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Block-triangular matrix algebras and factorable ideals of graded polynomial identities ☆

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Abstract

We study the graded polynomial identities of block-triangular matrix algebras with respect to the grading defined by an abelian group G. In particular, we describe conditions for the T_G-ideal of a such algebra to be factorable as a product of T_G-ideals corresponding to the algebras defining the diagonal blocks. Moreover, for the factorable T_2-ideal of a superalgebra we give a formula for computing its sequence of graded cocharacters once given the sequences of cocharacters of the T_2-ideals that factorize it. We finally apply these results to a specific example of block-triangular matrix superalgebra.

Keywords: Graded polynomial identities; Block-triangular matrices; Superalgebras; Cocharacters

1. Introduction

A variety of associative algebras over a field F is associated to an ideal of the free associative algebra $F\langle X\rangle$ that is invariant under all the endomorphisms of $F\langle X\rangle$. Such ideals are called “T-ideals” and they are the ideals of the polynomial identities satisfied by any algebra of the variety. For the study of the T-ideals over a field of characteristic zero, a fundamental tool is given by the representation theory of the linear and symmetric groups. Moreover, the Kemer’s theorems about the classification of the T-ideals of $F\langle X\rangle$ show that

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the notion of grading of an algebra defined by a group is another key ingredient for such study. In particular, one has that any proper T-ideal of $F\langle X\rangle$ is the ideal of the polynomial identities satisfied by the Grassmann envelope of a suitable $\mathbb{Z}_2$-graded algebra (also called superalgebra) of finite dimension. Recently, the work of Giambruno and Zaicev [11,12] has contributed to clarify why the notion of PI-exponent is crucial for a classification of the T-ideals in terms of growth of the sequence of their codimensions. Recall that the nth codimension of a T-ideal is defined as the degree of the representation of the group $\mathbb{S}_n$ on the vector space of the multilinear polynomials of $F\langle X\rangle$ of degree n modulo the considered T-ideal. In [12] the authors prove that the minimal varieties with respect to a fixed exponent are determined by the T-ideals of the Grassmann envelope of the so-called "minimal superalgebras". Over an algebraic closed field, such superalgebras can be realized as graded subalgebras of block-triangular matrix algebras equipped with a suitable $\mathbb{Z}_2$-grading. Precisely, the blocks along the main diagonal are simple superalgebras of finite dimension. Then, by the Lewin's Theorem [16] one has that the T-ideals of the identities satisfied by the minimal superalgebras and their Grassmann envelopes are products of the T-ideals corresponding to the diagonal blocks. Such results allow hence to solve in the positive a conjecture due to Drensky [6,7] about the factorability of the T-ideals of minimal varieties as a product of verbally prime T-ideals. Moreover, Berele and Regev [4] proved a formula that relates the sequence of ordinary cocharacters of a product of T-ideals to the sequences of cocharacters of these ideals.

The present paper intends to contribute to this line of research by studying the graded structure of the mentioned algebras. In particular, over an infinite field, we consider block-triangular matrix algebras endowed with an elementary grading defined by any finite abelian group G. Precisely, in Section 2 we summarize the basic definitions about graded algebras and their polynomial identities. In Section 3 we recall the Lewin's Theorem and we show how it can be applied for studying the T_G-ideals of the graded identities of block-triangular matrix algebras. In Section 4, for any graded subalgebra A of a complete matrix algebra we describe the notion of "G-regularity" in terms of suitable projections defined on the graded generic algebra associated to A. For the block-triangular matrix algebras of type:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}$$

where A, B are graded subalgebras of matrix algebras, we prove: $T_G(R) = T_G(A)T_G(B)$, provided that at least one of the algebras A, B is G-regular. In Section 5 we give an effective characterization of the property of G-regularity for complete matrix algebras. For instance, for the superalgebra $A = M_{k,l}(F)$ it holds that A is $\mathbb{Z}_2$-regular if and only if $k = l$. For suitable groups G and for A, B complete matrix algebras we prove also that the G-regularity of A or B is a necessary condition for the ideal $T_G(R)$ to be factorable. In Section 6, assuming $\mathrm{char}(F) = 0$, we prove a formula that allows to compute the sequence of graded cocharacters of a superalgebra R such that $T_2(A) = T_2(A)T_2(B)$ starting from the corresponding sequences of A and B. Such formula is based on the notion of convolution of two sequences of characters. We apply these results for computing the graded cocharacters of a concrete example.

2. *G*-graded structures

Let F be an infinite field and $(G,+)$ an abelian group. Let A be an associative F-algebra. We say that A is a *G-graded algebra* if $A=\bigoplus_{g\in G} A_g$, where $A_g\subset A$ are subspaces and $A_gA_h\subset A_{g+h}$ holds for any $g,h\in G$. The subspace A_g is called the *homogeneous component of A of degree g*. We say that the elements $a\in A_g$ are *homogeneous of degree g* and we denote their degrees as $|a|=g$. One defines G-graded: subspaces of A, A-modules, homomorphisms and so on, in a standard way, see for example [1].

Let now $X=\{x_1,x_2,\ldots\}$ be a countable set of variables. We denote by $F\langle X\rangle$ the free associative algebra generated by X. Given a map $|\,|: X\to G$, we can define a G-grading on $F\langle X\rangle$ by putting $|w|=|x_{j_1}|+\cdots+|x_{j_n}|$ for any monomial $w=x_{j_1}\cdots x_{j_n}\in F\langle X\rangle$. Then, the homogeneous component $F\langle X\rangle_g\subset F\langle X\rangle$ is the subspace spanned by all monomials of degree g. If the group G is finite, we assume that the fibers of the map $|\,|$ are all infinite.

If A is a G-graded algebra, we denote by $T_G(A)$ the intersection of the kernels of all G-graded homomorphisms $F\langle X\rangle\to A$. Then $T_G(A)$ is a graded two-sided ideal of $F\langle X\rangle$ and its elements are called *G-graded polynomial identities* of the algebra A. Note that $T_G(A)$ is stable under the action of any G-graded endomorphism of the algebra $F\langle X\rangle$. Any G-graded ideal of $F\langle X\rangle$ which verifies such property is said to be a *T_G-ideal*. Clearly, any T_G-ideal I is the ideal of the G-graded polynomial identities of the graded algebra $F\langle X\rangle/I$. Note also that for a G-graded algebra A, the quotient algebra $F\langle X\rangle/T_G(A)$ is the relatively free algebra for the variety of graded algebras generated by A.

If the algebra A is graded by the group $G=\mathbb{Z}^m$ and any homogeneous element $a\in A$ has degree $\partial(a)\in\mathbb{N}^m$ we say that A is *m-multigraded*. In this case, we can define the *Hilbert–Poincaré series of A* as the power series $\mathrm{HP}_m(A)=\sum_{\bar\alpha}\dim_F(A_{\bar\alpha})t_1^{\alpha_1}\cdots t_m^{\alpha_m}$ where the m-tuple $\bar\alpha=(\alpha_1,\ldots,\alpha_m)$ ranges over $\mathbb{N}^m$.

Let $X_m=\{x_{i_1},\ldots,x_{i_m}\}$ be any ordered subset of X with m elements and let $\partial: X_m\to\mathbb{Z}^m$ be any map. An m-multigrading on $F\langle X_m\rangle$ is defined by putting $\partial(w)=\partial(x_{j_1})+\cdots+\partial(x_{j_n})$ for all monomials $w=x_{j_1}\cdots x_{j_n}\in F\langle X_m\rangle$. In what follows we always assume that ∂ is the *natural multigrading* that is we put $\partial(x_{i_1})=(1,0,\ldots,0)$, $\partial(x_{i_2})=(0,1,\ldots,0)$, etc.

