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Boundary Effect on Asymptotic Behaviour of Solutions to the p-System with Linear Damping

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We consider the asymptotic behaviour of solutions to the p-system with linear damping on the half-line $\mathbf{R}_+ = (0, \infty)$,

$$v_t - u_x = 0, \qquad u_t + p(v)_x = -\alpha u,$$

with the Dirichlet boundary condition $u|_{x=0} = 0$ or the Neumann boundary condition $u_x|_{x=0} = 0$. The initial date $(v_0, u_0)(x)$ has the constant state (v_+, u_+) at $x = \infty$. L. Hsiao and T.-P. Liu [*Commun. Math. Phys.* **143** (1992), 599–605] have shown that the solution to the corresponding Cauchy problem behaves like diffusion wave, and K. Nishihara [*J. Differential Equations* **131** (1996), 171–188; **137** (1997), 384–395] has proved its optimal convergence rate. Our main concern in this paper is the boundary effect. In the case of null-Dirichlet boundary condition on u, the solution (v, u) is proved to tend to $(v_+, 0)$ as t tends to infinity. Its optimal convergence rate is also obtained by using the Green function of the diffusion equation with constant coefficients. In the case of null-Neumann boundary condition on u, $v(0, t)$ is conservative and $v(0, t) \equiv v_0(0)$ by virtue of the first equation, so that $v(x, t)$ is expected to tend to the diffusion wave $\bar{v}(x, t)$ connecting $v_0(0)$ and v_+. In fact the solution $(v, u)(x, t)$ is proved to tend to $(\bar{v}(x, t), 0)$. In the special case $v_0(0) = v_+$, the optimal convergence rate is also obtained. However, this is not known in the case $v_0(0) \neq v_+$. © 1999 Academic Press

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1. INTRODUCTION

In this paper we consider the initial-boundary value problem for the p-system with linear damping

$$\begin{aligned} &v_t - u_x = 0 \\ &u_t + p(v)_x = -\alpha u, \qquad x \in \mathbf{R}_+ = (0, \infty), \quad t > 0, \end{aligned} \tag{1.1}$$

with the initial data

$$(v, u)(x, 0) = (v_0, u_0)(x) \to (v_+, u_+), \quad v_+ > 0, \qquad \text{as} \quad x \to \infty, \tag{1.2}$$

and with the Dirichlet boundary or the Neumann boundary condition. Eq. (1.1) models a one-dimensional compressible flow through porous media. Here, $v > 0$ is the specific volume; u is the velocity; the pressure p is a smooth function of v with $p > 0$, $p' < 0$; and α is a positive constant.

For the Cauchy problem to (1.1), the solutions were shown by Hsiao and Liu [4, 5] to time-asymptotically behave like those of Darcy's law,

$$\begin{aligned} &\bar{v}_t - \bar{u}_x = 0 \\ &p(\bar{v})_x = -\alpha \bar{u}, \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} &\bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx} \\ &p(\bar{v})_x = -\alpha \bar{u}. \end{aligned} \tag{1.3$'$}$$

A better convergence rate and the optimal convergence rate when $v(\infty, 0) = v(-\infty, 0)$ were obtained by Nishihara [14, 15] by the enegy method and the pointwise estimate. For a related problem, see [3, 6] and references therein. See also the book [2] by Hsiao.

Although the initial-boundary value problems on $\mathbf{R}_+$ to the equations of viscous conservation laws have been recently investigated by several authors [7–9, 12, 13, 16], there are few works on (1.1) as far as we know. Our results discussed below show that even for the case with boundary condition, the Dirichlet or the Neumann boundary condition at $x = 0$, the solutions of (1.3) capture the time-asymptotic behaviour of the solutions to (1.1). In the case of the Dirichlet boundary condition

$$u(0, t) = 0, \tag{1.4}$$

we show that the solution $(v, u)(x, t)$ converges to $(v_+, 0)$ as $t \to \infty$. Futhermore, since the solution converges to a constant state, the analysis of [14, 15] can be applied and the optimal convergence rate is obtained. In the case of the Neumann boundary condition

$$u_x(0, t) = 0, \tag{1.5}$$

$(1.1)_1$ (the first equation of (1.1)) heuristically yields $(d/dt)\, v(0, t) = 0$ and $v(0, t) = v_0(0)$. Hence, when $v_0(0) \neq v_+$, the solution $(v, u)(x, t)$ will be shown to converge to the profile $(\bar{v}, \bar{u})$ of (1.3) in the form of $\bar{v} = \psi(\xi)$, $\xi = x/\sqrt{t+1}$, with $\psi(+\infty) = v_+$ and $\psi(0) = v_0(0)$. Eventually, if $v_0(0) = v_+$, $\bar{v}(x, t) \equiv v_+$, then the analysis in [14, 15] can also be applied and the optimal convergence rate is obtained.

Both problems are reformulated to the perturbed problems from the diffusion wave $(\bar{v}, \bar{u})(x, t)$ and the auxiliary function $(\hat{v}, \hat{u})(x, t)$, which are defined in a similar fashion to those in Hsiao and Liu [4]. These will be stated in later sections.

Here, we briefly mention the condition (1.5), which corresponds to the Dirichlet condition $v(0, t) = v_-$ (given constant) on v from the discussion above. Recently (1.1), with (1.2) and $v(0, t) = g(t)$, $g(t) \to v_+$ has been considered by Marcati and Mei [10]. However, the case $g(t) \equiv v_-(\neq v_+)$ or $g(t) \to v_-(\neq v_+)$ is not treated there.

The content of our paper is as follows. After we state the notations, in Section 2 the problem with the Dirichlet boundary condition is reformulated and the results will be stated. In Subsection 2.2 the proofs of theorems will be given; much of that subsection is based on the papers [14, 15]. In Section 3 the Neumann boundary problem will be considered.

