z) = z - \psi^2(z), & q = 1 \\ \psi(0) = 0, \end{cases} \tag{20}$$

$$\begin{cases} 16z\psi'(z) - 2\psi(z) = z - \psi^2(z), & q = -1 \\ \psi(0) = 0 \end{cases} \tag{21}$$

where the holomorphic solution $\psi(z)$ exists uniquely and converges in a neighborhood of the origin.

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