Consider now A be an algebra graded by any abelian group G. The ideal $T_G(A)\cap F\langle X_m\rangle$ is m-multigraded and so is the quotient algebra:

$$\tilde A_{X_m}=F\langle X_m\rangle/\big(T_G(A)\cap F\langle X_m\rangle\big).$$

Then, we can define $\mathrm{HP}_{X_m}(A)=\mathrm{HP}_m(\tilde A_{X_m})$ and we call such series the *Hilbert–Poincaré series of the G-graded algebra A associated to the set of variables X_m*. Given two finite subsets $X_m,X'_m\subset X$, note that the series $\mathrm{HP}_{X_m}(A)$ and $\mathrm{HP}_{X'_m}(A)$ coincide if there is a bijection $\sigma: X_m\to X'_m$ such that $|\sigma(x_{i_k})|=|x_{i_k}|$, for all $x_{i_k}\in X_m$.

If $G=\mathbb{Z}_2$, the $\mathbb{Z}_2$-graded algebras A are usually called *superalgebras*. In this case, the notation $T_2(A)$ is used for denoting the ideal of the $\mathbb{Z}_2$-graded polynomial identities of A. Moreover, note that the Hilbert–Poincaré series $\mathrm{HP}_{X_m}(A)$ associated to a subset $X_m\subset X$ is uniquely determined by the pair of integers k,l that counts the number of variables in X_m of degree respectively 0,1. We denote such series by $\mathrm{HP}_{k,l}(A)$.

Since the ideal $T_2(A)$ is stable under $\mathbb{Z}_2$-graded endomorphisms, a natural action of the product group $\mathrm{GL}_k(F) \times \mathrm{GL}_l(F)$ is defined for the quotient algebra $\tilde{A}_{X_m}$. Assuming $\mathrm{char}(F) = 0$, it follows that the series $\mathrm{HP}_{k,l}(A)$ can be decomposed as a sum of products of Schur functions defined on distinct sets of variables:

$$\mathrm{HP}_{k,l}(A) = \sum_{\mu,\nu} m_{\mu,\nu} s_\mu(\xi_1, \ldots, \xi_k) s_\nu(\eta_1, \ldots, \eta_l) \tag{1}$$

where the heights of the partitions μ, ν are bounded respectively by the integers k, l. Note now that the group $\mathbb{S}_k \times \mathbb{S}_l$ acts on the multilinear component of the algebra $\tilde{A}_{X_m}$. We denote by $\chi_{k,l}(A)$ the character of such representation and we call it the *$\mathbb{Z}_2$-graded cocharacter of the superalgebra A* or equivalently *of the ideal $T_2(A)$*. Denote χ_μ and χ_ν ($\mu \vdash k, \nu \vdash l$) the irreducible characters of the groups respectively $\mathbb{S}_k$ and $\mathbb{S}_l$. For simplifying the notation, we put $\chi_{\mu,\nu} = \chi_\mu \otimes \chi_\nu$. Then, if Eq. (1) holds, we have also (see [3,8]):

$$\chi_{k,l}(A) = \sum_{\mu,\nu} m_{\mu,\nu} \chi_{\mu,\nu}. \tag{2}$$

Let $E = E_0 \oplus E_1$ be the Grassmann (or exterior) algebra of a vector space of countable dimension equipped with its natural $\mathbb{Z}_2$-grading. For any superalgebra A, the *Grassmann envelope* of A is defined as the following superalgebra $G(A) = (A_0 \otimes E_0) \oplus (A_1 \otimes E_1)$. The relationship between the graded identities of the superalgebras $A, G(A)$ is described in [15] by means of an involution $I \mapsto I^*$ defined on the lattice of the T_2-ideals of the free superalgebra $F\langle X \rangle$. Note that this map satisfies also the property $(IJ)^* = I^* J^*$. Using the language of the representation theory, one has the following relationship between the sequences of graded cocharacters of A and $G(A)$: $\chi_{k,l}(A) = \sum_{\mu,\nu} m_{\mu,\nu} \chi_{\mu,\nu}$ if and only if $\chi_{k,l}(G(A)) = \sum_{\mu,\nu} m_{\mu,\nu} \chi_{\mu,\nu'}$, where $\nu' \vdash l$ is the conjugate partition of ν. These results, together with the classification of the simple superalgebras of finite dimension, allow us to reduce the study in this paper to the matrix algebras with entries in the field F.

3. Lewin's Theorem for G-graded algebras

Let A, B be G-graded algebras and U be a G-graded A–B-bimodule. We denote by R the block-triangular matrix algebra defined as follows:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}.$$

Typically, we may consider $A = M_m$, $B = M_n$ the complete matrix algebras and $U = M_{m \times n}$ the vector space of $m \times n$ rectangular matrices. In this case, R is the algebra

$\mathrm{UT}(m,n)$ of block-triangular matrices. The algebra R is G-graded in a natural way by putting:

$$R_g = \begin{bmatrix} A_g & U_g \\ 0 & B_g \end{bmatrix}$$

for any $g \in G$ and we have $T_G(A)T_G(B) \subset T_G(R)$. One of the main results of this paper consists in describing suitable conditions for the structures of A, B, U so that $T_G(A)T_G(B) = T_G(R)$. For this purpose, one of the main tools is the Lewin's Theorem [16].

Let I and J be any two-sided ideals of $F\langle X\rangle$. Consider the quotient algebras $F\langle X\rangle/I$, $F\langle X\rangle/J$ and let U be a $F\langle X\rangle/I$-$F\langle X\rangle/J$-bimodule. We define:

$$R = \begin{bmatrix} F\langle X\rangle/I & U \\ 0 & F\langle X\rangle/J \end{bmatrix}.$$

Fix $\{u_i\}$ a countable set of elements of U. Then $\varphi : x_i \mapsto a_i$ defines an algebra homomorphism, where:

$$a_i = \begin{bmatrix} x_i + I & u_i \\ 0 & x_i + J \end{bmatrix}.$$

If $f(x_1,\dots,x_n) \in F\langle X\rangle$ one has that $f(x_1,\dots,x_n) \mapsto f(a_1,\dots,a_n)$, where:

$$f(a_1,\dots,a_n) = \begin{bmatrix} f(x_1,\dots,x_n) + I & \delta(f) \\ 0 & f(x_1,\dots,x_n) + J \end{bmatrix}$$

and $\delta(f)$ is some element of U. Then $IJ \subset \ker(\varphi) = I \cap J \cap \ker(\delta)$ and $\delta : F\langle X\rangle \to U$ is an F-derivation.

Theorem 3.1 (Lewin [16]). *If $\{u_i\}$ is a countable free set of elements of the bimodule U then for the homomorphism φ defined by $\{u_i\}$, we have* $\ker(\varphi) = IJ$.

Suppose now that the free algebra $F\langle X\rangle$ is G-graded by some map $|\ |$. Consider I, J two T_G-ideals and let U be a G-graded $F\langle X\rangle/I$-$F\langle X\rangle/J$-bimodule. Clearly $IJ \subset T_G(R)$. Moreover, if the free elements $u_i \in U$ are all homogeneous and such that $|x_i| = |u_i|$ for all $i \geqslant 1$, then $\varphi : F\langle X\rangle \to R$ is a G-graded homomorphism. Hence $T_G(R) \subset \ker(\varphi)$ and by the Lewin's Theorem we have that $\ker(\varphi) = IJ$. We conclude:

Corollary 3.2. *If the G-graded bimodule U contains a countable free set $\{u_i\}$ of homogeneous elements such that $|x_i| = |u_i|$ for any $i \geqslant 1$, then $T_G(R) = IJ$.*

4. G-regularity and factorable T_G-ideals

In this section we will consider gradings over matrix algebras. In [1,2] all G-gradings over $M_m(F)$ are classified for the case that F is an algebraically closed field. In particular, the so-called *elementary gradings* are proved to be very important.