Notations. We denote several positive constants depending on $a, b, \ldots$ by $C_{a, b, \ldots}$ or only by C without confusion. For function spaces, $L^p = L^p(\mathbf{R}_+)$ $(1 \leqslant p \leqslant \infty)$ is a usual Lebesgue space with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}_+} |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leqslant p < \infty \qquad \text{and} \qquad \|f\|_{L^\infty} = \sup_{\mathbf{R}_+} |f(x)|.$$

The L^2-norm on $\mathbf{R}_+$ is simply denoted by $\|\cdot\|$. H^l $(l \geqslant 0)$ denotes the usual lth order Sobolev space on $\mathbf{R}_+$ with its norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{1/2}, \qquad \|\cdot\| = \|\cdot\|_0 = \|\cdot\|_{L^2}.$$

2. THE CASE OF THE DIRICHLET BOUNDARY CONDITION

2.1. *Reformulation of the Problem and Theorems*

We first reformulate the problem (1.1), (1.2) with the Dirichlet boundary condition (1.4). Expecting

$$(v, u)(x, t) \to (v_+, 0), \qquad t \to \infty, \tag{2.1}$$

we put $u_t \equiv 0$ to have (1.3) or (1.3)′ with $u(0, t) = v_x(0, t) = 0$, $v(+\infty, t) = v_+$. Approximating this by the solution $\bar{v}(x, t)$ of

$$\bar{v}_t - \kappa \bar{v}_{xx} = 0, \qquad \bar{v}_x(0, t) = 0, \qquad \bar{v}(+\infty, t) = v_+, \tag{2.2}$$

or explicitly,

$$\bar{v}(x, t) = v_+ + \frac{\delta_0}{\sqrt{4\kappa\pi(t+1)}} \exp\left(-\frac{x^2}{4\kappa(t+1)}\right), \tag{2.3}$$

where $\kappa := -p'(v_+)/\alpha > 0$ and δ_0 is defined by

$$\delta_0 = 2\left(\int_0^\infty (v_0(x) - v_+)\, dx - \frac{u_+}{\alpha}\right). \tag{2.4}$$

We set

$$\bar{u}(x, t) = -\frac{p'(v_+)}{\alpha} \bar{v}_x(x, t) = \kappa \bar{v}_x(x, t)$$

so that $\bar{u}|_{x=0} = 0$ because $\bar{v}_x|_{x=0} = 0$.

Thus, $(\bar{v}, \bar{u})(x, t)$, called the diffusion wave, satisfies

$$\begin{aligned} \bar{v}_t - \bar{u}_x &= 0 \\ p'(v_+)\, \bar{v}_x &= -\alpha \bar{u} \\ \bar{u}|_{x=0} = 0, \qquad (\bar{v}, \bar{u})|_{x=\infty} &= (v_+, 0). \end{aligned} \tag{2.5}$$

Next, expecting $u(+\infty, t) = u_+ e^{-\alpha t}$, we define the auxiliary function $(\hat{v}, \hat{u})(x, t)$ by

$$(\hat{v}, \hat{u})(x, t) = \left(\frac{u_+ m_0(x)}{-\alpha} e^{-\alpha t}, u_+ \int_0^x m_0(y)\, dy \cdot e^{-\alpha t}\right), \tag{2.6}$$

where m_0 is a smooth function with compact support such that

$$\int_0^\infty m_0(y)\, dy = 1, \qquad \operatorname{supp} m_0 \subset \mathbf{R}_+. \tag{2.7}$$

Therefore, $(\hat{v}, \hat{u})(x, t)$ satisfies

$$\begin{aligned} &\hat{v}_t - \hat{u}_x = 0 \\ &\hat{u}_t = -\alpha \hat{u} \\ &\hat{u}|_{x=0} = 0, \qquad (\hat{v}, \hat{u})|_{x=\infty} = (0, u_+ e^{-\alpha t}). \end{aligned} \tag{2.8}$$

Combining (1.1) with (2.5) and (2.8) we have

$$\begin{aligned} &(v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0 \\ &(u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x = -\alpha(u - \bar{u} - \hat{u}) - \bar{u}_t + (p'(v_+) - p'(\bar{v}))\, \bar{v}_x. \end{aligned} \tag{2.9}$$

By virtue of $(2.9)_1$ and (2.4),

$$\int_0^\infty (v - \bar{v} - \hat{v})(y, t)\, dy = \int_0^\infty (v_0(x) - v_+)\, dx - \frac{\delta_0}{2} - \frac{u_+}{\alpha} = 0,$$

and hence we reach the setting of perturbation

$$\begin{aligned} V(x, t) &= -\int_x^\infty (v - \bar{v} - \hat{v})(y, t)\, dy \\ z(x, t) &= u(x, t) - \bar{u}(x, t) - \hat{u}(x, t) \end{aligned} \tag{2.10}$$

and the reformulated problem, after the integration of $(2.9)_1$ once over (x, ∞),

$$\begin{aligned} &V_t - z = 0 \\ &z_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \alpha z = -\bar{u}_t + (p'(v_+) - p'(\bar{v}))\, \bar{v}_x \\ &(V, z)|_{t=0} = (V_0, z_0)(x) \\ &\quad := \left(-\int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0))\, dy,\ u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0) \right) \\ &z|_{x=0} = 0, \end{aligned} \tag{RP}$$

or the linearized problem around $\bar{v}$

$$\begin{aligned} &V_t - z = 0 \\ &z_t + (p'(\bar{v})\, V_x)_x + \alpha z = -F \\ &(V, z)|_{t=0} = (V_0, z_0)(x), \qquad z|_{x=0} = 0, \end{aligned} \tag{LP}$$

where

$$F = \frac{p'(v_+)}{\alpha} \bar{v}_{xt} - (p'(v_+) - p'(\bar{v}))\, \bar{v}_x + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})\, V_x)_x. \tag{2.11}$$

Noting that, by (2.4),

$$|\delta_0| \leqslant 2 \left(\|v_0 - v_+\|_{L^1} + \frac{|u_+|}{\alpha} \right), \tag{2.12}$$

we obtain the following first theorem.

