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Some multi-step iterative methods for solving nonlinear equations

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ABSTRACT

In this paper, we develop some new iterative methods for solving nonlinear equations by using the techniques introduced in Golbabai and Javidi (2007) [1] and Rafiq and Rafiullah (2008) [20]. We establish the convergence analysis of the proposed methods and then demonstrate their efficiency by taking some test problems.

1. Introduction

During the last few years, numerical techniques for solving nonlinear equations have been applied successfully. Recently, there has been some development on iterative methods with higher-order convergence that required the computation of derivatives of as low order as possible (see for example [1–9]).

In this paper, some numerical methods based on the modified homotopy perturbation method are proposed to solve nonlinear equations. The proposed methods are than applied to solve some problems in order to assess their validity and accuracy.

We need the following results.

Theorem 1 ([10]). *Suppose f is continuous on $[a, b]$ and differentiable in (a, b). Then there exists a point $\delta \in (a, b)$ such that*

$$f(y) = f(x) + (y - x)f'(x) + \frac{1}{2}(y - x)^2 f''(\delta).$$

Proof. The proof can be completed by applying Rolle's theorem for the function

$$\phi(t) = f(y) - f(t) - (y - t)f'(t) - (y - t)^2\theta,$$

where θ is a constant. □

In the fixed point iterative method for solving the nonlinear equation $f(x) = 0$, the equation is usually rewritten as

$$x = g(x), \tag{1.1}$$

where

(i) $\exists [a, b]$ s.t. $\forall x \in [a, b]$; $g(x) \in [a, b]$,

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(ii) $\exists L > 0$ s.t. $\forall x \in (a, b)$; $|g'(x)| \le L < 1$.

Considering the following iterative scheme:

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

and starting with a suitable initial approximation x_0, we build up a sequence of approximations, say $\{x_n\}$, for the solution of the nonlinear equation, say α. The scheme will converge to the root α, provided that

(i) the initial approximation x_0 is chosen in the interval $[a, b]$,
(ii) g has a continuous derivative on (a, b),
(iii) $|g'(x)| < 1$ for all $x \in [a, b]$,
(iv) $a \le g(x) \le b$ for all $x \in [a, b]$ (see [11]).

The order of convergence for the sequence of approximations derived from an iterative method is defined in the literature as:

Definition 2. Let $\{x_n\}$ converge to α. If there exist an integer constant p and real positive constant C such that

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C,$$

then p is called the order and C the constant of convergence.

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of $g(x_n)$:

$$g(x_n) = g(x) + \frac{g'(x)}{1!}(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + \cdots + \frac{g^{(k)}(x)}{k!}(x_n - x)^k + \cdots. \tag{1.3}$$

Using (1.1) and (1.2) in (1.3), we have

$$x_{n+1} - x = g'(x)(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + \cdots + \frac{g^{(k)}(x)}{k!}(x_n - x)^k + \cdots,$$

and state the following theorem:

Theorem 3 ([4]). *Suppose $g \in C^p[a, b]$. If $g^{(k)}(x) = 0$, for $k = 1, 2, \ldots, p - 1$ and $g^{(p)}(x) \ne 0$, then the sequence $\{x_n\}$ is of order p.*

2. Modified homotopy perturbation method

The homotopy perturbation method (HPM) was established by He in 1999 [12] and systematical description in 2000 [13] which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [13]. This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities [14], nonlinear wave equations [15], asymptotology [16], boundary value problem [17], limit cycle and bifurcation of nonlinear problems [18] and many other subjects. Thus He's method is a universal one which can solve various kinds of nonlinear equation. Subsequently, many researchers have applied the method to various linear and nonlinear problems (see for example [1,2,19]).

Consider the nonlinear equation

$$f(x) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

We assume that w is a simple zero of $f(x)$ and α is an initial guess sufficiently close to w. Using Theorem 1 around α for (2.1), we have

$$f(\alpha) + (x - \alpha)f'(\alpha) + \frac{1}{2}(x - \alpha)^2 f''(\delta) = 0, \tag{2.2}$$

where δ lies between x and α.