From now on, we assume that the abelian group G is finite. Let $M_m = M_m(F)$ be the algebra of matrices of order m with entries in F and fix a map $|\ |: \{1, 2, \ldots, m\} \to G$. Then $|\ |$ induces a grading on M_m by setting $|e_{ij}| = |j| - |i|$, for all matrix units $e_{ij} \in M_m$. We leave to the reader the verification that this is indeed the elementary grading defined by $(|1|, \ldots, |m|) \in G^m$. We write $(M_m, |\ |)$ for the matrix algebra M_m endowed with the G-grading defined by the map $|\ |: \{1, 2, \ldots, m\} \to G$. For $G = \mathbb{Z}_2$, the superalgebra $(M_m, |\ |)$ is simply denoted as $M_{k,l}(F)$ if $|i| = 0$ for $1 \leqslant i \leqslant k$ and $|i| = 1$ for $k + 1 \leqslant i \leqslant k + l = m$.

Let $(M_m, |\ |_m)$ and $(M_n, |\ |_n)$ be two G-graded matrix algebras. Define the map $|\ |: \{1, 2, \ldots, m + n\} \to G$ by putting $|i| = |i|_m$ for $i \leqslant m$ and $|i| = |i - m|_n$ for $i > m$. We consider then the matrix algebra M_{m+n} endowed with the G-grading defined by the map $|\ |$. Let $U = M_{m\times n}$ and let A, B be G-graded subalgebras respectively of M_m, M_n. Then

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix} \subset (M_{m+n}, |\ |)$$

is a G-graded subalgebra. We will prove that under suitable assumptions for the algebra A or B it holds $T_G(R) = T_G(A)T_G(B)$. The notion of "generic algebra" is very useful for this purpose.

Let Ω be any G-graded algebra. We denote by $\mathrm{Gen}_G(\Omega)$ each G-graded algebra isomorphic to $F\langle X\rangle / T_G(\Omega)$ and we call it a *G-graded generic algebra associated to Ω*. In particular, this implies that $T_G(\Omega) = T_G(\mathrm{Gen}_G(\Omega))$.

If Ω has finite dimension, one has a canonical way to define a graded generic algebra. Let $\{e_1, \ldots, e_n\}$ be an F-linear basis of Ω whose elements are all homogeneous. Denote $P(\Omega) = F[t_i^{(h)} \mid 1 \leqslant i \leqslant n, h \geqslant 1]$ the polynomial ring in the countable set of commuting variables $t_i^{(h)}$. We call $P(\Omega)$ the *polynomial ring associated to the finite dimensional algebra Ω*. Note that the tensor product $\Omega \otimes P(\Omega) = \bigoplus_{g\in G} \Omega_g \otimes P(\Omega)$ over the field F is a G-graded algebra such that:

$$T_G\big(\Omega \otimes P(\Omega)\big) = T_G(\Omega).$$

If x_h are the variables of the G-graded free algebra $F\langle X\rangle$, then we consider in $\Omega \otimes P(\Omega)$ the graded subalgebra Ω' generated, for all $h \geqslant 1$, by the homogeneous elements $a_h = \sum_{|e_i|=|x_h|} t_i^{(h)} e_i$ where the index i ranges over $1 \leqslant i \leqslant n$. We can easily prove:

$$\Omega' = \mathrm{Gen}_G(\Omega).$$

Note that if $\Omega = M_m$ then we choose canonically as F-linear basis the one given by the matrix units e_{ij} (for the non-graded case, see for instance [19]).

We consider now as an homogeneous linear basis of R the disjoint union of some bases of A, B and the canonical basis $\{e_{ij}\}$ ($1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n$) of U. If $P = P(R)$ then the algebra $R \otimes P$ contains $A' = \mathrm{Gen}_G(A)$, $B' = \mathrm{Gen}_G(B)$ and $R' = \mathrm{Gen}_G(R)$. Denote by $\overline{R} \subset R \otimes P$ the following G-graded subalgebra:

$$\overline{R} = \begin{bmatrix} A' & U' \\ 0 & B' \end{bmatrix} \tag{3}$$

where U' is the G-graded A'–B'-bimodule contained in $R \otimes P$ and generated, for all $h \geqslant 1$, by the following homogeneous elements:

$$u_h = \sum_{|e_{ij}|=|x_h|} t_{ij}^{(h)} e_{ij} \tag{4}$$

with $1 \leqslant i \leqslant m$, $m+1 \leqslant j \leqslant m+n$. We have then:

Proposition 4.1.

$$T_G(R') = T_G(R) = T_G(\overline{R}).$$

Proof. It is sufficient to note that $T_G(R') = T_G(R) = T_G(R \otimes P)$ and moreover $R' \subset \overline{R} \subset R \otimes P$. $\square$

Now we want to show that the homogeneous elements u_h defined in (4) form a free set of the bimodule U' under suitable conditions. For this purpose we introduce the notion of "G-regularity" of a matrix subalgebra.

Let A be any G-graded subalgebra of $(M_m, |\ |)$. Denote $P = P(A)$ the polynomial ring associated to A. For any $g \in G$ we consider the F-linear map $\pi_g : M_m \otimes P \to M_m \otimes P$ defined as follows:

$$\sum_{i,j} a_{ij} e_{ij} \mapsto \sum_{|i|=g,\, j} a_{ij} e_{ij}$$

where $1 \leqslant i, j \leqslant m$. Since $\mathrm{Gen}_G(A) \subset A \otimes P \subset M_m \otimes P$ we define also the map $\hat{\pi}_g : \mathrm{Gen}_G(A) \to M_m \otimes P$ as the restriction of π_g to $\mathrm{Gen}_G(A)$. Define $\pi_g^* : M_m \otimes P \to M_m \otimes P$ the F-linear map $\sum_{i,j} a_{ij} e_{ij} \mapsto \sum_{i,|j|=g} a_{ij} e_{ij}$ and denote by $\hat{\pi}_g^*$ its restriction to $\mathrm{Gen}_G(A)$.

Proposition 4.2. *The maps $\hat{\pi}_g$ are all injective if and only if the maps $\hat{\pi}_g^*$ are such, for all $g \in G$.*

Proof. Put $A' = \mathrm{Gen}_G(A)$ and let $\varphi : F\langle X\rangle \to A'$ be the canonical G-graded epimorphism such that $\ker(\varphi) = T_G(A)$. Suppose that the map $\hat{\pi}_g$ is not injective that is there is a matrix $a' \neq 0$ of A' such that $\pi_g(a') = 0$. Since the map π_g is graded, we can assume that the element a' is homogeneous of degree $h \in G$. We want to prove that $\pi^*_{g+h}(a') = 0$ that is $\hat{\pi}^*_{g+h}$ is not a monomorphism.

Let $f \in F\langle X\rangle$, $f \notin T_G(A)$ be an homogeneous polynomial of degree $|f| = h$ such that $\varphi(f) = a'$. Denote by $\bar{\pi}_g$ and $\bar{\pi}^*_g$ the F-linear maps defined on M_m which correspond respectively to π_g and π^*_g. The condition $\pi_g(a') = 0$ is equivalent to $\bar{\pi}_g(a) = 0$, where $a = \nu(f)$ and $\nu : F\langle X\rangle \to A$ is any G-graded evaluation. We prove that one has also $\bar{\pi}^*_{g+h}(a) = 0$ that is $a_{ij} = 0$ for any $i, j = 1, 2, \ldots, m$ with $|j| = g + h$. In fact, if $|j| - |i| \neq h$ then $a_{ij} = 0$ since the matrix $a \in A$ is homogeneous of degree h. Otherwise, if $|j| - |i| = h$ and hence $|i| = g$, then $a_{ij} = 0$ since $\bar{\pi}_g(a) = 0$. As ν varies over the evaluations, we get $\pi^*_{g+h}(a') = 0$. □

Definition 4.3. A G-graded subalgebra $A \subset M_m$ is said to be *G-regular* if the maps $\hat{\pi}_g$ (or equivalently the $\hat{\pi}^*_g$) are all injective, for any $g \in G$.