THEOREM 2.1 (Dirichlet boundary). *Suppose that* $v_0 - v_+$ *is in* L^1, $(V_0, z_0) \in H^3 \times H^2$ *and that both* $\|v_0 - v_+\|_{L^1} + \|V_0\|_3 + \|z_0\|_2$ *and* $|u_+|$ *are sufficiently small. Then there exists a unique time-global solution* $(V, z)(x, t)$ *of* (RP), *which satisfies*

$$V \in C^i([0, \infty); H^{3-i}), \qquad i = 0, 1, 2, 3$$

$$z \in C^i([0, \infty); H^{2-i}), \qquad i = 0, 1, 2$$

and moreover

$$\sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2$$
$$+ \int_0^t \left[\sum_{j=1}^{3} (1+\tau)^{j-1} \|\partial_x^j V(\cdot, \tau)\|^2 + \sum_{j=0}^{2} (1+\tau)^{j+1} \|\partial_x^j z(\cdot, \tau)\|^2 \right] d\tau$$
$$\leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|), \tag{2.13}$$

and

$$(1+t)^4 \|z_t(\cdot, t)\|^2 + (1+t)^5 (\|z_{xt}(\cdot, t)\|^2 + \|z_{tt}(\cdot, t)\|^2)$$
$$+ \int_0^t [(1+\tau)^4 \|z_{xt}(\cdot, \tau)\|^2 + (1+\tau)^5 \|z_{tt}(\cdot, \tau)\|^2]\, d\tau$$
$$\leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|). \tag{2.14}$$

The solution (V, z) obtained in Theorem 2.1 satisfies $V|_{x=0} = 0$ by the first equation of (RP) and the boundary condition $z|_{x=0} = 0$, and hence (RP) or (LP) can be rewritten as the problem to the second order wave equation of V with linear damping

$$\begin{aligned} &V_{tt} + (p'(\bar{v})\, V_x)_x + \alpha V_t = -F \\ &(V, V_t)|_{t=0} = (V_0, z_0)(x), \qquad V|_{x=0} = 0. \end{aligned} \tag{2.15}$$

Moreover, we rewrite $(2.15)_1$ to the linearized parabolic problem around v_+,

$$V_t - \kappa V_{xx} = -\frac{1}{\alpha}(V_{tt} + \tilde{F}) + \frac{1}{\alpha}(p'(v_+) - p'(\bar{v}))\,\bar{v}_x, \qquad \kappa = -\frac{p'(v_+)}{\alpha}, \tag{2.16}$$

to use the Green function of the parabolic equation with null-Dirichlet boundary

$$E(x, t; y) = \frac{1}{\sqrt{4\pi\kappa t}}(e^{-(x-y)^2/4\kappa t} - e^{-(x+y)^2/4\kappa t}), \tag{2.17}$$

where

$$\tilde{F} = \frac{p'(v_+)}{\alpha}\bar{v}_{xt} + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})\,V_x)_x - ((p'(v_+) - p'(\bar{v}))\,V_x)_x. \tag{2.18}$$

Hence we have the explicit formula of V:

$$\begin{aligned} V(x, t) &= \int_0^\infty E(x, t; y)\,V_0(y)\,dy \\ &\quad -\frac{1}{\alpha}\int_0^t\int_0^\infty E(x, t-\tau; y)(V_{tt} + \tilde{F})(y, \tau)\,dy\,d\tau \\ &\quad +\frac{1}{\alpha}\int_0^t\int_0^\infty E(x, t-\tau; y)(p'(v_+) - p'(\bar{v}))\,\bar{v}_x(y, \tau)\,dy\,d\tau. \end{aligned} \tag{2.19}$$

Define $\phi(x, t)$ by

$$\begin{aligned} \phi(x, t) &= \int_0^\infty E(x, t; y)\left(V_0(y) + \frac{1}{\alpha}z_0(y)\right)dy \\ &\quad +\frac{1}{\alpha}\int_0^t\int_0^\infty E(x, t-\tau; y)(p'(v_+) - p'(\bar{v}))\,\bar{v}_x(y, t)\,dy\,d\tau \end{aligned} \tag{2.20}$$

or the solution of

$$\begin{cases} \phi_t - \kappa\phi_{xx} = \dfrac{1}{\alpha}(p'(v_+) - p'(\bar{v}))\,\bar{v}_x, & (x, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ \phi(x, 0) = V_0(x) + \dfrac{1}{\alpha}z_0(x), & \phi(0, t) = 0. \end{cases} \tag{2.20$'$}$$

Then we have the asymptotic profile ϕ of V as $t \to \infty$ in the sense of the following theorem.

THEOREM 2.2 (Asymptotic Profile). *Define* ϕ *by* (2.20) *or* (2.21) *and suppose that* $(V_0, z_0) \in L^1 \times L^1$. *Then the solution* (V, z) *of* (RP) *obtained in Theorem* 2.1 *satisfies*

$$\|(V-\phi, (V-\phi)_x, (V-\phi)_t)(\cdot, t)\|_{L^\infty} = O(t^{-1}\ln t, t^{-3/2}\ln t, t^{-2}\ln t) \quad (2.21)$$

as $t \to \infty$.

Remark 2.1. Since ϕ satisfies

$$\|(\phi, \phi_x, \phi_t)(\cdot, t)\|_{L^\infty} = O(t^{-1/2}, t^{-1}, t^{-3/2}), \quad (2.22)$$

ϕ is generally an asymptotic profile of V as $t \to \infty$, which is on the same line of assertions in [15]. However, in the present case we have the slightly worse term $-((p'(v_+) - p'(\bar{v}))\, V_x)_x$ in $\tilde{F}$ and hence $\ln t$ in (2.21) are added.

Remark 2.2. All results are obtained under the condition that any data are small. For large data the singularity will generally develop after a finite time and the weak solution must be considered. In such cases the asymptotic behaviour of the solutions of (1.1) is unknown in general even for Cauchy problem.

2.2. *Proofs of Theorems*

First, applying the L^2-energy method we prove Theorem 2.1, which is established by the combination of the local existence result with a priori estimates. For the local existence of the solution (V, z) to (RP) see, e.g., Matsumura [11] and references therein.

