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Stability and regularity of weak solutions for a generalized thin film equation

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Abstract

In this paper we establish a stability result and an error estimate of weak solutions for the initial-boundary value problem of a generalized thin film equation and also obtain some higher regularity results for weak solutions.

Keywords: Stability; Error; Regularity; Fourth-order

1. Introduction

Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let p be a positive number with $p > 1$. Assume $q > 1$ and let q' be its conjugate Hölder exponent which satisfies $1/q + 1/q' = 1$.

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In this paper we consider the following fourth-order parabolic initial-boundary value problem

$$\begin{cases} u_t + \operatorname{div}\big(|\nabla\Delta u|^{p-2}\nabla\Delta u\big) = 0, & \text{in } Q, \\ u = 0, \quad \Delta u = 0, & \text{on } \Gamma, \\ u(x,0) = u_0(x), & \text{on } \Omega, \end{cases} \tag{1.1}$$

where the cylinder $Q \equiv \Omega \times (0,T)$, the lateral surface $\Gamma \equiv \partial\Omega \times (0,T)$. In particular, we may take $T = \infty$.

The fourth-order parabolic partial differential equations have drawn great interest of the people in the fields such as materials science, engineering, biological mathematics, image analysis, etc. For the background of problem (1.1), we refer to [1–3,7,8,10,12] for details. The thin film equation

$$h_t + \operatorname{div}\big(h^n \nabla\Delta h\big) = 0$$

models thin viscous flows on solid surfaces. When $N = 1$, Eq. (1.1) is a generalized thin film equation in [7], which has been extensively studied recently. For example, Bernoff and Witelski in [1] investigated the stability of compactly-supported source-type self-similar solutions.

When $p = 2$, Eq. (1.1) is known as a modified version of the Cahn–Hilliard equation (see [2])

$$u_t + \operatorname{div}\left[M(u)\nabla\left(K\Delta u - \frac{\partial f}{\partial u}\right)\right] = 0,$$

which originally describes the evolution of a conserved concentration field during phase separation. The Cahn–Hilliard equation has become a pillar of materials science and engineering. The Cahn–Hilliard equation is also used to improve the sharpness of vague images in image analysis. Myers in [11] used a general thin film equation

$$u_t + \operatorname{div}\big(f(u)\nabla\Delta u\big) = 0$$

to model the surface tension dominated motion of thin viscous film and spreading droplets.

Liu [9] studied the finite speed of propagation of perturbations and regularity of weak solutions for problem (1.1) in one space dimension with $p > 2$. In [12] we established the global existence and uniqueness of weak solutions for problem (1.1) and obtained some higher regularity with respect to the spatial variable for weak solutions of problem (1.1).

In this paper we will establish the stability and regularity of problem (1.1) by proving a stability theorem, an error estimate, and a higher regularity result with respect to the time variable for weak solutions of problem (1.1). In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Main results

In this paper we assume that

$$u_0 \in H_0^1(\Omega). \tag{2.1}$$

For the convenience of the readers, let us first recall the definition of weak solutions for problem (1.1) and the main results in [12].

Definition 2.1. A function $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$ is a weak solution of problem (1.1) if the following conditions are satisfied:

(1) $u \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$ with $\Delta u \in L^p(0,T; W_0^{1,p}(\Omega))$;

(2) For any $\varphi \in C^1(\bar{Q}) \cap L^p(0,T; W_0^{1,p}(\Omega))$ with $\varphi(\cdot,T) = 0$, we have

$$-\int_\Omega u_0(x)\varphi(x,0)\,dx - \int_0^T\!\!\int_\Omega \left[u\varphi_t + |\nabla\Delta u|^{p-2}\nabla\Delta u \cdot \nabla\varphi\right] dx\,d\tau = 0. \tag{2.2}$$

Theorem 2.2. *(See [12, Theorem 2.4].) Under the assumption* (2.1)*, the initial-boundary value problem* (1.1) *admits a unique weak solution.*