We can rewrite (2.2) in the following form:

$$x = c + N(x), \tag{2.3}$$

where

$$c = \alpha - \frac{f(\alpha)}{f'(\alpha)}, \tag{2.4}$$

and

$$N(x) = -\frac{1}{2}(x - \alpha)^2 \frac{f''(\delta)}{f'(\alpha)}. \tag{2.5}$$

Now we construct a homotopy (mainly due to Golbabai and Javidi [1]) $\ominus : (\mathbb{R} \times [0, 1]) \times \mathbb{R} \to \mathbb{R}$ for (2.3), which satisfies

$$\ominus(\varpi, \beta, \theta) = \varpi - c - \beta N(\varpi) + \beta(1-\beta)\theta = 0, \quad \theta, \varpi \in \mathbb{R} \text{ and } \beta \in [0, 1], \tag{2.6}$$

where θ is unknown real number and β is embedding parameter. It is obvious that

$$\ominus(\varpi, 0, \theta) = \varpi - c = 0, \tag{2.7}$$

$$\ominus(\varpi, 1, \theta) = \varpi - c - \beta N(\varpi) = 0. \tag{2.8}$$

The embedding parameter β increases monotonically from zero to unity as the trivial problem $\ominus(\varpi, 0, \theta) = \varpi - c = 0$ is continuously deformed to the original problem $\ominus(\varpi, 1, \theta) = \varpi - c - \beta N(\varpi) = 0$. The modified HPM uses the homotopy parameter β as an expanding parameter to obtain (see [12] and the references therein)

$$\varpi = x_0 + \beta x_1 + \beta^2 x_2 + \cdots \tag{2.9}$$

The approximate solution of (2.1), therefore, can be readily obtained:

$$w = \lim_{\beta \to 1} \varpi = x_0 + x_1 + x_2 + \cdots \tag{2.10}$$

The convergence of the series (2.10) has been proved by He in his paper [12].

For the application of the modified HPM to (2.1), we can write (2.3) as follows, by expanding $N(\varpi)$ into a Taylor series around x_0:

$$\varpi - c - \beta \left\{ N(x_0) + (\varpi - x_0)\frac{N'(x_0)}{1!} + (\varpi - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \right\} + \beta(1-\beta)\theta = 0. \tag{2.11}$$

Substituting (2.9) into (2.11) yields

$$\begin{aligned} x_0 + \beta x_1 + \beta^2 x_2 + \cdots - c - \beta \Big\{ & N(x_0) + (x_0 + \beta x_1 + \beta^2 x_2 + \cdots - x_0)\frac{N'(x_0)}{1!} \\ & + (x_0 + \beta x_1 + \beta^2 x_2 + \cdots - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \Big\} + \beta(1-\beta)\theta = 0. \end{aligned} \tag{2.12}$$

By equating terms with identical powers of β, we have

$$\beta^0 : x_0 = c, \tag{2.13}$$

$$\beta^1 : x_1 = N(x_0) - \theta, \tag{2.14}$$

$$\beta^2 : x_2 = x_1 N'(x_0) + \theta, \tag{2.15}$$

$$\beta^3 : x_3 = x_2 N'(x_0) + \frac{1}{2} x_1^2 N''(x_0), \tag{2.16}$$

.

.

.,

where

$$N(x_0) = -\frac{[f(\alpha)]^2 f''(\delta)}{2\,[f'(\alpha)]^3}, \tag{2.17}$$

$$N'(x_0) = \frac{f(\alpha) f''(\delta)}{[f'(\alpha)]^2}, \tag{2.18}$$

and

$$N''(x_0) = -\frac{f''(\delta)}{f'(\alpha)}. \tag{2.19}$$

Substituting (2.14) into (2.15) and letting $x_2 = 0$, we obtain

$$\theta = \frac{N(x_0) N'(x_0)}{N'(x_0) - 1}. \tag{2.20}$$

From (2.4), (2.5) and (2.13)–(2.20), we have

$$x_0 = \alpha - \frac{f(\alpha)}{f'(\alpha)}, \tag{2.21}$$

$$x_1 = -\frac{1}{2}\frac{[f(\alpha)]^2 f''(\delta)}{[f'(\alpha)]^3 - f(\alpha)f'(\alpha)f''(\delta)}, \tag{2.22}$$

$$x_2 = 0, \tag{2.23}$$

$$x_3 = -\frac{1}{8\,[f'(\alpha)]^3}\frac{[f(\alpha)]^4\left[f''(\delta)\right]^3}{\left([f'(\alpha)]^2 - f(\alpha)f''(\delta)\right)^2}, \tag{2.24}$$

$$\vdots$$

Substituting (2.21)–(2.24) in (2.10), we can obtain the solution of (2.1) as follows:

$$w = \alpha - \frac{f(\alpha)}{f'(\alpha)} - \frac{1}{2}\frac{[f(\alpha)]^2 f''(\delta)}{[f'(\alpha)]^3 - f(\alpha)f'(\alpha)f''(\delta)} - \frac{1}{8\,[f'(\alpha)]^3}\frac{[f(\alpha)]^4\left[f''(\delta)\right]^3}{\left([f'(\alpha)]^2 - f(\alpha)f''(\delta)\right)^2} + \cdots. \tag{2.25}$$

This formulation allows us to suggest the following iterative method for solving nonlinear equation (2.1).