We now devote ourselves to the a priori estimates of the solution $(V, z)(x, t)$, $0 < t < T$, to the linearized equation (LP) under the a priori assumption

$$N(T) := \sup_{0<t<T} \left\{ \sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2 \right\} \leqslant \varepsilon. \quad (2.23)$$

Since it suffices to establish the estimates for sufficiently smooth solution, the equations in (RP) and $z|_{x=0} = 0$ give the following boundary conditions for higher order derivatives:

$$V(0, t) = V_{xx}(0, t) = V_t(0, t) = V_{txx}(0, t) = 0, \qquad \text{etc.}$$

Therefore, estimates obtained below are formally quite similar to those in Section 3 of [14]. The difference between (LP) in [14] and (LP) in this paper is the second term of F:

$$h(x, t) := -(p'(v_+) - p'(\bar{v}))\, \bar{v}_x. \quad (2.24)$$

Since $h(x, t) = O(1)(\bar{v} - v_+)\,\bar{v}_x$, the decay properties

$$\int_0^\infty |h(x,t)|^2\,dx \leqslant C\delta_0^4(1+t)^{-5/2}$$
$$\int_0^\infty |h_x(x,t)|^2\,dx \leqslant C\delta_0^4(1+t)^{-7/2}, \qquad \text{etc.,} \tag{2.25}$$

hold, the decay rates of which are the same as those in (LP) of [14]. Note that the first term decays faster than those in (2.25). Hence, we briefly repeat the lemmas.

Multiplying $(\mathrm{LP})_2$ by $z + \lambda V (0 < \lambda \ll 1)$ and using $(\mathrm{LP})_1$, we have the first lemma.

LEMMA 2.1. *If $N(T) \leqslant \varepsilon$ and $|\delta_0|$ are small, then*

$$\|V(t)\|_1^2 + \|z(t)\|^2 + \int_0^t (\|V_x(\tau)\|^2 + \|z(\tau)\|^2)\,d\tau \leqslant C(\|V_0\|_1^2 + \|z_0\|^2 + |\delta_0|).$$

Multiplying $(\mathrm{LP})_2$ by $(1+t)\,z$ and applying Lemma 2.1, we have the second lemma.

LEMMA 2.2. *If $\varepsilon + |\delta_0| \ll 1$, then*

$$(1+t)(\|V_x(t)\|^2 + \|z(t)\|^2) + \int_0^t (1+\tau)\,\|z(\tau)\|^2\,d\tau \leqslant C(\|V_0\|_1^2 + \|z_0\|^2 + |\delta_0|).$$

Next, differentiate $(\mathrm{LP})_2$ with respect to x to obtain

$$z_{xt} + (p'(\bar{v})\,V_x)_{xx} + \alpha z_x = -F_x. \tag{2.26}$$

Multiplying (2.26) by $(1+t)^k\,(z_x - \lambda V_{xx})$ $(0 < \lambda \ll 1)$, $k = 0, 1$, we have

$$(1+t)(\|V_x(t)\|_1^2 + \|z_x(t)\|^2) + \int_0^t (1+\tau)(\|V_{xx}(\tau)\|^2 + \|z_x(\tau)\|^2)\,d\tau$$
$$\leqslant C(\|V_0\|_2^2 + \|z_0\|_1^2 + |\delta_0|). \tag{2.27}$$

Again, multiplying (2.26) by $(1+t)^2\,z_x$ and applying (2.27), we have

$$(1+t)^2\,(\|V_{xx}(t)\|^2 + \|z_x(t)\|^2) + \int_0^t (1+\tau)^2\,\|z_x(\tau)\|^2\,d\tau$$
$$\leqslant C(\|V_0\|_2^2 + \|z_0\|_1^2 + |\delta_0|), \tag{2.28}$$

which gives the third lemma together with (2.27).

LEMMA 2.3. *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^2 (\|V_{xx}(t)\|^2 + \|z_x(t)\|^2) + \int_0^t [(1+\tau) \|V_{xx}(\tau)\|^2 + (1+\tau)^2 \|z_x(\tau)\|^2] \, d\tau \leqslant C(\|V_0\|_2^2 + \|z_0\|_1^2 + |\delta_0|).$$

A similar procedure applied to the equation obtained by differentiating (2.26) with respect to x once more yields

LEMMA 2.4. *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^3 (\|V_{xxx}(t)\|^2 + \|z_{xx}(t)\|^2) + \int_0^t [(1+\tau)^2 \|V_{xxx}(\tau)\|^2 + (1+\tau)^3 \|z_{xx}(\tau)\|^2] \, d\tau \leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta|_0).$$

Applying the same procedure as above to

$$z_{tt} + (p'(\bar{v}) \, V_x)_{xt} + \alpha z_t = -F_t, \tag{2.29}$$

we have the following two lemmas.

LEMMA 2.5. *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^2 \|z(t)\|^2 + (1+t)^3 (\|z_x(t)\|^2 + \|z_t(t)\|^2) + \int_0^t [(1+\tau)^2 \|z_x(\tau)\|^2 + (1+\tau)^3 \|z_t(\tau)\|^2] \, d\tau \leqslant C(\|V_0\|_2^2 + \|z_0\|_1^2 + |\delta_0|).$$

LEMMA 2.6. *If* $\varepsilon + |\delta_0| \ll 1$, *then*

$$(1+t)^4 (\|z_{xx}(t)\|^2 + \|z_{xt}(t)\|^2) + \int_0^t [(1+\tau)^3 \|z_{xx}(\tau)\|^2 + (1+\tau)^4 \|z_{xt}(\tau)\|^2] \, d\tau \leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|).$$

The estimates obtained in the series of the above six lemmas show (2.13). To obtain (2.14), we differentiate (2.29) with respect to t once more:

$$z_{ttt} + (p'(\bar{v}) \, V_x)_{xtt} + \alpha z_{tt} = -F_{tt}$$

or

$$z_{ttt}+(p'(\bar{v})\,z_{xt})_x+\alpha z_{tt}=-F_{tt}-(2p''(\bar{v})\,\bar{v}_t z_x+(p''(\bar{v})\,\bar{v}_{tt}+p'''(\bar{v})\,\bar{v}_t^2)\,V_x)_x$$
$$:=-F_{tt}-\tilde{P}_x. \tag{2.30}$$