If the map $|\ | : \{1, 2, \ldots, m\} \to G$ is not surjective, clearly all the graded subalgebras of M_m are not G-regular since there is some $\pi_g = 0$. For this reason, from now on we assume that the map $|\ |$ is surjective and thus the finite group G has order $\leqslant n$. Moreover note that for the ordinary case, that is for $G = \{0\}$, all the subalgebras of M_m are regular. With the notation of definitions (3) and (4), we have:

Proposition 4.4. *Let A, B be G-graded subalgebras respectively of M_m, M_n. If one of these subalgebras is G-regular then the homogeneous elements u_h of the graded A'–B'-bimodule U' form a countable free set such that $|u_h| = |x_h|$, for all $h \geqslant 1$.*

Proof. We assume that B is a G-regular subalgebra of M_n. Since the non-zero entries of the matrices u_h are distinct variables for all the indices h, clearly it is sufficient to prove that each element u_h is torsion-free. Then, let $\sum_s a_s u b_s = 0$ with $a_s \in A'$, $b_s \in B'$. Suppose that the matrices b_s are linearly independent and that $a_s \neq 0$ for any index s. For any pair of indices (i, q) we have:

$$\sum_s \sum_{j,p} (a_s)_{ij} u_{jp} (b_s)_{pq} = 0.$$

Note that $u_{jp} \neq 0$ if and only if $|p| - |j| = |u|$. Moreover, the entries $u_{jp} \neq 0$ are variables that are distinct from those of the polynomials $(a_s)_{ij}$ and $(b_s)_{pq}$. It follows that $\sum_s (a_s)_{ij}(b_s)_{pq} = 0$, for any quadruple of indices (i, j, p, q) such that $|p| - |j| = |u|$. Since $a_1 \neq 0$, there are indices i_1, j_1 such that $(a_1)_{i_1,j_1} \neq 0$. By putting $g = |j_1| + |u|$ we have then $\sum_s (a_s)_{i_1 j_1}(b_s)_{pq} = 0$, for any indices p, q with $|p| = g$. By multiplying now this equation for e_{pq} and by summing over the indices p, q, we finally obtain:

$$\sum_s (a_s)_{i_1 j_1} \hat{\pi}_g(b_s) = 0.$$

Note that the matrices $\hat{\pi}_g(b_s)$ are linearly independent since $\hat{\pi}_g$ is a monomorphism. Since $(a_1)_{i_1 j_1} \neq 0$, we get then a contradiction. We argue in a similar way if A is a G-regular subalgebra of M_m. □

Theorem 4.5. *Let R be the G-graded block-triangular matrix algebra defined as follows*:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}$$

where $A \subset M_m$, $B \subset M_n$ are graded subalgebras and $U = M_{m\times n}$. If one of A and B is G-regular then the T_G-ideal $T_G(R)$ factorizes as:

$$T_G(R) = T_G(A)T_G(B).$$

Proof. By Proposition 4.1 we have $T_G(R) = T_G(\overline{R})$. Since $A' \approx F\langle X\rangle/T_G(A)$ and $B' \approx F\langle X\rangle/T_G(B)$, by Corollary 3.2 and Proposition 4.4 we obtain $T_G(\overline{R}) = T_G(A')T_G(B') = T_G(A)T_G(B)$. □

5. *G*-regularity for complete matrix algebras

In this section we give some examples of G-regular graded subalgebras and characterize such notion for complete matrix algebras in terms of fibers of the map $|\ |$ that defines the G-grading. Moreover, for a block-triangular matrix algebra R whose blocks along the diagonal are complete matrix algebras, under some assumptions for the group G we prove that the ideal $T_G(R)$ is factorable if and only if some of the algebras on the diagonal is G-regular.

Proposition 5.1. *Let $(M_m, |\ |_m)$ be a G-graded matrix algebra. If $|G| = n$ then there is a G-grading of M_{mn} such that the following monomorphism is graded*:

$$\varphi : M_m \to M_{mn}, \quad a \mapsto \operatorname{diag}(a,\dots,a) = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & a \end{bmatrix}.$$

Moreover, if A is any G-graded subalgebra of M_m then $\varphi(A)$ is G-regular as graded subalgebra of M_{mn}.

Proof. Say $G = \{g_0, g_1, \dots, g_{n-1}\}$. The required map $|\ | : \{1,2,\dots,mn\} \to G$ is defined as $|i + km| = |i|_m + g_k$, for all $i = 1,2,\dots,m$ and $k = 0,1,\dots,n-1$. Let now $A \subset M_m$ be a G-graded subalgebra and put $P = P(A) = P(\varphi(A))$. Note that for the generic algebra $\operatorname{Gen}_G(\varphi(A))$ one has the following chain of immersions:

$$\operatorname{Gen}_G\big(\varphi(A)\big) \subset \varphi(A) \otimes P \subset \varphi(M_m) \otimes P.$$

Therefore, it is sufficient to note that the restrictions of the F-linear maps $\pi_{g_j} : M_{mn} \otimes P \to M_{mn} \otimes P$ to the subspace:

$$\varphi(M_m) \otimes P = \big\{\operatorname{diag}(a,\dots,a) \mid a \in M_m \otimes P\big\}$$

are injective since all the entries of a matrix $a \in M_m \otimes P$ occur in the rows of degree g_j of $\mathrm{diag}(a, \ldots, a)$, for any $g_j \in G$. $\square$

Proposition 5.2. *Let M_m be any matrix algebra and $G = \mathbb{Z}_n$ the cyclic group of order n. Consider the G-graded algebra $B = \bigoplus_{i=0}^{n-1} t^i M_m$, where $t^n = 1$ and $t^i M_m$ is the homogeneous component of degree $i \in G$. Let $\varphi : B \to M_{mn}$ be the monomorphism defined as follows*:

$$\sum_{i=0}^{n-1} t^i a_i \mapsto \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \ldots & a_{n-1} & a_0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_0 & \ldots & a_{n-3} & a_{n-2} \end{bmatrix}.$$

Then, a G-grading of M_{mn} is given such that φ is graded and if $A \subset B$ is any G-graded subalgebra then $\varphi(A)$ is G-regular as graded subalgebra of M_{mn}.

Proof. The required grading of M_{mn} is defined by the map $|i + km| = k$, for all $i = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, n-1$. The remaining part of the proof is similar to the corresponding part of the proof of the previous proposition. $\square$

By the classification of the finite dimensional simple superalgebras over an algebraically closed field of characteristic different from 2 (see [21]), it holds that there are exactly two classes of such superalgebras up to isomorphisms: $M_{k,l}(F)$ with $k \geqslant l \geqslant 0, k \neq 0$ and $M_m \oplus tM_m$ with $m > 0, t^2 = 1$. From the previous proposition we have that the latter superalgebra is $\mathbb{Z}_2$-regular as embedded in M_{2m}. We prove now that the superalgebra $M_{k,l}(F)$ is $\mathbb{Z}_2$-regular as subalgebra of itself if and only if $k = l$. Actually, we prove a general result for complete matrix algebras graded by any finite abelian group G.

Proposition 5.3. *Let $A = (M_{mn}, |\ |)$ be a G-graded matrix algebra. Assume that $|G| = n$ and each fiber of $|\ |$ has exactly m elements. Put $P = P(A) = F[t_{ij}^{(h)} \mid 1 \leqslant i, j \leqslant mn, h \geqslant 1]$ and fix any integer $1 \leqslant i \leqslant mn$. Define the F-linear map*: $\rho_i : A \otimes P \to A \otimes P$, $a \mapsto \sum_j a_{ij} e_{ij}$, *where $1 \leqslant j \leqslant mn$. Then, the restriction $\hat{\rho}_i : \mathrm{Gen}_G(A) \to A \otimes P$ is an injective map.*

Proof. Put $A' = \mathrm{Gen}_G(A)$ and let $\varphi : F\langle X\rangle \to A'$ be the canonical G-graded epimorphism such that $\ker(\varphi) = T_G(A)$. Let a' be a matrix of A' such that $\rho_i(a') = 0$. Moreover, let $f \in F\langle X\rangle$ be a polynomial such that $\varphi(f) = a'$. We have to prove that $f \in T_G(A)$ that is $\nu(f) = 0$, for any G-graded evaluation $\nu : F\langle X\rangle \to A$. Fix an evaluation ν and put $a = \nu(f)$. Since $\rho_i(a') = 0$ we have that $a_{ij} = 0$, for any $j = 1, 2, \ldots, mn$.