Remark 2.3. Since T is arbitrary, we obtain a unique global weak solution $u \in C([0,\infty); L^2(\Omega)) \cap L^\infty(0,\infty; H_0^1(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$ with $\Delta u \in L^p(0,\infty; W_0^{1,p}(\Omega))$ for problem (1.1). By choosing a test function Δu in Definition 2.1 (indeed we may use the Steklov averages

$$[v]_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,\tau)\,d\tau$$

of the function $v(x,t)$ to replace the corresponding function, and then pass to the limits $h \to 0$), we obtain an energy type equality

$$\frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2 dx + \int_0^t\!\!\int_\Omega |\nabla\Delta u|^p\,dx\,d\tau = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx, \quad \text{for } t > 0. \tag{2.3}$$

Furthermore, we conclude from (2.1) and (2.3) that

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} \leqslant \|u_0\|_{H_0^1(\Omega)}, \qquad \|\Delta u\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant C\|u_0\|_{H_0^1(\Omega)}^{2/p}. \tag{2.4}$$

Now we state our main results as follows.

The first theorem is about the stability of the problem.

Theorem 2.4. *Let u be a weak solution of problem* (1.1)*.*

(1) *If* $\max\{1, \frac{2N}{N+4}\} \leqslant p < 2$*, then there exist two positive numbers C and T_0 with*

$$T_0 \leqslant C\|u_0\|_{H_0^1(\Omega)}^{2-p}$$

such that

$$u(x,t) = 0, \quad \textit{for } x \in \Omega,\ t \geqslant T_0. \tag{2.5}$$

(2) *If $p = 2$, then there exists a positive constant C such that*

$$\left\|u(t)\right\|_{H_0^1(\Omega)} \leqslant e^{-Ct}\|u_0\|_{H_0^1(\Omega)}, \quad \textit{for } t > 0. \tag{2.6}$$

(3) *If $p > 2$, then there exists a positive constant C such that*

$$\left\|u(t)\right\|_{H_0^1(\Omega)} \leqslant \left[Ct + \|u_0\|_{H_0^1(\Omega)}^{2-p}\right]^{\frac{1}{2-p}}, \quad \textit{for } t > 0. \tag{2.7}$$

Therefore, when $p > \max\{1, \frac{2N}{N+4}\}$, *we have*

$$\lim_{t\to\infty} \|u(t)\|_{H_0^1(\Omega)} = 0. \tag{2.8}$$

The second theorem is an error estimate.

Theorem 2.5. *Let the functions u and v be respectively weak solutions of problem* (1.1) *corresponding to the initial data* u_0 *and* v_0. *Then we have*

$$\|u - v\|_{L^\infty(0,T;H_0^1(\Omega))} \leqslant \|u_0 - v_0\|_{H_0^1(\Omega)} \tag{2.9}$$

and

$$\|u - v\|_{L^p(0,T;W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))} + \|\Delta u - \Delta v\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant \begin{cases} C\|u_0 - v_0\|_{H_0^1(\Omega)}^{2/p}, & \text{for } p \geqslant 2, \\ C\big(\|u_0\|_{H_0^1(\Omega)} + \|v_0\|_{H_0^1(\Omega)}\big)^{\frac{2-p}{p}} \|u_0 - v_0\|_{H_0^1(\Omega)}, & \text{for } 1 < p < 2, \end{cases} \tag{2.10}$$

where C depends only on p, N and Ω.

The third theorem is about the higher regularity of the weak solution with respect to the time variable.

Theorem 2.6. *Let the function u be a weak solution of problem* (1.1) *with the initial data* u_0. *In addition, if* u_0 *satisfies*

$$\nabla \Delta u_0 \in \big(L^p(\Omega)\big)^N, \tag{2.11}$$

then we have

$$D_t \nabla u \in \big(L^2(Q)\big)^N, \qquad \nabla \Delta u \in \big(L^\infty\big(0, T; L^p(\Omega)\big)\big)^N, \tag{2.12}$$

where $D_t \equiv \frac{\partial}{\partial t}$.

3. Proof of Theorem 2.4

Before proving Theorem 2.4, we first prove a lemma which claims the $W^{2,p}$-norm of a function in $W^{2,p} \cap W_0^{1,p}$ can be controlled by the L^p-norm of its Laplacian.

