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# A study of the transient fluid flow around a semi-infinite crack

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ABSTRACT

Applying the implicit finite difference approximation of the time derivative term, the diffusion equation governing fluid-flow around a crack in a fluid-infiltrated undeformable porous medium is transformed into a non-homogeneous modified Helmholtz's equation. Then, Vekua's theory regarding the solution of linear, second order, elliptical partial differential equations is employed for its solution and the corresponding Riemann function is found. Subsequently, the general solution of the Dirichlet initial-boundary value problem for a prescribed arbitrary distribution of pressure acting along a semi-infinite crack is found in the form of a Cauchy singular integral equation of the second kind. A numerical Gauss–Chebyshev quadrature scheme is proposed to solve this singular integral equation that is first applied to the steady-state problem and then to the transient problem. It is shown that the density of the Cauchy integral of the transient problem $\hat{\mu}$ bears a simple similarity relationship with the steady-state problem $\hat{\mu}_0$ of the form $\hat{\mu}(x) \approx (1-\lambda/0.4)\hat{\mu}_0(x)$ for $0 \leqslant x < \infty,\ y = 0$, wherein $\lambda = 1/\sqrt{D \cdot t}$, with $D$ denoting the diffusivity coefficient and $t$ the time. This solution is the first step towards the solution of transient fluid flow around multiple cracks and then of the coupled problem of a crack or cracks in deformable porous media and for the study of fluid-driven cracks in poroelastic media.



## 1. Introduction

Pre-existing cracks have generally a great influence on global transport properties (of hydraulic, thermal or diffusive nature), as well as on deformability and strength of porous materials exhibiting double porosity. Thus, the theoretical study of either fluid flow or diffusion or heat conduction processes in fractured rocks or other porous deformable engineering materials has many significant technological applications such as hydraulic fracturing in boreholes in petroleum, natural gas and geothermal reservoirs in order to increase the permeability of the formation, contaminant transport in jointed porous rocks, stress-assisted diffusion of corrosive substances around cracks, investigations on the conditions for initiation of cracks and faults in porous solids and in the Earth's crust, respectively, and many others.

A general complex potential formulation with illustrating example cases of the plane steady-state heat conduction or fluid flow in a fully-saturated porous body containing an arbitrary number of either non-intersecting or intersecting curvilinear cracks which are represented by branch cuts (line discontinuities) subjected to Dirichlet, Neumann or mixed boundary conditions was given by Liolios and Exadaktylos (2006). Although in the geometric model the fractures have a zero thickness, their physical thickness is not zero and they can contain some fluid mass quantity. Subsequently, the same authors have proposed the appropriate numerical scheme for an arbitrary number of non-intersecting curvilinear cracks subjected to any type of the above boundary conditions. This formulation has been recently elaborated by Pouya and Ghabezloo (2010) to investigate the basic problem of a single straight crack in an infinite body submitted to a pressure gradient at infinity. A closed form solution was presented for the case of void cracks (infinite conductivity along and across the cracks), as well as a semi-analytical solution for the case of cracks with Poiseuille type conductivity assuming a linear relation between the discharge and the pressure gradient along the crack. These solutions, derived first for an isotropic matrix, were then extended to anisotropic matrices using a general transformation lemma. Finally, using the solution obtained for a single crack, a closed-form estimation of the effective permeability of microcracked porous materials with weak crack density was derived from a self-consistent upscaling scheme. In a subsequent publication (Pouya, 2012) has presented the governing equations for flow in three-dimensional heterogeneous and anisotropic porous media containing cracks with infinite transverse permeability leading to pore pressure continuity across the crack was assumed and thus excluding the case of impervious fractures. Fractures have been modeled as zero thickness inclusions with the possibility of multiple intersections. It has been assumed that flow obeys Darcy's law in the porous matrix and a Poiseuille type law in fractures. In this manner a general potential solution, based on singular integral equations, has been established for the steady state flow in an