Let now $g \in G$. We call a permutation $\sigma \in \mathbb{S}_{mn}$ *homogeneous of degree g* if $|\sigma(k)| = |k| + g$. Such a permutation induces a G-graded conjugation automorphism $\hat{\sigma} : A \to A$, $e_{pq} \mapsto e_{\sigma(p)\sigma(q)}$. Since $\hat{\sigma}\nu$ is still a graded evaluation we have $a_{\sigma(i)j} = 0$ for any $j = 1, 2, \ldots, mn$.

Because all fibers of $|\ |$ have the same number of elements, for any couple of indices $1 \leqslant h, k \leqslant mn$ there exists an homogeneous permutation σ such that $\sigma(h) = k$. Then, the index $\sigma(i)$ can assume all the values $1, 2, \ldots, mn$ and therefore $\nu(f) = a = 0$ since all the rows of this matrix have zero entries. □

It is convenient to recall here the definitions of *standard polynomial* and *Capelli polynomial* which are respectively:

$$s_n(u_1, \ldots, u_n) = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) u_{\sigma(1)} \cdots u_{\sigma(n)},$$

$$d_n(u_1, \ldots, u_n, v_1, \ldots, v_{n+1}) = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) v_1 u_{\sigma(1)} v_2 \cdots v_n u_{\sigma(n)} v_{n+1}.$$

We define also the function $\omega : G \to \mathbb{N}$, where $\omega(g)$ is the cardinality of the fiber of $g \in G$ for the map $|\;| : \{1, \ldots, m\} \to G$.

Theorem 5.4. *Let $A = (M_m, |\;|)$ be a G-graded complete matrix algebra. Then A is G-regular if and only if the map $|\;|$ is surjective and all its fibers are equipotent.*

Proof. The sufficient condition follows a fortiori by Proposition 5.3. For the necessary condition, note that for the homogeneous component A_0 we have the following decomposition:

$$A_0 = \bigoplus_{g \in G} A_0^{(g)} \tag{5}$$

where the subspace $A_0^{(g)} = \langle e_{ij} : |i| = |j| = g \rangle$ is canonically isomorphic to $M_{\omega(g)}$. Assume now that there is $g \in G$ such that $\omega(g) < \omega(g')$, for some $g' \in G$. Put $d = \omega(g)$ and consider the polynomial $s_{2d}(y_1, \ldots, y_{2d}) \in F\langle X \rangle$, where the y_i are variables of degree 0 of the set X. Fix $\nu : F\langle X \rangle \to A$ an arbitrary G-graded evaluation. Then, the matrix $\nu(s_{2d})$ is homogeneous of degree 0, and the Amitsur–Levitzki theorem implies that $\nu(s_{2d})$ has zero component in $A_0^{(g)}$ as direct summand of the decomposition (5). Moreover, the same theorem provides that there exists a graded evaluation ν' such that $\nu'(s_{2d})$ is a matrix of degree 0 which has non-zero component in $A_0^{(g')}$. Thus, s_{2d} defines a matrix $a' \in \operatorname{Gen}_G(A)$, $a' \neq 0$ such that $\hat{\pi}_g(a') = 0$. □

Note that the G-regularity of $A = (M_m, |\;|)$ is verified in particular when the order of G is exactly m and the map $|\;|$ is bijective. This is the case, for instance, when we consider the natural $\mathbb{Z}_m$-grading of M_m (see [20]).

We want to prove now that, under some assumptions, the G-regularity is a necessary condition for block-triangular matrix algebras to have a factorable T_G-ideal. For this purpose, we need the following lemmas.

Lemma 5.5. *Let $a_k = (a_{ij}^{(k)})$ be matrices of M_m, for any $k = 1, 2, \ldots, n$. Then, for any index j and pairs of indices $(i_1, j_1), \ldots, (i_n, j_n)$ of the set $\{1, 2, \ldots, m\}$, we have*:

$$d_n(a_1, \ldots, a_n, e_{ji_1}, e_{j_1 i_2}, \ldots, e_{j_{n-1} i_n}, e_{j_n j}) = \det(b) e_{jj}$$

where $b \in M_n$ is defined as $b_{hk} = a^{(k)}_{i_h j_h}$. In particular, if $a_k = e_{i_k j_k}$ are distinct matrix units for any k, then $\det(b) = 1$.

Proof. Straightforward computation. $\square$

Lemma 5.6. *The standard polynomial $s_t(u_1, \dots, u_t)$ is a polynomial identity for the block-triangular matrix algebra* $\mathrm{UT}(m, n)$ *if and only if* $t \geqslant 2(m + n)$.

Proof. Put $k = m + n$. If $t \geqslant 2k$ the polynomial $s_t(u_1, \dots, u_t)$ is an identity of $\mathrm{UT}(m, n)$ by the Amitsur–Levitski theorem. Moreover, the algebra $\mathrm{UT}(m, n)$ contains the staircase $\{e_{11}, e_{12}, \dots, e_{k-1k}, e_{kk}\}$ and therefore it has no polynomial identity of degree strictly less than $2k$. $\square$

With the notation of the decomposition (5), it holds:

Proposition 5.7. *Let G be a finite group of prime order and consider $A = (M_m, |\ |)$ a G-graded matrix algebra which is not G-regular. Fix $t \in G$ such that $\omega(t) = \max \omega(G)$. Then, there is a multilinear polynomial $f \in F\langle X\rangle$, $f \notin T_G(A)$ such that $\nu(f) \in A_0^{(t)}$ for any G-graded evaluation $\nu : F\langle X\rangle \to A$.*

Proof. Let $k = \omega(t) = \max \omega(G)$ and put $T = \{g \in G \mid \omega(g) = k\}$. Consider also $s_{2k-1}(u_1, \dots, u_{2k-1})$ the standard polynomial of degree $2k - 1$. Assuming that $|u_i| = 0$, then for each graded evaluation ν on A one has:

$$\nu(s_{2k-1}) \in \bigoplus_{\omega(g)=k} A_0^{(g)} \subset A. \tag{6}$$

In fact, the polynomial s_{2k-1} is an identity for any matrix algebra of order $< k$ and hence for $A_0^{(g)}$, when $\omega(g) < k$. If $T = \{t_1, t_2, \dots, t_r\}$, for any $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, 2k-1$ let u_{ij} be distinct variables whose G-degree is zero. By putting $c_i = s_{2k-1}(u_{i1}, \dots, u_{i2k-1})$ we claim that the required polynomial is $f = w_1 c_1 z_1 \cdots w_r c_r z_r$, where w_i and z_i are variables of degree $|w_i| = -|z_i| = t_i - t$, for any i. Say $t = t_1$ and denote $f_i = w_i c_i z_i$. For each evaluation ν, by (6) one has $\nu(f_i) \in \bigoplus_g A_0^{(g)}$ where the elements $g \in G$ are such that $t_i - t_1 + g \in T$. Therefore, it holds:

$$\nu(f) = \nu(f_1) \cdots \nu(f_r) \in \bigoplus_g A_0^{(g)}$$

where g satisfy $t_i - t_1 + g \in T$ for all $i = 1, 2, \dots, r$, that is $T - t_1 = T - g$. Then $T = T + (t_1 - g)$ and thus T is union of cosets of the subgroup $H = \langle t_1 - g\rangle \subset G$. Since A is not G-regular we have that $T \neq G$ and hence $H \neq G$. Because the order of G is prime, it follows that $H = 0$ that is $g = t_1 = t$ and hence $\nu(f) \in A_0^{(t)}$. We prove finally that $f \notin T_G(A)$.