Algorithm 1. For a given ω_0, calculate the approximation solution ω_{n+1}, by the iterative scheme

$$\omega_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)}; \quad f'(\omega_n) \neq 0.$$

Algorithm 2. For a given ω_0, calculate the approximation solution ω_{n+1}, by the iterative scheme

$$\omega_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)} - \frac{1}{2}\frac{[f(\omega_n)]^2 f''(y_n)}{[f'(\omega_n)]^3 - f(\omega_n)f'(\omega_n)f''(y_n)},$$
$$y_n = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)}; \qquad f'(\omega_n) \neq 0.$$

Algorithm 3. For a given ω_0, calculate the approximation solution ω_{n+1}, by the iterative scheme

$$\omega_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)} - \frac{1}{2}\frac{[f(\omega_n)]^2 f''(y_n)}{[f'(\omega_n)]^3 - f(\omega_n)f'(\omega_n)f''(y_n)} - \frac{1}{8\left[f'(\omega_n)^3\right]}\frac{[f(\omega_n)]^4\left[f''(y_n)\right]^3}{\left([f'(\omega_n)]^2 - f(\omega_n)f''(y_n)\right)^2},$$
$$y_n = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)}; \quad f'(\omega_n) \neq 0.$$

3. Convergence analysis

Now we discuss the convergence analysis of Algorithm 2.

Theorem 4. *Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Algorithm 2, the convergence is at least of order 3.*

Proof. For the iteration function defined as in Algorithm 2, which has a fixed point w, let us define

$$y = x - \frac{f(x)}{f'(x)}, \tag{3.1}$$

and

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\frac{[f(x)]^2 f''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)}. \tag{3.2}$$