We apply the same procedure as above to (2.30), that is, multiply (2.30) by $(1+t)^k\,(z_{tt}+\lambda z_t)$ $(0<\lambda\ll 1)$, $k=0, 1, ..., 4$ and use Lemmas 2.1–2.6. Then we have

$$(1+t)^4\,(\|z_t(t)\|_1^2+\|z_{tt}(t)\|^2)+\int_0^t(1+\tau)^4\,(\|z_{xt}(\tau)\|^2+\|z_{tt}(\tau)\|^2)\,d\tau$$
$$\leqslant C(\|V_0\|_3^2+\|z_0\|_2^2+|\delta_0|). \tag{2.31}$$

Since $\int_0^t(1+\tau)^5\int_0^\infty|F_{tt}+\tilde{P}_x|^2\,dx\,d\tau\leqslant C(\|V_0\|_3^2+\|z_0\|_2^2+|\delta_0|)$ is shown by (2.31) after tedious calculations, multiplying of (2.30) by $(1+t)^5\,z_{tt}$ and using of (2.31) yield

$$(1+t)^5\,(\|z_{xt}(t)\|^2+\|z_{tt}(t)\|^2)+\int_0^t(1+\tau)^5\,\|z_{tt}(\tau)\|^2\,d\tau$$
$$\leqslant C(\|V_0\|_3^2+\|z_0\|_2^2+|\delta_0|),$$

which shows (2.14) together with (2.31). Thus we have completed the proof of Theorem 2.1.

We now turn to the L^∞-estimate assuming that $(V_0, z_0)\in L^1\times L^1$. The proof is very similar to that in [15].

First we show (2.22). The first term of right-hand side in (2.20) clearly satisfies (2.22) since $(V_0, z_0)\in L^1\times L^1$. The last term is estimated by (2.3) as follows:

$$|\text{the last term in (2.21)}|$$
$$\leqslant\left|\frac{1}{\alpha}\int_0^{t/2}\int_0^\infty E(x, t-\tau;\, y)(p'(v_+)\bar{v}-\int_{v_+}^{\bar{v}}p'(s)\,ds)_y\,(y,\tau)\,dy\,d\tau\right|$$
$$+C\int_{t/2}^t\int_0^\infty E(x, t-\tau;\, y)\,|\bar{v}-v_+|\,|\bar{v}_x|\,dy\,d\tau$$
$$\leqslant C\int_0^{t/2}\|E(t-\tau)\|_{L^\infty}\,\|(\bar{v}-v_+)(\tau)\|^2\,d\tau$$
$$+C\int_{t/2}^t\|E(t-\tau)\|_{L^\infty}\,\|\bar{v}_x\|_{L^\infty}\,\|\bar{v}-v_+)(\tau)\|_{L^1}\,d\tau$$
$$\leqslant C\left(\int_0^{t/2}(t-\tau)^{-1}\,(1+\tau)^{-1/2}\,d\tau+\int_{t/2}^t(t-\tau)^{-1/2}\,(1+\tau)^{-1}\,d\tau\right)$$
$$\leqslant C(1+t)^{-1/2}.$$

Derivatives of ϕ are also estimated similarly.

Since

$$-\frac{1}{\alpha}\int_0^{t/2}\int_0^{\infty} E(x, t-\tau; y)\, V_{tt}(y, \tau)\, dy\, d\tau$$

$$= \frac{1}{\alpha}\int_0^{\infty} E(x, t; y)\, z_0(y)\, dy - \frac{1}{\alpha}\int_0^{\infty} E(x, t/2; y)\, z(y, t/2)\, dy$$

$$-\frac{1}{\alpha}\int_0^{t/2}\int_0^{\infty} E_t(x, t-\tau; y)\, z(y, \tau)\, dy\, d\tau, \tag{2.32}$$

(2.19) and (2.20) with (2.32) give the expression

$$(V-\phi)(x, t) = -\frac{1}{\alpha}\int_0^{\infty} E(x, t/2; y)\, z(y, t/2)\, dy$$

$$-\frac{1}{\alpha}\int_0^{t/2}\int_0^{\infty} E_t(x, t-\tau; y)\, z(y, \tau)\, dy\, d\tau$$

$$-\frac{1}{\alpha}\int_{t/2}^{t}\int_0^{\infty} E(x, t-\tau; y)\, z_t(y, \tau)\, dy\, d\tau$$

$$-\frac{1}{\alpha}\left(\int_0^{t/2} + \int_{t/2}^{t}\right)\int_0^{\infty} E(x, t-\tau; y)\, \tilde{F}(y, \tau)\, dy\, d\tau$$

$$:= I + II + III + (IV_1 + IV_2). \tag{2.33}$$

Since the Green kernel E is given by (2.17), the following estimates hold:

$$|I| \leqslant C\, \|E(t/2)\|\, \|z(t/2)\| \leqslant C(1+t)^{-1/4-1},$$

$$|II| \leqslant \int_0^{t/2} \|E(t-\tau)\|\, \|z(\tau)\|\, d\tau$$

$$\leqslant \int_0^{t/2} (1+t-\tau)^{-5/4}\, (1+\tau)^{-1}\, d\tau \leqslant C(1+t)^{-5/4} \ln(2+t)$$

$$|III| \leqslant C\int_{t/2}^{t} \|E(t-\tau)\|\, \|z_t(\tau)\|\, d\tau$$

$$\leqslant C\int_{t/2}^{t} (1+t-\tau)^{-1/4}\, (1+\tau)^{-2}\, d\tau \leqslant C(1+t)^{-2+3/4}. \tag{2.34}$$