Lemma 3.1. *Suppose that* $p > 1$. *Then there exists a positive constant C depending only on* p, N, Ω *such that, for every* $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,

$$\|v\|_{W^{2,p}(\Omega)} \leqslant C\|\Delta v\|_{L^p(\Omega)}. \tag{3.1}$$

Proof. It follows from the standard $W^{2,p}$-estimate that there exists a positive constant C such that

$$\|v\|_{W^{2,p}(\Omega)} \leqslant C\big(\|\Delta v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\big). \tag{3.2}$$

(See [6, Chapter 2] or [4, Chapter 3].)

In order to prove (3.1), we only need to show that

$$\|v\|_{W^{1,p}(\Omega)} \leqslant C\|\Delta v\|_{L^p(\Omega)}. \tag{3.3}$$

If (3.3) is violated, then there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\|v_n\|_{W^{1,p}(\Omega)} > n\|\Delta v_n\|_{L^p(\Omega)}. \tag{3.4}$$

Without loss of generality, we assume that

$$\|v_n\|_{W^{1,p}(\Omega)} = 1. \tag{3.5}$$

Then it follows from (3.2) and (3.4) that

$$\|v_n\|_{W^{2,p}(\Omega)} \leqslant C, \qquad \|\Delta v_n\|_{L^p(\Omega)} \leqslant \frac{1}{n}.$$

We draw a subsequence (we still denote it by $\{v_n\}_{n=1}^{\infty}$) and a function $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } W^{2,p}(\Omega),$$

which implies that

$$v_n \to v \quad \text{strongly in } W_0^{1,p}(\Omega).$$

Therefore, we deduce from (3.5) that

$$\|v\|_{W^{1,p}(\Omega)} = 1. \tag{3.6}$$

On the other hand, by the weak convergence of $D^2 v_n$ we know that

$$\|\Delta v\|_{L^p(\Omega)} \leqslant \liminf_{n\to\infty} \|\Delta v_n\|_{L^p(\Omega)} = 0,$$

which implies that $\Delta v = 0$. As we know that $v \in W_0^{1,p}(\Omega)$, we conclude that $v = 0$ a.e. in Ω. This is a contradiction to (3.6). Thus (3.3) holds and then (3.1) is true. □

Proof of Theorem 2.4. It follows from (2.3) that $\|\nabla u(t)\|^2_{L^2(\Omega)}$ is a non-increasing function with respect to t. Therefore, its derivative with respect to t exists almost everywhere on $(0,\infty)$. After differentiating (2.3) with respect to t, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla u(x,t)\right|^2 dx + \int_\Omega \left|\nabla \Delta u(x,t)\right|^p dx = 0, \quad \text{a.e. } t \in (0,\infty).$$

Denote $v = \nabla u$. It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|v(x,t)\right|^2 dx + \int_\Omega \left|\Delta v(x,t)\right|^p dx = 0, \quad \text{a.e. } t \in (0,\infty). \tag{3.7}$$

Recalling $\Delta u \in L^p(0,\infty; W_0^{1,p}(\Omega))$, we obtain $\Delta u \in L^p(0,\infty; L^{p^*}(\Omega))$, where p^* is the Sobolev embedding exponent with respect to p. By Lemma 3.1 we have

$$\|u\|_{W^{2,p^*}(\Omega)} \leqslant C\|\Delta u\|_{L^{p^*}(\Omega)} \leqslant C\|\nabla \Delta u\|_{L^p(\Omega)}.$$

Using Sobolev embedding theorem, we have

$$\|v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^{p^{**}}(\Omega)} \leqslant C\|\nabla u\|_{W^{1,p^*}(\Omega)} \leqslant C\|\Delta v\|_{L^p(\Omega)}$$

for $p \geqslant \max\{1, \frac{2N}{N+4}\}$, where p^{**} is the Sobolev embedding exponent with respect to p^*.