Now consider

$$\begin{aligned} g'(x) &= \frac{f(x)f''(x)}{[f'(x)]^2} - \frac{f(x)f'(x)f''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)} - \frac{[f(x)]^3 f''(x)f'''(y)}{2\,[f'(x)]^2\,([f'(x)]^3 - f(x)f'(x)f''(y))} \\ &\quad + \frac{[f(x)]^2 f''(y)\left[3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right]}{2([f'(x)]^3 - f(x)f'(x)f''(y))^2}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} g''(x) &= \frac{f''(x)}{f'(x)} - \frac{2f(x)\left[f''(x)\right]^2}{[f'(x)]^3} - \frac{\left[f'(x)\right]^2 f''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)} \\ &\quad - \frac{f(x)f''(x)f''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)} + \frac{f(x)f'''(x)}{[f'(x)]^2} - \frac{5\,[f(x)]^2 f''(x)f'''(y)}{2f'(x)([f'(x)]^3 - f(x)f'(x)f''(y))} \\ &\quad + \frac{[f(x)]^3\left[f''(x)\right]^2 f'''(y)}{[f'(x)]^3\,([f'(x)]^3 - f(x)f'(x)f''(y))} - \frac{[f(x)]^3 f'''(x)f'''(y)}{2\,[f'(x)]^2\,([f'(x)]^3 - f(x)f'(x)f''(y))} \\ &\quad + \frac{2f(x)f'(x)f''(y)\left(3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right)}{([f'(x)]^3 - f(x)f'(x)f''(y))^2} \\ &\quad + \frac{[f(x)]^3 f''(x)f'''(y)\left(3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right)}{[f'(x)]^2\,([f'(x)]^3 - f(x)f'(x)f''(y))^2} \\ &\quad - \frac{[f(x)]^2 f''(y)\left(3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right)^2}{([f'(x)]^3 - f(x)f'(x)f''(y))^3} \\ &\quad - \frac{[f(x)]^4\left[f''(x)\right]^2 f''''(y)}{2([f'(x)]^4\,([f'(x)]^3 - f(x)f'(x)f''(y)))} \\ &\quad + \frac{[f(x)]^2 f''(y)\left(6f'(x)\left[f''(x)\right]^2 - 3f'(x)f''(x)f''(y) + 3\left[f'(x)\right]^2 f'''(x) - f(x)f''(y)f'''(x)\right)}{2([f'(x)]^3 - f(x)f'(x)f''(y))^2} \\ &\quad - \frac{[f(x)]^2 f''(y)\left(3f(x)f''(x)f'''(y) + \frac{[f(x)]^2 f'''(x)f'''(y)}{f'(x)} + \frac{f(x)^3 f''(x)^2 f''''(y)}{f'(x)^3}\right)}{2([f'(x)]^3 - f(x)f'(x)f''(y))^2}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} g'''(x) &= \frac{f(x)f^{(4)}(x)}{[f'(x)]^2} + \frac{6f(x)\left[f''(x)\right]^3}{[f'(x)]^4} + \frac{3f^{(4)}(x)f^{(4)}(y)\left[f''(x)\right]^3}{[f'(x)]^5\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]} \\ &\quad - \frac{1}{2}\frac{[f(x)]^5\left[f^{(5)}(y)\right]^2\left[f''(x)\right]^3}{[f'(x)]^6\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]} - \frac{3\left[f''(x)\right]^2}{[f'(x)]^2} \\ &\quad - \frac{3\,[f(x)]^3 f'''(y)\left[f''(x)\right]^2}{[f'(x)]^3\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]^2}\Big[3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) \\ &\quad - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\Big] + \frac{9}{2}\frac{[f(x)]^2\left[f''(x)\right]^2 f'''(y)}{[f'(x)]^2\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]} \\ &\quad + \frac{2f'''(x)}{f'(x)} - \frac{6f(x)f''(x)f'''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)} - \frac{6f(x)f''(x)f'''(x)}{[f'(x)]^3} \\ &\quad - \frac{3f(x)f''(x)f''(y)}{[f'(x)]^3 - f(x)f'(x)f''(y)} + \frac{3\left[f'(x)\right]^2 f''(y)}{\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]^2} \\ &\quad \times \left(3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{f(x)f'''(x)f''(y)}{[f'(x)]^3-f(x)f'(x)f''(y)}+\frac{1}{2}\frac{[f(x)]^2f''(y)}{\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\\
&\times\Bigg[6\left[f''(x)\right]^3+18f'(x)f''(x)f'''(x)+3\left[f'(x)\right]^2f^{(4)}(x)-3\left[f''(x)\right]^2f''(y)\\
&-\frac{3f(x)\left[f''(x)\right]^2f'''(y)}{f'(x)}-4f'(x)f'''(x)f''(y)-\frac{6\,[f(x)]^2\left[f''(x)\right]^2f^{(4)}(y)}{[f'(x)]^2}\\
&-3f'(x)f''(x)f'''(y)-5f(x)f'''(x)f'''(y)-f(x)f^{(4)}(x)f''(y)\\
&+\frac{3\,[f(x)]^3\left[f''(x)\right]^3f^{(4)}(y)}{[f'(x)]^4}-\frac{[f(x)]^4\left[f''(x)\right]^3f^{(5)}(y)}{[f'(x)]^5}\\
&-\frac{3\,[f(x)]^3f''(x)f'''(x)f^{(4)}(y)}{[f'(x)]^3}-\frac{[f(x)]^2f^{(4)}(x)f'''(y)}{f'(x)}\Bigg]+\frac{3\,[f(x)]^2f''(y)}{[f'(x)-f(x)f'(x)f''(y)]^4}\\
&\times\left[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)-f(x)f''(x)f''(y)-\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)}\right]^3\\
&+\frac{3}{2}\frac{[f(x)]^4\left[f''(x)\right]^2f^{(4)}(y)}{[f'(x)]^4\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\\
&\times\left[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)-f(x)f''(x)f''(y)-\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)}\right]\\
&-\frac{3}{2}\frac{[f(x)]^4f''(x)f'''(x)f^{(4)}(y)}{[f'(x)]^4\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]}-\frac{4\,[f(x)]^2f'''(x)f'''(y)}{f'(x)\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]}\\
&+\frac{15}{2}\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\\
&\times\left[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)-f(x)f''(x)f''(y)-\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)}\right]\\
&-\frac{9}{2}\frac{[f(x)]^3\left[f''(x)\right]^2f^{(4)}(y)}{[f'(x)]^3\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]}-\frac{6f(x)f'(x)f''(y)}{\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^3}\\
&\times\left[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)-f(x)f''(x)f''(y)-\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)}\right]^2\\
&+\frac{3f(x)f''(x)f''(y)}{\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\\
&\times\left[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)-f(x)f''(x)f''(y)-\frac{[f(x)]^2f''(x)f'''(y)}{f'(x)}\right]\\
&+\frac{3f(x)f''(x)f''(y)}{\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\Bigg[6f'(x)\left[f''(x)\right]^2+3\left[f'(x)\right]^2f'''(x)\\
&-3f'(x)f''(x)f''(y)-3f(x)f''(x)f'''(y)-f(x)f'''(x)f''(y)-\frac{[f(x)]^3\left[f''(x)\right]^2f^{(4)}(y)}{[f'(x)]^3}\\
&-\frac{[f(x)]^2f'''(x)f'''(y)}{f'(x)}\Bigg]-\frac{3\,[f(x)]^3\left[f''(x)\right]^3f'''(y)}{[f'(x)]^4\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]}\\
&+\frac{3\,[f(x)]^3f''(x)f'''(x)f'''(y)}{[f'(x)]^3\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]}-\frac{1}{2}\frac{[f(x)]^3f^{(4)}(x)f'''(y)}{[f'(x)]^2\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\\
&+\frac{3}{2}\frac{[f(x)]^3f'''(x)f'''(y)}{[f'(x)]^2\left[[f'(x)]^3-f(x)f'(x)f''(y)\right]^2}\Bigg[3\left[f'(x)\right]^2f''(x)-\left[f'(x)\right]^2f''(y)
\end{aligned}
$$