For IV_1 and IV_2 we recall that $\tilde{F}$ given by (2.18) has the form $\tilde{F}=f_x$. Hence

$$
\begin{aligned}
|IV_1| &\leqslant C\int_0^{t/2}\int_0^{\infty}|E_y(x,t-\tau;y)|\,|f(y,\tau)|\,dy\,d\tau\\
&\leqslant C\int_0^{t/2}(t-\tau)^{-1}\,(\|\bar{v}_t(\tau)\|_{L^1}+\|\hat{v}(\tau)\|_{L^1}+\|V_x(\tau)\|^2+\|\hat{v}(\tau)\|^2\\
&\quad+\|V_x(\tau)\|L^{\infty}\,\|(\bar{v}-v_+)(\tau)\|_{L^1})\,d\tau\\
&\leqslant Ct^{-1}\int_0^{t/2}((1+\tau)^{-1}+e^{-\alpha\tau}+(1+\tau)^{-3/4})\,d\tau\leqslant C(1+t)^{-3/4}
\end{aligned}\tag{2.35}
$$

and

$$
\begin{aligned}
|IV_2| &\leqslant C\int_{t/2}^{t}\{\|E(t-\tau)\|\,(\|\bar{v}_{xt}(\tau)\|+\|\hat{v}_x(\tau)\|+\|(V_xV_{xx}+\hat{v}\hat{v}_x)(\tau)\|)\\
&\quad+\|E_y(t-\tau)\|\,\|V_x(\tau)\|_{L^\infty}\,\|(\bar{v}-v_+)(\tau)\|\}\,d\tau\\
&\leqslant C\int_{t/2}^{t}(t-\tau)^{-1/4}\,((1+\tau)^{-7/4}+e^{-\alpha\tau})+(t-\tau)^{-3/4}\,(1+\tau)^{-3/4-1/4})\,d\tau\\
&\leqslant C(1+t)^{-3/4}.
\end{aligned}\tag{2.36}
$$

Combining (2.33) with (2.34)–(2.36) shows that $\|(V-\phi)(t)\|_{L^\infty}=O(t^{-3/4})$. Estimates of $\|(V-\phi)_x\,(t)\|_{L^\infty}=O(t^{-5/4})$ and $\|(V-\phi)_t\,(t)\|_{L^\infty}=O(t^{-7/4})$ are obtained in a similar fashion to the above. In particular, $\|(V-\phi)_x\,(t)\|_{L^\infty}=O(t^{-5/4})$ and (2.22) show that $\|V_x(t)\|_{L^\infty}=O(t^{-1})$. Applying this to (2.35) and (2.36) again, we have

$$|IV_1|\leqslant C(1+t)^{-1}\ln(2+t) \tag{2.35$'$}$$

and

$$|IV_2|\leqslant C(1+t)^{-1}, \tag{2.36$'$}$$

which gives the estimate $\|(V-\phi)(t)\|_{L^\infty}=O(t^{-1}\ln t)$. Derivatives of $V-\phi$ are also obtained, which yields the desired estimate (2.21).

3. THE CASE OF THE NEUMANN BOUNDARY CONDITION

We now turn to the problem with the Neumann boundary condition (1.5)

$$\begin{aligned} &v_t - u_x = 0, && (x, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ &u_t + p(v)_x = -\alpha u \\ &(v, u)|_{t=0} = (v_0, u_0)(x), && u_x|_{x=0} = 0. \end{aligned} \tag{3.1}$$

As in the preceding section we first reformulate (3.1). Heuristically, $(3.1)_1$ yields $(d/dt)\, v(0, t) = u_x(0, t) = 0$ and $v(0, t) = v_0(0)$ for any $t > 0$. Hence, we can expect that

$$(v, u)(x, t) \to (\bar{v}, 0)(x, t) \qquad \text{as} \quad t \to \infty, \tag{3.2}$$

where $\bar{v}(x, t)$ is a diffusion wave connecting $v_0(0)$ and v_+.

In the case of $v_0(0) \neq v_+$, putting $u_t = 0$ in $(3.1)_2$ we have

$$u = -\frac{1}{\alpha} p(v)_x \qquad \text{and} \qquad v_t + \frac{1}{\alpha} p(v)_{xx} = 0. \tag{3.3}$$

To construct the diffusion wave $(\bar{v}, \bar{u})$, it is known that for any constant $v_- > 0$ we have a self-similar solution $\tau = \psi(x/\sqrt{t+1})$ satisfying

$$\begin{aligned} &\tau_t + \frac{1}{\alpha} p(\tau)_{xx} = 0, \qquad x \in \mathbf{R} = (-\infty, \infty), \quad t > 0 \\ &\tau|_{x = \pm\infty} = v_\pm. \end{aligned} \tag{3.4}$$

Therefore, for $v_0(0) > 0$ between v_- and v_+, there exists a unique $\bar{v}(x, t)$ in the form of $\psi(x/\sqrt{t+1})|_{x \geqslant 0}$ satisfying

$$\begin{aligned} &\bar{v}_t + \frac{1}{\alpha} p(\bar{v})_{xx} = 0, \qquad (x, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ &\bar{v}|_{x=0} = v_0(0), \qquad \bar{v}|_{x=\infty} = v_+. \end{aligned} \tag{3.5}$$

For these results see [1]. Moreover, $\bar{u}$ is defined by

$$\bar{u}(x, t) = -\frac{1}{\alpha} p(\bar{v})_x$$

so that

$$\bar{u}_x|_{x=0} = \bar{v}_t|_{x=0} = \psi'(x/\sqrt{t+1}) \left(-\frac{x}{2\sqrt{t+1}\,(t+1)} \right) \Bigg|_{x=0} = 0. \tag{3.6}$$

Thus we have had

$$\begin{aligned} &\bar{v}_t - \bar{u}_x = 0 \\ &p(\bar{v})_x = -\alpha \bar{u} \\ &(\bar{v}, \bar{u}_x)|_{x=0} = (v_0(0), 0), \qquad (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{aligned} \tag{3.7}$$