Therefore we obtain from (3.7) that

$$\frac{d}{dt}\int_\Omega \left|v(x,t)\right|^2 dx + C\left(\int_\Omega \left|v(x,t)\right|^2 dx\right)^{\frac{p}{2}} \leqslant 0, \quad \text{a.e. } t \in (0,\infty). \tag{3.8}$$

Denote

$$G(t) = \int_\Omega \left|v(x,t)\right|^2 dx.$$

Then we get from (3.8) that

$$G'(t) + CG^{\frac{p}{2}}(t) \leqslant 0, \quad \text{a.e. } t \in (0,\infty). \tag{3.9}$$

Case I: $\max\{1, \frac{2N}{N+4}\} \leqslant p < 2$.

Denote

$$T_0 = \sup\{t \in (0,\infty) \mid G(t) > 0\}.$$

Then it follows from (3.9) that

$$\left[G^{1-\frac{p}{2}}\right]' + C \leqslant 0, \quad \text{a.e. } t \in (0,T_0). \tag{3.10}$$

Integrating (3.10) over $(0,t)$ with $t \in (0,T_0)$, we have

$$\left[G(t)\right]^{1-\frac{p}{2}} \leqslant \left[G(0)\right]^{1-\frac{p}{2}} - Ct,$$

as long as the right-hand side is nonnegative. Therefore, we conclude that

$$T_0 \leqslant C\|u_0\|^{2-p}_{H^1_0(\Omega)}$$

and

$$G(t) = 0, \quad t \geqslant T_0,$$

which implies that

$$\int_\Omega u^2\,dx \leqslant C\int_\Omega |\nabla u|^2\,dx = 0, \quad t \geqslant T_0.$$

That is,

$$u(x,t) = 0, \quad \text{for } x \in \Omega,\ t \geqslant T_0.$$

Case II: $p = 2$.

It follows from (3.9) that

$$G'(t) + CG(t) \leqslant 0, \quad \text{a.e. } t \in (0,\infty).$$

This implies that

$$\left[G(t)e^{Ct}\right]' \leqslant 0, \quad \text{a.e. } t \in (0,\infty).$$

Integrating the above inequality over $(0,t)$ with $t \in [0,\infty)$, we have

$$G(t) \leqslant e^{-Ct}G(0),$$

which implies that

$$\|u(t)\|_{H_0^1(\Omega)} \leqslant e^{-Ct}\|u_0\|_{H_0^1(\Omega)}, \quad \text{for } t > 0.$$

Case III: $p > 2$.

It follows from (3.9) that

$$\left[G^{1-\frac{p}{2}}\right]' + \frac{2-p}{2}C \geqslant 0, \quad \text{a.e. } t \in (0,\infty). \tag{3.11}$$

Integrating (3.11) over $(0,t)$ with $t \in [0,\infty)$, we have

$$\left[G(t)\right]^{1-\frac{p}{2}} \geqslant Ct + \left[G(0)\right]^{1-\frac{p}{2}},$$

which implies that

$$\|\nabla u(t)\|_{L^2(\Omega)} \leqslant \left[Ct + \|\nabla u_0\|_{L^2(\Omega)}^{2-p}\right]^{\frac{1}{2-p}}, \quad \text{for } t > 0.$$

Therefore, we obtain

$$\|u(t)\|_{H_0^1(\Omega)} \leqslant \left[Ct + \|u_0\|_{H_0^1(\Omega)}^{2-p}\right]^{\frac{1}{2-p}}, \quad \text{for } t > 0.$$

Thus we complete the proof of the theorem. □

4. Proof of Theorem 2.5

Now, we begin to prove Theorem 2.5.

Proof of Theorem 2.5. We choose $\Delta(u-v)$ as a test function for problem (1.1) corresponding to the data u_0 and v_0 respectively (see Remark 2.3), subtract the equations, and obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u - \nabla v|^2(t)\,dx \\
&\quad + \int_0^t\int_\Omega \left[|\nabla\Delta u|^{p-2}\nabla\Delta u - |\nabla\Delta v|^{p-2}\nabla\Delta v\right]\cdot(\nabla\Delta u - \nabla\Delta v)\,dx\,d\tau \\
&= \frac{1}{2}\int_\Omega |\nabla u_0 - \nabla v_0|^2\,dx.
\end{aligned} \tag{4.1}$$

It is obvious that (4.1) implies (2.9). Now we will divide the proof into two cases.