$$-f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x) f'''(y)}{f'(x)}\Bigg] - \frac{3[f(x)]^3 f''(x)f'''(y)}{[f'(x)]^2\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]^3}$$
$$\times\left[3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right]^2$$
$$+\frac{3}{2}\frac{[f(x)]^3 f''(x)f'''(y)}{[f'(x)]^2\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]^2}\Bigg[6f'(x)\left[f''(x)\right]^2 + 3\left[f'(x)\right]^2 f'''(x)$$
$$-3f'(x)f''(x)f'''(y) - 3f(x)f''(x)f'''(y) - f(x)f'''(x)f''(y) - \frac{[f(x)]^3\left[f''(x)\right]^2 f^{(4)}(y)}{[f'(x)]^3}$$
$$-\frac{[f(x)]^2 f'''(x)f'''(y)}{f'(x)}\Bigg] - \frac{3[f(x)]^2 f''(y)}{\left[[f'(x)]^3 - f(x)f'(x)f''(y)\right]^3}$$
$$\times\left[3\left[f'(x)\right]^2 f''(x) - \left[f'(x)\right]^2 f''(y) - f(x)f''(x)f''(y) - \frac{[f(x)]^2 f''(x)f'''(y)}{f'(x)}\right]$$
$$\times\Bigg[6f'(x)\left[f''(x)\right]^2 + 3\left[f'(x)\right]^2 f'''(x) - 3f'(x)f''(x)f''(y) - 3f(x)f''(x)f'''(y)$$
$$-f(x)f'''(x)f''(y) - \frac{[f(x)]^3\left[f''(x)\right]^2 f^{(4)}(y)}{[f'(x)]^3} - \frac{[f(x)]^2 f'''(x)f'''(y)}{f'(x)}\Bigg]. \tag{3.5}$$

Thus, one can easily see that

$$g(w) = w, \qquad g'(w) = 0 = g''(w), \qquad g'''(w) = 2\frac{f'''(w)}{f'(w)}. \tag{3.6}$$