Note that we have $s_{2k-1}(e_{11}, e_{12}, \dots, e_{k-1k}, e_{kk}) = e_{1k}$ in the matrix algebra M_k. For any $t_i \in T$, denote by a_i, b_i respectively the minimal and maximal element of the set

$\{j \mid 1 \leqslant j \leqslant m, |j| = t_i\}$. Then, there exists an evaluation μ_i on $A_0^{(t_i)} \approx M_k$ such that $\mu_i(c_i) = e_{a_i b_i}$. Fix $s \in \{1, 2, \ldots, m\}$ of degree $|s| = t$ and define $\bar{w}_i = e_{s a_i} \in A_{t_i - t}$, $\bar{z}_i = e_{b_i s} \in A_{t - t_i}$. Since $\{w_i, z_i\}$ and $\{u_{ij}\}$ are disjoint sets of variables, we have a graded evaluation ν on A such that $\nu(c_i) = \mu_i(c_i)$, $\nu(w_i) = \bar{w}_i$, $\nu(z_i) = \bar{z}_i$. Clearly we have:

$$\nu(f) = e_{ss}. \qquad \square \tag{7}$$

We can finally prove the following result:

Theorem 5.8. *Let R be the G-graded algebra defined as follows*:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}$$

where $A = (M_m, |\ |_m)$, $B = (M_n, |\ |_n)$ are G-graded complete matrix algebras and $U = M_{m \times n}$. If the finite group G has prime order then the T_G-ideal of R factorizes as $T_G(R) = T_G(A)T_G(B)$ if and only if one of the algebras A or B is G-regular.

Proof. Assume that the algebras A and B are both non-regular ones. We want to define a polynomial $f \in F\langle X\rangle$ such that $f \in T_G(R)$ but $f \notin T_G(A)T_G(B)$. Recall that the map $|\ | : \{1, 2, \ldots, m + n\} \to G$ that provides the G-grading of R is obtained as $|i| = |i|_m$ for $i \leqslant m$ and $|i| = |i - m|_n$ for $i > m$. Note that for any fixed element $g \in G$, we can obtain a new G-grading for the algebra R by defining a new map $|\ |^*$ as $|i|^* = |i|$ for $i \leqslant m$ and $|i|^* = |i| + g$ for $i > m$. We say that $|\ |^*$ is obtained *by translation of* $|\ |_n$ *by means of* g. We denote by R^* the graded algebra obtained by R using the G-grading defined by $|\ |^*$. Since R, R^* differ only for the degree of the matrix units in U one has that $T_G(A)T_G(B) \subset T_G(R^*)$. Hence, it is sufficient to prove that $f \in T_G(R) \setminus T_G(R^*)$.

Denote by ω_A and ω_B the functions corresponding to the maps $|\ |_m$ and $|\ |_n$. Owing to Theorem 5.4, we have that $h = \max \omega_A(G) > \min \omega_A(G)$ and $k = \max \omega_B(G) > \min \omega_B(G)$. We consider two cases. The first one is that there is no $g \in G$ such that $\omega_A(g) = h$ and $\omega_B(g) = k$. In this case, we have that $h + k > \omega_A(g) + \omega_B(g)$ for any $g \in G$. If we put $p = 2(h + k) - 1$, then the required polynomial is $f = s_p(y_1, \ldots, y_p)$ where the variables y_i have all G-degree equal to zero. In fact, the homogeneous component R_0 can be decomposed as the following direct sum of subalgebras:

$$R_0 = \bigoplus_{g \in G} R_0^{(g)}$$

where $R_0^{(g)} = \langle e_{ij}\colon |i| = |j| = g\rangle$. Such subalgebras are canonically isomorphic to the block-triangular matrix algebras $\mathrm{UT}(\omega_A(g), \omega_B(g))$ and hence $f \in \mathrm{T}_G(R)$ by virtue of the Lemma 5.6.

Let now $a, b \in G$ such that $\omega_A(a) = h$, $\omega_B(b) = k$ and put $c = a - b$. Let $|\ |^*$ be the map obtained by translation of $|\ |_n$ by means of c and denote by R^* the corresponding G-graded algebra. Similarly to what we have for R, for the decomposition of the homogeneous component R_0^* one has that the direct summand corresponding to the degree a is canonically isomorphic to $\mathrm{UT}(h,k)$. Then, still by Lemma 5.6 it follows that $f \notin T_G(R^*)$.

We consider now the case that there exists $t \in G$ such that $\omega_A(t) = h = \max \omega_A(G)$ and $\omega_B(t) = k = \max \omega_B(G)$. Then, by Proposition 5.7 there are multilinear polynomials $f_A, f_B \in F\langle X\rangle$ defined on disjoint sets of variables which verify the following properties:

(a) $f_A \notin T_G(A)$, $f_B \notin T_G(B)$;
(b) for any G-graded evaluations $\nu_A : F\langle X\rangle \to A$, $\nu_B : F\langle X\rangle \to B$ one has $\nu_A(f_A) \in A_0^{(t)}$, $\nu_B(f_B) \in B_0^{(t)}$.

Suppose now that d is a multilinear polynomial whose variables are different from those of f_A, f_B. Suppose also that the G-degree of d is different from zero and $\nu(d) \in U$, for any graded evaluation $\nu : F\langle X\rangle \to R$. We will prove later that such a polynomial exists. In this case, we can define the polynomial $f = f_A d f_B$ and prove that it belongs to the ideal $T_G(R)$ but not to $T_G(R^*)$. By contradiction, assume that there is a G-graded evaluation $\nu : F\langle X\rangle \to R$ such that $\nu(f) \neq 0$. Because f is multilinear we can assume that all variables are evaluated by ν at matrix units. Since we have:

$$AB = BA = BU = UA = U^2 = 0$$

and $\nu(d) \in U$, this implies that ν maps the variables of f_A and f_B respectively into elements of A and B. Thus, by means of the property (b) we have that $\nu(f_A) \in A_0^{(t)}$, $\nu(f_B) \in B_0^{(t)}$.

Consider now any $a \in A_0^{(t)} \subset R$ and $b \in B_0^{(t)} \subset R$, and let $e_{ij} \in U$. Note that if the product $a e_{ij} b \neq 0$ then $|i| = |j| = t$ and hence $|e_{ij}| = 0$. Since $\nu(d)$ is an element of U whose G-degree is different of zero, by $\nu(f) \neq 0$ we get a contradiction and therefore $f \in T_G(R)$.

We define now the required polynomial d. Fix $g \in G$, $g \neq 0$ and consider the homogeneous components A_g, B_g. If we put $p = \max(\dim_F A_g, \dim_F B_g) + 1$ then d is a Capelli polynomial $d_p(u_1, \ldots, u_p, v_1, \ldots, v_{p+1})$ whose variables are different from those of f_A, f_B, the total G-degree is different from zero and $|u_i| = g$ for any i. Let ν be an arbitrary G-graded evaluation on R. We show that $\nu(d) \in U$.

Since d is multilinear we can assume that all variables are evaluated at matrix units. Note that if $\nu(u_i) \in A$ or $\nu(u_i) \in B$ for all i then $\nu(d) = 0$, since d is alternating for the p variables u_i. Thus, if $\nu(d) \neq 0$ then $\nu(d) \in U$ since $AB = BA = 0$ and U is an ideal of R.