Case I: $p \geqslant 2$.

Recalling an elementary inequality

$$|\xi-\eta|^p \leqslant C\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta), \quad \text{for } \xi,\eta \in \mathbb{R}^N,$$

we obtain from (4.1) that

$$\|\nabla\Delta u - \nabla\Delta v\|_{L^p(Q)}^p \leqslant C\|u_0 - v_0\|_{H_0^1(\Omega)}^2.$$

Noting that $u, v = 0$ on the boundary $\partial\Omega$, we conclude that

$$\|u-v\|_{L^p(0,T;W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))}^p \leqslant C\|u_0 - v_0\|_{H_0^1(\Omega)}^2,$$

which implies that (2.10) is true in this case.

Case II: $1 < p < 2$.

It follows from (2.4) that

$$\|\nabla\Delta u\|_{L^p(Q)} \leqslant C\|u_0\|^{2/p}_{H^1_0(\Omega)}, \qquad \|\nabla\Delta v\|_{L^p(Q)} \leqslant C\|v_0\|^{2/p}_{H^1_0(\Omega)}. \tag{4.2}$$

Using an elementary inequality

$$\left(|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|^2 \leqslant C\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta), \quad \text{for } \xi,\eta\in\mathbb{R}^N,$$

we obtain from (4.1) and (4.2) that

$$\begin{aligned}
&\int_0^t\int_\Omega |\nabla\Delta u-\nabla\Delta v|^p\,dx\,d\tau \\
&\quad= \int_0^t\int_\Omega \left(|\nabla\Delta u|+|\nabla\Delta v|\right)^{\frac{p(2-p)}{2}}\left[\frac{|\nabla\Delta u-\nabla\Delta v|^2}{(|\nabla\Delta u|+|\nabla\Delta v|)^{2-p}}\right]^{\frac{p}{2}}dx\,d\tau \\
&\quad\leqslant \left[\int_0^t\int_\Omega \left(|\nabla\Delta u|+|\nabla\Delta v|\right)^p dx\,d\tau\right]^{\frac{2-p}{2}}\left[\int_0^t\int_\Omega \frac{|\nabla\Delta u-\nabla\Delta v|^2}{(|\nabla\Delta u|+|\nabla\Delta v|)^{2-p}}\,dx\,d\tau\right]^{\frac{p}{2}} \\
&\quad\leqslant C\left(\|u_0\|_{H^1_0(\Omega)}+\|v_0\|_{H^1_0(\Omega)}\right)^{2-p} \\
&\qquad\times\left[\int_0^t\int_\Omega \left(|\nabla\Delta u|^{p-2}\nabla\Delta u-|\nabla\Delta v|^{p-2}\nabla\Delta v\right)\cdot(\nabla\Delta u-\nabla\Delta v)\,dx\,d\tau\right]^{\frac{p}{2}}.
\end{aligned}$$

Making use of (4.1), we conclude that (2.10) is true in this case by the same method.

Therefore we complete the proof of the theorem. □

Indeed, we can obtain an error estimate for the following general problem

$$\begin{cases} u_t+\operatorname{div}\left(|\nabla\Delta u|^{p-2}\nabla\Delta u\right)=f-\operatorname{div} g, & \text{in } Q,\\ u=0, \quad \Delta u=0, & \text{on } \Gamma,\\ u(x,0)=u_0(x), & \text{on } \Omega, \end{cases} \tag{4.3}$$

where the data (u_0, f, g) satisfy

$$u_0\in H^1_0(\Omega), \qquad f\in L^{p'}\left(0,T;L^{(p^*)'}(\Omega)\right), \qquad g\in\left(L^{p'}(Q)\right)^N. \tag{4.4}$$

Remark 4.1. For problem (4.3), we can obtain a similar error estimate. Specifically, if the functions u and v, respectively, are weak solutions of problem (4.3) corresponding to the data (u_0, f, g) and $(v_0, \tilde{f}, \tilde{g})$, then we have