But $g'''(w) \neq 0$ provides that

$$2\frac{f'''(w)}{f'(w)} \neq 0,$$

and so, in general, the Algorithm 2 is of order 3. Based on the result (3.6), by using Taylor's series expansion of $g(x_n)$ about w and with the help of (1.2), we obtain the error equation as follows:

$$\begin{aligned} x_{n+1} &= g(x_n) \\ &= g(w) + g'(w)(x_n - w) + \frac{g''(w)}{2!}(x_n - w)^2 + \frac{g'''(w)}{3!}(x_n - w)^3 + O(x_n - w)^4. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned} x_{n+1} &= w + \frac{g'''(w)}{3!}e_n^3 + O(e_n^4) \\ &= w + \frac{1}{6}\left(2\frac{f'''(w)}{f'(w)}\right)e_n^3 + O(e_n^4) \\ &= w + \frac{1}{3}\frac{f'''(w)}{f'(w)}e_n^3 + O(e_n^4) \\ &= w + 2\frac{f'''(w)}{3!f'(w)}e_n^3 + O(e_n^4) \\ &= w + 2c_3 e_n^3 + O(e_n^4), \end{aligned}$$

where $e_n = x_n - w$ and $c_3 = \frac{f'''(w)}{3!f'(w)}$. Thus,

$$e_{n+1} = 2c_3 e_n^3 + O(e_n^4). \quad \square$$

4. Applications

Now we present some examples to illustrate the efficiency of the developed method, namely Algorithm 2. The list of examples used is given in Table 1. We compare the classical Newton–Raphson (NR), Weerakoon and Fernando (WF) [9], Özban (OZ) [8], and Homeier (HH) [7] methods with Algorithm 2, as shown in Table 2.

Table 1

Sample number	$f(x)$	Initial approximation
1.	$t^5 + t^4 + 4t^2 - 15$	1.5
2.	$\sin(t) - \frac{1}{3}t$	1.9
3.	$10t\,e^{-t^2} - 1$	0
4.	$e^{-t^2+t+2} - 1$	2.1
5.	$\ln(t^2 + t + 2) - t + 1$	1
6.	$\cos(t) - t$	1
7.	$e^{-t} + \cos(t)$	1.75
8.	$\sin^{-1}(t^2 - 1) - \frac{1}{2}t + 1$	0.3

Table 2

Method	Iteration	Obtained solution	Function value
1.			
NR	4	1.34742809896837	2.451372438372346e−012
WF	87	1.34742809897041	7.796430168127699e−011
OZ	3	1.34742809896831	−3.552713678800501e−015
HM	3	1.34742809896831	−3.552713678800501e−015
ALGO2	3	1.34742809896831	−3.552713678800501e−015
2.			
NR	5	2.27886266007583	0
WF	77	2.27886265998297	9.134282219491752e−011
OZ	3	2.27886266007583	0
HM	3	2.27886266007583	0
ALGO2	3	2.27886266007583	0
3.			
NR	3	0.10102584831565	−3.071987109137808e−013
WF	80	0.10102584830550	−9.872347384032310e−011
OZ	2	0.1010258483088	−6.596834190020218e−011
HM	3	0.10102584831569	0
ALGO2	3	0.10102584831569	0
4.			
NR	4	2	9.325873406851315e−015
WF	76	2.00000000003244	−9.733280847967762e−011
OZ	3	2	0
HM	2	2.00000000000111	−3.324007735727719e−012
ALGO2	3	2	0
5.			
NR	5	4.15259073675716	−1.776356839400251e−015
WF	83	4.15259073661573	8.517941907371096e−011
OZ	3	4.15259073661478	8.575007370836829e−011
HM	3	4.15259073675755	−2.349231920106831e−013
ALGO2	4	4.15259073675716	0
6.			
NR	4	0.73908513321516	0
WF	78	0.73908513326208	−7.852074546121912e−011
OZ	2	0.73908513325605	−6.843992039762270e−011
HM	2	0.73908513322115	−1.002142813177898e−011
ALGO2	3	0.73908513321516	1.110223024625157e−016
7.			
NR	2	1.74613953040724	8.901490655688349e−013
WF	62	1.74613953047727	−8.027192799353600e−011
OZ	2	1.74613953040801	−1.110223024625157e−016
HM	2	1.74613953040801	−1.110223024625157e−016
ALGO2	2	1.74613953040801	1.387778780781446e−016
8.			
NR	4	0.59481096839837	0
WF	77	0.59481096832697	−7.559819437119586e−011
OZ	2	0.59481096839886	5.223599330861362e−013
HM	3	0.59481096839837	0
ALGO2	3	0.59481096839837	0

5. Conclusions

The modified homotopy perturbation method due to Golbabai and Javidi [1] is used to develop some numerical methods for solving nonlinear equations. With the help of some examples, comparison of the obtained results with existing methods, such as the classical Newton–Raphson (NR), Weerakoon and Fernando (WF) [9], Özban (OZ) [8], and Homeier (HH) [7] methods, is also given.

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