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infinite fractured body with uniform and isotropic matrix permeability. The main unknown variable in the equations was the pressure acting on the crack surfaces, reducing thus from three to two the dimension of the numerical problem. A general transformation lemma was then given that allowed the extension of the solution to matrices with anisotropic permeability. The results have lead to a simple and efficient numerical method for modeling steady-state flow in three-dimensional undeformable fractured porous bodies. Pouya and Vu (2012a) used the solutions derived by (Pouya, 2012) to establish a semi-analytical solution for the case of an elliptical disc-shaped crack. It was shown that this solution takes a closed-form expression for the case of superconducting circular cracks. It has been also shown that the flow solution for an elliptical disc-shaped crack obeying the Poiseuille's law is different from that obtained as the limiting case of flattened ellipsoidal inclusions. The results were then used to establish dilute Mori–Tanaka and self-consistent estimates of the effective permeability of porous media containing Poiseuille's type elliptical cracks. Furthermore Pouya and Vu (2012b) have been first completed the work of Pouya and Ghabezloo (2010) by the mass balance equation at crack intersection points. Then a numerical method has been developed to solve the general system of singular equations for the case of an infinite body containing a dense family of curvilinear and intersecting cracks. This method has been based on the resolution of the equations for a finite number of *collocation points*. A special choice of collocation points has been given to simplify the computation. All the elementary integral terms have been explicitly presented that lead to a highly efficient and fast calculation method. After presenting the method, a successful validation has been first presented by comparing the numerical results obtained for a single *superconductive* crack with the closed-form solution for this case. Subsequently, the flow has been modeled around several curvilinear and intersecting cracks and the mass balance has been checked carefully at intersection points. The same authors have also presented the improvements achieved of the method presented by Liolios and Exadaktylos (2006), namely the numerical solution of intersecting cracks (although in the paper by Liolios and Exadaktylos (2006) the mathematical formulation permitted the consideration of either intersection or non-intersecting cracks as was mentioned above) and the consideration of anisotropic hydraulic conductivity. In the sequel, the effective permeability of a material containing a random crack distribution, inspired from geological observations on a rock formation, has been studied for the illustration of their method. Finally, the effective permeability of a periodic crack network has been calculated with this method and the result is compared to those obtained by theoretical methods. Seyedi et al. (2011), have also studied the effects of $CO_2$ injection on the hydromechanical behavior of a fault running across a reservoir which is a significant problem in view of risk assessment of $CO_2$ storage. At the fault-zone scale, the equivalent permeability of the statistically distributed fractures has been modeled by the singular integral equations method. At the site scale, a fault has been modeled as a shallow layer of filling material surrounded on both sides by a series of hydromechanical joint elements and an equivalent porous media representing the damaged zone. The capacities of the model have been demonstrated through a large-scale simulation. The fluid flow across the fault due to the injection has been also studied.

Since the above presented theories have been concerned only with rigid porous media, Vernerey (2011) has proposed a unified mathematical framework based on mixtures theory to describe the macroscopic mechanical behavior of porous continua with embedded interfaces filled with an inviscid fluid. Interfaces whose thickness is significantly smaller than the typical length-scale of applications (size of solid, characteristic length scale of physical processes), have been modeled as two-dimensional surfaces across which fields such as displacement, strain, fluid pressure and pressure gradient are discontinuous. This paper showed the need for future research activities aiming at deriving material constants from homogenization techniques, as well as deriving a numerical strategy to obtain a solution of the proposed formulation for the general case of curved interfaces in elastic porous bodies.

When interfaces are present, significant fluid velocity gradients may develop in the vicinity of the interface, giving rise to boundary layers. The characteristic length-scale of the boundary layers is very sensitive to the fluid viscosity a phenomenon that is well captured by considering Darcy–Brinkman equation. Vernerey (2012) has considered the microscopic problem of Darcy–Brinkman flow near a microscopic interface of finite thickness, and he investigated the nature of microscopic interface fluxes in terms of various interface and bulk properties. The information obtained from this study was then utilized in conjunction with the concept of thickness averaging so that a relationship between microscopic and macroscopic flows could be established. This subsequently enabled the derivation of macroscopic permeabilities in terms of microscopic interface properties.

Furthermore, in order to understand pore-fluid driven mineralization in porous crack-like inclusions and fault, theoretical studies have been conducted to derive analytical solutions for *steady-state pore-fluid flow* and related *heat transfer* within and around cracks and geological faults in fluid-saturated porous media (Zhao et al., 2006a, 2006b, 2008a, 2008b). When the heat transfer (i.e. heat flow) process is considered in geothermal reservoirs and the fracture of Earth's crust, convective pore-fluid flow can take place in large cracks and geological faults (Zhao et al., 2004, 2008b).