It remains to be proved that $f \notin T_G(R^*)$ for some map $|\ |^*$. We show that for a suitable choice of G-degrees for the variables $v_1, \ldots, v_{p+1}$ of $d = d_p(u_1, \ldots, u_p, v_1, \ldots, v_{p+1})$, there is a graded evaluation $\nu : F\langle X\rangle \to R^*$ such that $\nu(f) \neq 0$.

Say $\dim_F A_g \geqslant \dim_F B_g$ that is $p - 1 = \dim_F A_g$. Denote by $a_1, \ldots, a_{p-1}$ the F-linear basis of A_g given by matrix units. Owing to Lemma 5.5, for any $i = 1, 2, \ldots, m$ there are suitable matrix units $a'_1, \ldots, a'_p$ in A such that $d_{p-1}(a_1, \ldots, a_{p-1}, a'_1, \ldots, a'_p) = e_{ii}$. For $j > m$, we put $a_p = e_{ij}$ and $a'_{p+1} = e_{jj}$. Since a_p is an element of U one has that $d_p(a_1, \ldots, a_p, a'_1, \ldots, a'_{p+1}) = e_{ij}$. In particular, we may choose indices i, j such that $|i| = |j| = t$. Consider now the map $|\ |^*$ obtained by translating $|\ |_n$ by means of g and let R^* be the corresponding G-graded algebra. In R^* we have then $|a_q|^* = |a_q| = g$ for $q = 1, 2, \ldots, p-1$ and $|a'_q|^* = |a'_q|$ for $q = 1, 2, \ldots, p$. Moreover, it holds:

$$|a_p|^* = |j|^* - |i|^* = t + g - t = g.$$

By putting $|v_q| = |a'_q|^*$ for any $q = 1, 2, \ldots, p+1$, a graded evaluation ν on R^* is clearly given such that $\nu(d_p) = e_{ij}$. By using Eq. (7) with $s = i, s = j$ for the polynomials f_A, f_B respectively, we conclude that $\nu(f) \neq 0$. □

6. Cocharacters of superalgebras with factorable T_2-ideal

In this section we assume that the field F has characteristic zero. We generalize to the $\mathbb{Z}_2$-graded algebras the results of Berele–Regev [4] about the sequence of the cocharacters of ordinary PI-algebras whose T-ideal is factorable as a product of two T-ideals. Such generalization can be further extended to any G-grading, but we avoid to do this to keep the notations reasonably simple and since we apply these results just to superalgebras. We start with the following basic result:

Theorem 6.1 (Formanek [9,13]). *Let I, J be m-multigraded ideals of the free algebra $\mathcal{P} = F\langle X_m\rangle$. Then, we have*:

$$\mathrm{HP}_m(\mathcal{P}/IJ) = \mathrm{HP}_m(\mathcal{P}/I) + \mathrm{HP}_m(\mathcal{P}/J) - \frac{\mathrm{HP}_m(\mathcal{P}/I)\mathrm{HP}_m(\mathcal{P}/J)}{\mathrm{HP}_m(\mathcal{P})}.$$

In what follows it is useful to note that the Hilbert–Poincaré series of the algebra $F\langle X_m\rangle$ is $\mathrm{HP}_m(F\langle X_m\rangle) = 1/(1 - t_1 - \cdots - t_m)$. Moreover, in terms of Schur functions s_λ one has:

$$\begin{aligned} -1/\mathrm{HP}_m\big(F\langle X_m\rangle\big) &= t_1 + \cdots + t_m - 1 = \xi_1 + \cdots + \xi_k + \eta_1 + \cdots + \eta_l - 1 \\ &= s_{(1)}(\xi_1, \ldots, \xi_k)s_{(0)}(\eta_1, \ldots, \eta_l) + s_{(0)}(\xi_1, \ldots, \xi_k)s_{(1)}(\eta_1, \ldots, \eta_l) - 1 \end{aligned}$$

where we denote by (i) the row-partition of length i.

Let now χ'_k, χ''_k ($k \geqslant 0$) be two sequences of characters of the group $\mathbb{S}_k$. We define $(\chi' \circ \chi'')_k$ to be the following sequence of characters obtained *by convolution* from χ'_k, χ''_k:

$$\big(\chi' \circ \chi''\big)_k = \sum_{i=0}^{k} \chi'_i \mathbin{\hat{\otimes}} \chi''_{k-i}$$

where $\hat{\otimes}$ denotes the outer tensor product of the characters of $\mathbb{S}_k$. In a similar way, if $\chi'_{k,l}, \chi''_{k,l}$ ($k, l \geqslant 0$) are two sequences of characters of the product group $\mathbb{S}_k \times \mathbb{S}_l$, we define $(\chi' \circ \chi'')_{k,l}$ to be the following sequence of characters:

$$\big(\chi' \circ \chi''\big)_{k,l} = \sum_{i=0}^{k} \sum_{j=0}^{l} \chi'_{i,j} \mathbin{\hat{\otimes}} \chi''_{k-i,l-j}$$

where $\hat{\otimes}$ is now the outer tensor product of the characters of $\mathbb{S}_k \times \mathbb{S}_l$. Explicitly, for the irreducible characters $\chi_{\mu',\nu'}$, $\chi_{\mu'',\nu''}$ one has that $\chi_{\mu',\nu'} \hat{\otimes} \chi_{\mu'',\nu''} = (\chi_{\mu'} \hat{\otimes} \chi_{\mu''}) \otimes (\chi_{\nu'} \hat{\otimes} \chi_{\nu''})$. We are ready to prove the following result:

Theorem 6.2. *Let I, J be T_2-ideals of the superalgebras A, B respectively. Denote by R any superalgebra whose T_2-ideal factorizes as the product IJ. Then, the $\mathbb{Z}_2$-graded cocharacters $\chi_{k,l}(R)$ of this algebra verify*:

$$\chi_{k,l}(R) = \chi_{k,l}(A) + \chi_{k,l}(B) + \chi_{(1),(0)} \hat{\otimes} \big(\chi(A) \circ \chi(B)\big)_{k-1,l} + \chi_{(0),(1)} \hat{\otimes} \big(\chi(A) \circ \chi(B)\big)_{k,l-1} - \big(\chi(A) \circ \chi(B)\big)_{k,l}. \tag{8}$$

Proof. By (1) and (2) it is sufficient to argue for the Hilbert–Poincaré series $\mathrm{HP}_{k,l}(R)$. By Theorem 6.1 we get:

$$\begin{aligned}
\mathrm{HP}_{k,l}(R) &= \sum_{\mu,\nu} m_{\mu,\nu}(R) s_\mu(\xi) s_\nu(\eta) \\
&= \alpha + \beta \times \sum_{\mu',\nu'} m_{\mu',\nu'}(A) s_{\mu'}(\xi) s_{\nu'}(\eta) \times \sum_{\mu'',\nu''} m_{\mu'',\nu''}(B) s_{\mu''}(\xi) s_{\nu''}(\eta) \\
&= \alpha + \beta \times \sum_{\mu,\nu} \Bigg[\sum_{\mu',\mu'',\nu',\nu''} m_{\mu',\nu'}(A) m_{\mu'',\nu''}(B) c^{\mu}_{\mu',\mu''} c^{\nu}_{\nu',\nu''} \Bigg] s_\mu(\xi) s_\nu(\eta)
\end{aligned}$$

where $\alpha = \sum_{\mu,\nu} [m_{\mu,\nu}(A) + m_{\mu,\nu}(B)] s_\mu(\xi) s_\nu(\eta)$, $\beta = -1/\mathrm{HP}_m(F\langle X_m \rangle)$ and $c^{\lambda}_{\lambda'\lambda''}$ are the Littlewood–Richardson numbers that is:

$$\begin{aligned}
s_{\mu'}(\xi_1, \dots, \xi_k) s_{\mu''}(\xi_1, \dots, \xi_k) &= \sum_{\mu} c^{\mu}_{\mu',\mu''} s_\mu(\xi_1, \dots, \xi_k), \\
s_{\nu'}(\eta_1, \dots, \eta_l) s_{\nu''}(\eta_1, \dots, \eta_l) &= \sum_{\nu} c^{\nu}_{\nu',\nu''} s_\nu(\eta_1, \dots, \eta_l).
\end{aligned}$$