(1) For $p \geqslant 2$,

$$\begin{aligned}\|u-v\|_{L^\infty(0,T;H^1_0(\Omega))} &\leqslant \|u_0-v_0\|_{H^1_0(\Omega)}+C\|f-\tilde{f}\|^{p'/2}_{L^{p'}(0,T;L^{(p^*)'}(\Omega))} \\ &\quad+C\|g-\tilde{g}\|^{p'/2}_{(L^{p'}(Q))^N}\end{aligned} \tag{4.5}$$

and

$$\|u-v\|_{L^p(0,T;W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))}+\|\Delta u-\Delta v\|_{L^p(0,T;W_0^{1,p}(\Omega))}$$
$$\leqslant \|u_0-v_0\|_{H_0^1(\Omega)}^{2/p}+C\|f-\tilde{f}\|_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}^{\frac{1}{p-1}}+C\|g-\tilde{g}\|_{(L^{p'}(Q))^N}^{\frac{1}{p-1}}. \tag{4.6}$$

(2) For $1<p<2$,

$$\|u-v\|_{L^\infty(0,T;H_0^1(\Omega))}\leqslant \|u_0-v_0\|_{H_0^1(\Omega)}$$
$$+C(\Lambda+\widetilde{\Lambda})^{1/2}\big(\|f-\tilde{f}\|_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}^{1/2}+\|g-\tilde{g}\|_{(L^{p'}(Q))^N}^{1/2}\big) \tag{4.7}$$

and

$$\|u-v\|_{L^p(0,T;W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))}+\|\Delta u-\Delta v\|_{L^p(0,T;W_0^{1,p}(\Omega))}$$
$$\leqslant C(\Lambda+\widetilde{\Lambda})^{\frac{2-p}{2}}\|u_0-v_0\|_{H_0^1(\Omega)}$$
$$+C(\Lambda+\widetilde{\Lambda})^{\frac{3-p}{2}}\big(\|f-\tilde{f}\|_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}^{1/2}+\|g-\tilde{g}\|_{(L^{p'}(Q))^N}^{1/2}\big), \tag{4.8}$$

where

$$\Lambda\equiv\|u_0\|_{H_0^1(\Omega)}^{2/p}+C\|f\|_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}^{1/(p-1)}+C\|g\|_{(L^{p'}(Q))^N}^{1/(p-1)},$$
$$\widetilde{\Lambda}\equiv\|v_0\|_{H_0^1(\Omega)}^{2/p}+C\|\tilde{f}\|_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}^{1/(p-1)}+C\|\tilde{g}\|_{(L^{p'}(Q))^N}^{1/(p-1)}.$$

Here C depends only on p, N and Ω.

Remark 4.2. The existence and uniqueness of weak solutions of problem (4.3) under (4.4) is presented in Theorem 2.4 in [12]. The uniqueness is a direct consequence of Remark 4.1. It follows from Remark 4.1 that the weak solution is small if the given data are small. This is a stability result, too. The proof of Remark 4.1 is the same as that of Theorem 2.5. Since the procedure is long and tedious, we omit the standard proof.

5. Proof of Theorem 2.6

In this section a general result (Theorem 5.1) for problem (4.3) will be proved. Theorem 2.6 for problem (1.1) is only a particular case.

For $q, r>1$, denote

$$W^{1,r}\big(0,T;L^q(\Omega)\big)=\big\{v(x,t)\mid v\in L^r\big(0,T;L^q(\Omega)\big),\ D_t v\in L^r\big(0,T;L^q(\Omega)\big)\big\}$$

(see [5, Chapter 5]). From Sobolev embedding theorem, we know that

$$W^{1,r}\big(0,T;L^q(\Omega)\big)\hookrightarrow L^\infty\big(0,T;L^q(\Omega)\big). \tag{5.1}$$

The general result is the following:

Theorem 5.1. *Let the function u be a weak solution of problem* (4.3) *under* (4.4) *corresponding to the data* (u_0, f, g).