The above studies provide very good models of fluid flows within porous solids containing cracks (and can be extended to arbitrary porous interfaces), with significant technological applications in petroleum engineering and $CO_2$ storage among many others. However, in the above works, the transient behavior or the interstitial fluid flow around cracks was not considered. Herein, the general solution of the transient fluid flow problem for a homogeneous and hydraulically isotropic fully-saturated porous medium with a semi-infinite straight crack lying along the positive *Ox*-axis as is illustrated in Fig. 1, is found. We are considering a semi-infinite crack because in a first place we want to study the state-of-affairs of the pore pressure and fluid discharge close to the crack tip. In order to further simplify our analysis in this first attempt, we consider that the solid skeleton is rigid, hence the lips of the crack are undeformable. Then, for illustration purposes the Dirichlet initial-boundary value problem for a prescribed arbitrary pore pressure

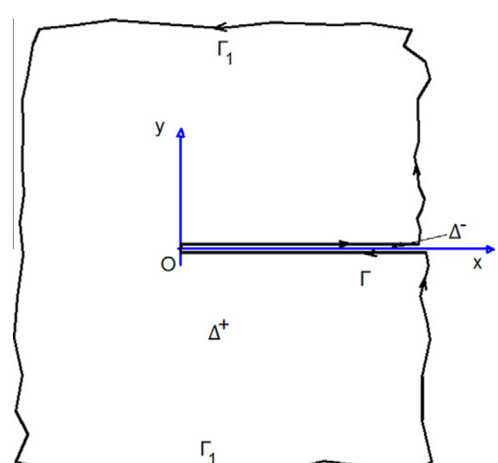


**Fig. 1.** Semi-infinite crack in an unbounded homogeneous porous medium and Cartesian coordinates with origin *O* at the tip.

along the crack is reduced to a Cauchy singular integral equation of the 2nd kind, which is subsequently solved by recourse to Gauss–Chebyshev numerical integration scheme. This solution is the first step towards the solution of transient fluid flow around multiple cracks and of the coupled poromechanical problem of a crack or cracks in deformable porous media. Generally, the derivation presented here serves as a closed form solution that can be used as benchmark for subsequent numerical solutions. This solution could be extended to an arbitrary number of finite cracks, but this is out of the scope of the present paper.

The theory of porous media, originally introduced by Biot (1941) in the context of consolidation has been very successful at describing the interactions between fluid flow and deformation of porous solids. In this context for isothermal conditions of fluid flow in fluid infiltrated and deformable isotropic porous bodies without internal fluid sinks or sources, the governing equation that may be derived from mass balance of the pore fluid, applicability of Darcy's law and the assumption that the porosity of the medium depends on pore pressure and effective hydrostatic stress (i.e. the stress exerted on the solid skeleton) both considered as state variables, has the following form in the plane $Oxy$

$$\beta\frac{\partial p}{\partial t}+\gamma\frac{\partial \sigma'}{\partial t}=k\nabla^2 p \tag{1a}$$