For the outer product of irreducible characters of $\mathbb{S}_k \times \mathbb{S}_l$ we have also that $\chi_{\mu',\nu'} \hat{\otimes} \chi_{\mu'',\nu''} = (\chi_{\mu'} \hat{\otimes} \chi_{\mu''}) \otimes (\chi_{\nu'} \hat{\otimes} \chi_{\nu''}) = \sum_{\mu,\nu} c^{\mu}_{\mu',\mu''} c^{\nu}_{\nu',\nu''} \chi_{\mu,\nu}$ and hence:

$$\big(\chi(A) \circ \chi(B)\big)_{k,l} = \sum_{\mu,\nu} \Bigg[\sum_{\mu',\mu'',\nu',\nu''} m_{\mu',\nu'}(A) m_{\mu'',\nu''}(B) c^{\mu}_{\mu',\mu''} c^{\nu}_{\nu',\nu''} \Bigg] \chi_{\mu,\nu}.$$

Since $\beta = s_{(1)}(\xi) s_{(0)}(\eta) + s_{(0)}(\xi) s_{(1)}(\eta) - 1$, we get finally Eq. (8). □

It is important to study how the convolution product of sequences of characters decomposes as sum of irreducibles (see for instance [18]). By the previous theorem such study is needed for computing the cocharacters of superalgebras whose T_2-ideal is factorable. In particular, the convolution of characters of row-partitions are relevant for the matrix algebras.

For a fixed integer $i \geqslant 0$, denote by $\chi_{(i)}$ the irreducible character of $\mathbb{S}_k$ associated to the row-partition (i). We write $\phi = \{\chi_{(i)}\}_{i \geqslant 0}$ for the corresponding sequence of characters. We define:

$$\mathrm{Y}_k^{(n)} = (\underbrace{\phi \circ \cdots \circ \phi}_{n})_k = \sum_{\mu} \chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_n)}$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is any n-tuple of integers $\mu_i \geqslant 0$ such that $\mu_1 + \cdots + \mu_n = k$. We call $\mathrm{Y}_k^{(n)}$ the *nth Young character* of the symmetric group $\mathbb{S}_k$. In the terminology used by Regev in [18], the sequence of characters $\mathrm{Y}_k^{(n)}$ ($k \geqslant 0$) is said to be *Young-derived n times* by the sequence of characters $\psi_0 = 1$ and $\psi_k = 0$ for $k > 0$. Clearly $\mathrm{Y}_k^{(n)}$ is the character of the representation of $\mathbb{S}_k$ on the tensor power $\mathrm{T}^{(k)}(V)$ of a vector space V of dimension n. We refer the reader to the books [10,14,17] for the knowledge on algebraic combinatorics and representation theory of symmetric and general linear groups.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be a n-tuple of non-negative integers whose sum is equal to k. Then, we may think of μ as the *content* in n letters of a semistandard tableaux of shape λ, where λ is any partition of the integer k. By definition, the *Kostka number* $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and content μ. By means of the Pieri–Young rule, such number is the multiplicity of the irreducible character χ_λ in the decomposition of the character $\chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_n)}$, that is:

$$\chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_n)} = \sum_{\lambda} K_{\lambda\mu} \chi_\lambda$$

where the partitions λ are all of height $\leqslant n$. By summing over all the k-contents μ in n letters, we have:

Proposition 6.3. *The decomposition of nth Young character $\mathrm{Y}_k^{(n)}$ is the following*:

$$\mathrm{Y}_k^{(n)} = \sum_{\lambda} K_\lambda^{(n)} \chi_\lambda$$

where λ ranges over all the partitions of k with height at most n and the multiplicity $K_\lambda^{(n)} = \sum_\mu K_{\lambda\mu}$ is equal to the total number of semistandard tableaux of shape λ and content in n letters.

It is well known that the number $K_\lambda^{(n)}$ is equal to the value of the Schur function $s_\lambda(t_1, \ldots, t_n)$ in n variables for $t_1 = \cdots = t_n = 1$. It can be proved (see [10, Chapter 6]) that:

$$K_\lambda^{(n)} = s_\lambda(1, \ldots, 1) = \prod_{1 \leqslant i < j \leqslant n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{9}$$

where λ is any partition of height at most n.

We apply now the previous results to compute the cocharacters of a superalgebra which has a factorable T_2-ideal.

Example 6.4. Consider the block-triangular matrix superalgebra:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}$$

where $A = M_{1,0}(F)$, $B = M_{1,1}(F)$ and $U = M_{1\times 2}$. Denote $m_{\mu,\nu}$ $(\mu \vdash k, \nu \vdash l)$ the non-zero multiplicities of the $\mathbb{Z}_2$-graded cocharacter $\chi_{k,l}(R)$. For $l > 1$, these multiplicities are given by the following table:

μ/ν	(u)	(u,v)	(u,u)	$(u,v,1)$
(a,b)	α	β	α	$\gamma(a,b,0)$
(a,b,c)	3γ	4γ	3γ	γ
$(a,b,c,1)$	γ	γ	γ	

where b can be equal to zero, the integers $c, v \neq 0$ and $u \neq v$ for the partition (u,v). Moreover, we denote by α, β, γ the following polynomials:

$$\alpha = 1/2 \cdot (a-b+1)(2ab+a+3b+2),$$

$$\beta = 1/2 \cdot (a-b+1)(3ab+2a+5b+4),$$

$$\gamma = K^{(3)}_{(a,b,c)} = 1/2 \cdot (a-b+1)(b-c+1)(a-c+2).$$

For $l = 0$, the table of the $m_{\mu,\nu}$ is the following:

μ/ν	(0)
(a)	1
(a,b)	α
$(a,b,1)$	α

where $b \neq 0$ and $\alpha = K^{(2)}_{(a,b)} = a - b + 1$. Finally, for $l = 1$ we have:

μ/ν	(1)
(a)	$\alpha(a,0)$
(a,b)	α
(a,b,c)	2β
$(a,b,c,1)$	β

where the integers $b, c \neq 0$ and the polynomials α, β are the following:

$$\alpha = 1/2 \cdot (a-b+1)(ab+b+2), \qquad \beta = K^{(3)}_{(a,b,c)}.$$

For computing the multiplicities $m_{\mu,\nu}$ note that $B = M_{1,1}(F)$ is a $\mathbb{Z}_2$-regular algebra and hence $T_2(R) = T_2(A)T_2(B)$ by Theorem 4.5. Then, we can apply the formula (8). For the superalgebra A we have clearly:

$$\chi_{k,l}(A) = \begin{cases} \Upsilon_k^{(1)} \otimes \chi_{(0)} & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the $\mathbb{Z}_2$-graded cocharacter of B has been computed in [5] as:

$$\chi_{k,l}(B) = \begin{cases} \Upsilon_k^{(1)} \otimes \chi_{(0)} & \text{if } l = 0, \\ \Upsilon_k^{(2)} \otimes \sum_\nu \chi_\nu & \text{otherwise,} \end{cases}$$

where $\nu \vdash l$ ranges over all the partitions with height $\leqslant 2$. By using the property $\Upsilon_k^{(i)} \circ \Upsilon_k^{(j)} = \Upsilon_k^{(i+j)}$ we obtain therefore:

$$\big(\chi(A) \circ \chi(B)\big)_{k,l} = \begin{cases} \Upsilon_k^{(2)} \otimes \chi_{(0)} & \text{if } l = 0, \\ \Upsilon_k^{(3)} \otimes \sum_\nu \chi_\nu & \text{otherwise.} \end{cases}$$

Then, we get the multiplicities $m_{\mu,\nu}$ by applying the formula (9).

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