Similar to that in the Dirichlet boundary problem, the auxiliary function $(\hat{v}, \hat{u})(x, t)$ is defined by

$$(\hat{v}, \hat{u})(x, t) = \left(\frac{u_0(0) - u_+}{\alpha} m_0(x) e^{-\alpha t}, \right.$$
$$\left. \left[(u_0(0) - u_+) \int_x^\infty m_0(y)\, dy + u_+ \right] e^{-\alpha t} \right), \tag{3.8}$$

where m_0 is a smooth function satisfying (2.7). Hence $(\hat{v}, \hat{u})$ satisfies

$$\begin{aligned} &\hat{v}_t - \hat{u}_x = 0 \\ &\hat{u}_t = -\alpha \hat{u} \\ &(\hat{u}, \hat{u}_x)|_{x=0} = (u_0(0)\, e^{-\alpha t}, 0), \qquad \hat{v}|_{x=0} = 0 \\ &(\hat{v}, \hat{u})|_{x=\infty} = (0, u_+ e^{-\alpha t}). \end{aligned} \tag{3.9}$$

Combining (3.1) with (3.7) and (3.9) we have

$$\begin{aligned} &(v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0 \\ &(u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x = -\alpha(u - \bar{u} - \hat{u}) - \bar{u}_t \\ &(u - \bar{u} - \hat{u})_x|_{x=0} = 0 \\ &(v - \bar{v} - \hat{v}, u - \bar{u} - \hat{u})|_{t=0} = (v_0, u_0)(x) - (\bar{v} + \hat{v}, \bar{u} + \hat{u})(x, 0). \end{aligned} \tag{3.10}$$

Defining the perturbation by

$$\begin{aligned} &V(x, t) = -\int_x^\infty (v - \bar{v} - \hat{v})(y, t)\, dy \\ &z(x, t) = (u - \bar{u} - \hat{u})(x, t), \end{aligned} \tag{3.11}$$

we have the reformulated problem, after the integration of $(3.10)_1$ once over (x, ∞),

$$
\begin{aligned}
& V_t - z = 0 \\
& z_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x = -\alpha z - \bar{u}_t \\
& z_x|_{x=0} = 0 \qquad (\text{or } V_x|_{x=0}) \qquad\qquad (\text{NRP}) \\
& (V, z)|_{t=0} = (V_0, z_0)(x) \\
& := \left(-\int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)\, dy, u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0) \right),
\end{aligned}
$$

or the second order wave equation of V with damping

$$
\begin{aligned}
& V_{tt} + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \alpha V_t = -\bar{u}_t \\
& V_x|_{x=0} = 0, \qquad (V, V_t)|_{t=0} = (V_0, z_0).
\end{aligned} \tag{3.12}
$$

Note that, if (V, z) is sufficiently smooth in x, t, (3.10) or (3.12) yields the boundary conditions at $x = 0$

$$
V_x = V_{tx} = V_{ttx} = p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) = (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_{xx} = 0, \quad \text{etc.}
$$

Therefore, once we have the smooth solutions, we can treat them formally the same as those in the Cauchy problem in [14]. The diffusion wave $\bar{v}$ defined in (3.6) has the same behaviour as the self-similar solution τ defined in (3.5). For the diffusion wave see [1] and [4, 14]. Hence, the same L^2-estimates for the local smooth solution (V, z) to (NRP) are obtained. Thus we have the following theorem.

THEOREM 3.1 (The Case of $v_0(0) \neq v_+$). *Suppose that $v_0 - v_+$ is in $L^1(\mathbf{R}_+)$ and both $\|V_0\|_3 + \|z_0\|_2$ and $\delta_1 := |(v_0(0) - v_+, u_+ - u_0(0))|$ are small. Then, there exists a unique time-global solution $(V, z)(x, t)$ of* (NRP), *which satisfies*

$$
\begin{aligned}
& V \in C^i([0, \infty); H^{3-i}), \qquad i = 0, 1, 2, 3 \\
& z \in C^i([0, \infty); H^{2-i}), \qquad i = 0, 1, 2
\end{aligned}
$$

and moreover

$$
\begin{aligned}
& \sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2 \\
& \quad + \int_0^t \left[\sum_{j=1}^{3} (1+\tau)^{j-1} \|\partial_x^j V(\cdot, \tau)\|^2 + \sum_{j=0}^{2} (1+\tau)^{j+1} \|\partial_x^j z(\cdot, \tau)\|^2 \right] d\tau \\
& \leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_1),
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
&(1+t)^4 \|z_t(\cdot,t)\|^2+(1+t)^5(\|z_{xt}(\cdot,t)^2+\|z_{tt}(\cdot,t)\|^2)\\
&\qquad+\int_0^t [(1+\tau)^4 \|z_{xt}(\cdot,\tau)\|^2+(1+\tau)^5 \|z_{tt}(\cdot,\tau)\|^2]\,d\tau\\
&\leqslant C(\|V_0\|_3^2+\|z_0\|_2^2+\delta_1).
\end{aligned}
\tag{3.14}
$$

The derivation of (3.14) is similar to that of (2.14) and so the proof of Theorem 3.1 is omitted.

Remark. The L^1-property of v_0-v_+ is a sufficient condition for the definition of $V_0(x)$. The decay rates in (3.13)–(3.14) will be optimal in L^2-setting. In L^1-setting the optimal decay rates are not known, different from Theorem 2.2. However, when $v_0(0)=v_+$, optimal decay rates are obtained as shown below.

We now treat the case of $v_0(0)=v_+$. Taking

$$(\bar v,\bar u)(x,t)\equiv(v_+,0) \tag{3.15}$$

and

$$
\begin{aligned}
(\hat v,\hat u)(x,t)=\Bigg(&\frac{u_0(0)-u_+}{\alpha}\,m_0(x)\,e^{-\alpha t},\\
&\left[(u_0(0)-u_+)\int_x^\infty m_0(y)\,dy+u_+\right]e^{-\alpha t}\Bigg),
\end{aligned}
\tag{3.16}
$$

we have

$$
\left.
\begin{aligned}
&(v-v_+-\hat v)_t-(u-\hat u)_x=0\\
&(u-\hat u)_t+(p(v)-p(v_+))_x=-\alpha(u-\hat u)\\
&(u-\hat u)_x|_{x=0}=0 \qquad (\text{or } (v-v_+-\hat v)|_{x=0}=0)\\
&(v-v_+-\hat v,u-\hat u)|_{t=0}=(v_0-v_+,u_0)(x)-(\hat v,\hat u)(x,0).
\end{aligned}
\right\}
\tag{3.17}
$$