(1) *If* (u_0, f, g) *also satisfy*

$$\nabla\Delta u_0\in\big(L^p(\Omega)\big)^N,\qquad \nabla(f-\operatorname{div} g)\in\big(L^2(Q)\big)^N,$$

then we have

$$D_t\nabla u\in\big(L^2(Q)\big)^N,\qquad \nabla\Delta u\in\big(L^\infty\big(0,T;L^p(\Omega)\big)\big)^N. \tag{5.2}$$

Moreover, we obtain

$$\|D_t\nabla u\|^2_{L^2(Q)}+\|\nabla\Delta u\|^p_{L^\infty(0,T;L^p(\Omega))} \leqslant C\|\nabla\Delta u_0\|^p_{L^p(\Omega)}+C\big\|\nabla(f-\operatorname{div}g)\big\|^2_{L^2(Q)}. \tag{5.3}$$

(2) *If* (u_0,f,g) *also satisfy*

$$\nabla\Delta u_0\in\big(L^p(\Omega)\big)^N,\qquad f\in W^{1,p'}\big(0,T;L^{(p^*)'}(\Omega)\big), \\ g\in\big(W^{1,p'}\big(0,T,L^{p'}(\Omega)\big)\big)^N, \tag{5.4}$$

then we have

$$D_t\nabla u\in\big(L^2(Q)\big)^N,\qquad \nabla\Delta u\in\big(L^\infty\big(0,T;L^p(\Omega)\big)\big)^N. \tag{5.5}$$

Moreover, we obtain

$$\|D_t\nabla u\|^2_{L^2(Q)}+\|\nabla\Delta u\|^p_{L^\infty(0,T;L^p(\Omega))}\leqslant C\|\nabla\Delta u_0\|^p_{L^p(\Omega)}+C\|\nabla u_0\|^2_{L^2(\Omega)} \\ +C\|f\|^{p'}_{W^{1,p'}(0,T;L^{(p^*)'}(\Omega))}+C\|g\|^{p'}_{(W^{1,p'}(0,T;L^{p'}(\Omega)))^N}. \tag{5.6}$$

Proof of Theorem 5.1. (1) Denote

$$\tilde f=f-\operatorname{div}g.$$

Choosing $D_t\Delta u$ as a test function (see Remark 2.3) for problem (4.3), we have

$$\int_0^t\int_\Omega|D_t\nabla u|^2\,dx\,d\tau+\frac1p\int_\Omega|\nabla\Delta u|^p(x,t)\,dx \\ =\frac1p\int_\Omega|\nabla\Delta u_0|^p\,dx-\int_0^t\int_\Omega\tilde f D_t\Delta u\,dx\,d\tau. \tag{5.7}$$

Since

$$\left|\int_0^t\int_\Omega\tilde f D_t\Delta u\,dx\,d\tau\right|=\left|-\int_0^t\int_\Omega\nabla\tilde f\cdot D_t\nabla u\,dx\,d\tau\right| \\ \leqslant\frac14\int_0^t\int_\Omega|D_t\nabla u|^2\,dx\,d\tau+\int_0^t\int_\Omega|\nabla\tilde f|^2\,dx\,d\tau,$$

we obtain (5.3).

(2) We choose the same test function $D_t \Delta u$ as (1) to obtain

$$\int_0^t \int_\Omega |D_t \nabla u|^2 \, dx \, d\tau + \frac{1}{p} \int_\Omega |\nabla \Delta u|^p(x,t) \, dx$$
$$= \frac{1}{p} \int_\Omega |\nabla \Delta u_0|^p \, dx - \int_0^t \int_\Omega [f D_t \Delta u + g \cdot D_t \nabla \Delta u] \, dx \, d\tau. \tag{5.8}$$

We first estimate the last term on the right-hand side. For the first term involving f, we have

$$\left| \int_0^t \int_\Omega f D_t \Delta u \, dx \, d\tau \right| = \left| \int_\Omega f \Delta u(x,t) \, dx - \int_\Omega f(0) \Delta u_0 \, dx - \int_0^t \int_\Omega D_t f \Delta u \, dx \, d\tau \right|$$
$$\leqslant \left| \int_\Omega f \Delta u(x,t) \, dx \right| + \left| \int_\Omega f(0) \Delta u_0 \, dx \right| + \left| \int_0^t \int_\Omega D_t f \Delta u \, dx \, d\tau \right|$$
$$\equiv I_1 + I_2 + I_3. \tag{5.9}$$

Using Hölder's, Sobolev's and Young's inequalities, we estimate I_1 to have

$$I_1 \leqslant \left(\int_\Omega \left| \Delta u(x,t) \right|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_\Omega |f|^{(p^*)'} dx \right)^{\frac{1}{(p^*)'}}$$
$$\leqslant C \left(\int_\Omega \left| \nabla \Delta u(x,t) \right|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega |f|^{(p^*)'} dx \right)^{\frac{1}{(p^*)'}}$$
$$\leqslant \epsilon \int_\Omega \left| \nabla \Delta u(x,t) \right|^p dx + C(\epsilon) \left\| f(t) \right\|^{p'}_{L^{(p^*)'}(\Omega)},$$

where a small positive number ϵ is to be determined later. Using Hölder's, Young's and Sobolev's inequalities, we estimate I_2 to get

$$I_2 \leqslant \|\Delta u_0\|^p_{L^{p^*}(\Omega)} + \left\| f(0) \right\|^{p'}_{L^{(p^*)'}(\Omega)}$$
$$\leqslant C \|\nabla \Delta u_0\|^p_{L^p(\Omega)} + \left\| f(0) \right\|^{p'}_{L^{(p^*)'}(\Omega)}.$$

Similarly, we estimate I_3 to obtain

$$I_3 \leqslant C \int_0^t \int_\Omega \left| \nabla \Delta u(x,t) \right|^p dx + C \|D_t f\|^{p'}_{L^{p'}(0,T;L^{(p^*)'}(\Omega))}$$
$$\leqslant C \|\nabla u_0\|^2_{L^2(\Omega)} + C \|f\|^{p'}_{L^{p'}(0,T;L^{(p^*)'}(\Omega))} + C \|g\|^{p'}_{(L^{p'}(Q))^N}$$
$$+ C \|D_t f\|^{p'}_{L^{p'}(0,T;L^{(p^*)'}(\Omega))},$$

where we use the energy type estimate in [12]. Thus, plugging the above three inequalities into (5.9), we have

$$\left|\int_0^t\int_\Omega f D_t\Delta u\,dx\,d\tau\right| \leqslant \epsilon\int_\Omega \left|\nabla\Delta u(x,t)\right|^p dx + C\|\nabla u_0\|_{L^2(\Omega)}^2$$
$$+ C\|\nabla\Delta u_0\|_{L^p(\Omega)}^p + C\|f\|_{W^{1,p'}(0,T;L^{(p^*)'}(\Omega))}^{p'} + C\|g\|_{(W^{1,p'}(0,T;L^{p'}(\Omega)))^N}^{p'}. \tag{5.10}$$

For the second term involving g on the right-hand side in (5.8), we have

$$\left|\int_0^t\int_\Omega g\cdot D_t\nabla\Delta u\,dx\,d\tau\right|$$
$$= \left|\int_\Omega g\cdot\nabla\Delta u(x,t)\,dx - \int_\Omega g(0)\cdot\nabla\Delta u_0\,dx - \int_0^t\int_\Omega D_t g\cdot\nabla\Delta u\,dx\,d\tau\right|$$
$$\leqslant \epsilon\int_\Omega \left|\nabla\Delta u(x,t)\right|^p dx + C\|\nabla\Delta u_0\|_{L^p(\Omega)}^p + C\|\nabla u_0\|_{L^2(Q)}^2$$
$$+ C\|f\|_{W^{1,p'}(0,T;L^{(p^*)'}(\Omega))}^{p'} + C\|g\|_{(W^{1,p'}(0,T;L^{p'}(\Omega)))^N}^{p'}. \tag{5.11}$$

By (5.8), (5.10) and (5.11), we obtain the desired result (5.6).

Thus, we complete the proof of Theorem 5.1. □

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