The definition

$$(V,z)(x,t)=\left(-\int_x^\infty (v-v_+-\hat v)(y,t)\,dy,\,u(x,t)-\hat u(x,t)\right) \tag{3.18}$$

gives the reformulated problem

$$
\begin{aligned}
V_t - z &= 0 \\
z_t + (p(V_x + v_+ + \hat{v}) - p(v_+))_x &= \alpha z \\
z_x|_{x=0} &= 0 \qquad (\text{or } V_x|_{x=0} = 0)
\end{aligned}
\tag{3.19}
$$

$$
(V, z)|_{t=0} = (V_0, z_0)(x) := \left(-\int_x^\infty (v_0(y) - v_+ - \hat{v}(y, 0))\, dy,\ u_0(x) - \hat{u}(x, 0) \right)
$$

or the linearized wave equation of V around v_+

$$
\begin{aligned}
v_{tt} + p'(v_+)\, V_{xx} + \alpha V_t &= -\tilde{F} \\
&:= -(p(V_x + v_+ + \hat{v}) - p(v_+) - p'(v_+)\, V_x)_x \\
V_x|_{x=0} = 0, \qquad & (V, z)|_{t=0} = (V_0, z_0)(x).
\end{aligned}
\tag{3.20}
$$

Therefore, if $(V_0, z_0) \in H^3 \times H^2$, then we can obtain the following theorem on the same line as Theorem 3.1.

THEOREM 3.2 (The Case of $v_0(0) = v_+$). *Suppose that* $v_0 - v_+$ *is in* $L^1(\mathbf{R}_+)$ *and both* $\|V_0\|_3 + \|z_0\|_2$ *and* $\delta_2 := |u_+ - u_0(0)|$ *are small. Then, there exists a unique time-global solution* $(V, z)(x, t)$ *of* (3.19), *which satisfies*

$$
\begin{aligned}
V &\in C^i([0, \infty); H^{3-i}), \qquad i = 0, 1, 2, 3 \\
z &\in C^i([0, \infty); H^{2-i}), \qquad i = 0, 1, 2
\end{aligned}
$$

and moreover

$$
\begin{aligned}
&\sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2 \\
&\qquad + \int_0^t \left[\sum_{j=1}^{3} (1+\tau)^{j-1} \|\partial_x^j V(\cdot, \tau)\|^2 + \sum_{j=0}^{2} (1+\tau)^{j+1} \|\partial_x^j z(\cdot, \tau)\|^2 \right] d\tau \\
&\quad \leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_2),
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
&(1+t)^4 \|z_t(\cdot, t)\|^2 + (1+t)^5 (\|z_{xt}(\cdot, t)\|^2 + \|z_{tt}(\cdot, t)\|^2) \\
&\qquad + \int_0^t [(1+\tau)^4 \|z_{xt}(\cdot, \tau)\|^2 + (1+\tau)^5 \|z_{tt}(\cdot, \tau)\|^2]\, d\tau \\
&\quad \leqslant C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta_2).
\end{aligned}
\tag{3.22}
$$

Moreover, the same line to Theorem 2.2 can be applied. In the Dirichlet problem, since $1/\alpha(p'(v_+)-p'(\bar{v}))\,\bar{v}_x$ in (2.16) has not enough decay order, ϕ was defined by (2.20)$'$ with a sourcing term. However, in the present case, since $\bar{v}\equiv v_+$, we define an asymptotic profile $\phi_1(x,t)$ by

$$\phi_1(x,t)=\int_0^\infty E_1(x,t;y)\Big(V_0(y)+\frac{1}{\alpha}z_0(y)\Big)\,dy \tag{3.23}$$

or the solution of the corresponding parabolic equation

$$\begin{aligned} &\phi_{1t}-\kappa\phi_{1xx}=0, && \kappa=-\frac{p'(v_+)}{\alpha} \\ &\phi_{1x}|_{x=0}=0, && \phi_1|_{t=0}=V_0(x)+\frac{1}{\alpha}z_0(x), \end{aligned} \tag{3.24}$$

where

$$E_1(x,t;y)=\frac{1}{\sqrt{4\kappa\pi t}}\big(e^{-(x+y)^2/4\kappa t}+e^{-(x-y)^2/4\kappa t}\big). \tag{3.25}$$

Then we have the expression

$$\begin{aligned} (V-\phi_1)(x,t)=&-\frac{1}{\alpha}\int_0^\infty E_1(x,t/2;y)\,z(y,t/2)\,dy \\ &-\frac{1}{\alpha}\int_0^{t/2}\int_0^\infty E_{1t}(x,t-\tau;y)\,z(y,\tau)\,dy\,d\tau \\ &-\frac{1}{\alpha}\int_{t/2}^{t}\int_0^\infty E_1(x,t-\tau;y)\,z_t(y,\tau)\,dy\,d\tau \\ &-\frac{1}{\alpha}\Big(\int_0^{t/2}+\int_{t/2}^{t}\Big)\int_0^\infty E_1(x,t-\tau;y)\,\tilde{F}_1(y,\tau)\,dy\,d\tau. \end{aligned} \tag{3.26}$$

We note that $\tilde{F}_1$ in (3.26) does not include the bad term likely $-((p'(v_+)-p'(\bar{v}))\,V_x)_x$ in (2.18). Thus, applying the decay properties obtained in Theorem 3.2, we can estimate (3.26) to reach the final theorem.

THEOREM 3.3 [Asymptotic Profile]. *Define ϕ_1 by* (3.23) *or* (3.24) *and suppose that $(V_0,z_0)\in L^1\times L^1$. Then, the solution (V,z) of* (3.19) *obtained in Theorem* 3.2 *satisfies*

$$\|(V-\phi_1,(V-\phi_1)_x,(V-\phi_1)_t)(\cdot,t)\|_{L^\infty}=O(t^{-1},t^{-3/2},t^{-2}).$$

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