wherein the symbol $p$ stands for the increment of fluid pressure $[\mathrm{F\,L^{-2}}]$, $\nabla^2$ for the Laplacian operator in the Cartesian coordinates system $Oxy$, $\beta=\partial n/\partial p$ denotes the *pore volume compressibility* with units $[\mathrm{F^{-1}\,L^2}]$, $n$ stands for the porosity of the medium, $\gamma=\partial n/\partial\sigma'$ with units $[\mathrm{F^{-1}\,L^2}]$ represents the change of porosity due to the change of the effective hydrostatic stress indicated by the symbol $\sigma'$, and $k$ with units $[\mathrm{L^4\,F^{-1}\,T^{-1}}]$ (L = length, F = force and T = time)[1] denotes the permeability of the porous medium and depends among other things on the viscosity of the interstitial fluid. It is related with the physical permeability $\kappa$ of the porous medium that has dimensions of surface and the dynamic viscosity of the fluid $\mu_d$ with units $[\mathrm{F\,L^{-2}\,T}]$ by the relation $k=\kappa/\mu_d$. The physical or intrinsic permeability stands as a measure of the cross-sectional area of the microscopic channels in the interconnected void space. The ratio $D=k/\beta$ is also used as "*diffusivity coefficient*" with units $[\mathrm{L^2\,T^{-1}}]$; for example for rocks it ranges from $10^{-4}$ cm$^2$/s for low porosity shales up to $10^4$ cm$^2$/s for very permeable sandstones (Detournay and Cheng, 1993). As is noted by the same authors, this means that for a core length of 5 cm, the time required for the pore pressure to reach equilibrium could thus vary from less than one second for a sandstone to the order of days for a shale. The l.h.s. of Eq. (1a) represents fluid storage, while the term on the r.h.s. represents the fluid transport. In the sequel, in order to simplify our analysis we assume an undeformable fluid infiltrated body with $\gamma\ll\beta$, hence Eq. (1a) reduces to a pure diffusion equation for $p$. It is worth noting that by substituting pore volume compressibility $\beta$ with $\rho c$, with $\rho$ denoting density of the medium and $c$ the specific heat, $k$ also denoting thermal conductivity, and instead of pressure $p$ putting temperature $T$, then Eq. (1a) (for $\gamma=0$ or $\partial\sigma'/\partial t=0$) represents the heat conduction equation obeying Fourier's law without a heat source or sink inside the body. The behavior of thermal stresses in the vicinity of the crack tips when a steady heat flow is disturbed by the presence of cracks, and large thermal stresses arise in the neighborhood of the crack tips and cause crack propagation in structural components has been studied by Sekine (1977) and Barzokas and Exadaktylos (1995), both by recourse to temperature and stresses complex potentials functions and singular integral equations theories.

The so-called specific discharge vector $q$ or discharge velocity [L/T] obeys Darcy's law that has been already assumed valid to derive Eq. (1a) and is given in the following complex form

$$q=-k\left(\frac{\partial p}{\partial x}+i\frac{\partial p}{\partial y}\right) \tag{1b}$$

where $i$ denotes the imaginary unit, i.e. $i\equiv\sqrt{-1}$.

A simple numerical approximation of first order differentials is the backward or implicit finite difference method, i.e.

$$\frac{\partial p}{\partial t}\approx\frac{p(t+\Delta t)-p(t)}{\Delta t}+o(\Delta t) \tag{2}$$

where $o(\cdot)$ denotes Landau's order-of-magnitude symbol and represents the error of the method. Hence by substituting Eq. (2) into Eq. (1a) we end-up with a non-homogeneous modified Helmholtz's equation that must be solved at each time step $t+\Delta t$, i.e.

$$\nabla^2 u(x,y)-\lambda^2 u(x,y)=f(x,y) \tag{3}$$

where we have set

$$\begin{aligned}&\lambda=\sqrt{\frac{\beta}{k\Delta t}}=\frac{1}{\sqrt{D\Delta t}}\\&u=p^{[t+\Delta t]}(x,y)\\&f=-\lambda^2p^{[t]}(x,y)\end{aligned} \tag{4}$$

The superscript $t$ in brackets i.e. [t] denotes time with units [T], and $\lambda$ has units $[\mathrm{L^{-1}}]$. That is to say, $u$, stands for the current pore pressure at $t+\Delta t$. In order to find the solution of the parabolic partial differential equation Eq. (1a) in the space and time domains, the initial and boundary conditions should be known a priori. In order to get reasonable accuracy it is needed to choose $\Delta t$ sufficiently small because the global error is proportional to $\Delta t$; however, regarding stability, any $\Delta t$ may be used without the solution being blown-up.

## 2. Construction of the Riemann function

An elegant technique has been developed by Vekua (1967) for the solution of the non-homogeneous second order elliptical linear partial differential equations (pdes) of the form

$$E(u)=\nabla^2u+\alpha(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=f(x,y) \tag{5}$$

where $u(x,y)$ is the desired solution of the problem. By introducing the complex variables

$$z=x+iy\quad\text{and}\quad\bar z=x-iy \tag{6a}$$

we have

$$x=\frac{z+\bar z}{2}\wedge y=\frac{z-\bar z}{2i} \tag{6b}$$

By considering the following differential operators

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y},\quad 2\frac{\partial}{\partial \bar z}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y} \tag{7}$$

then Eq. (5) takes the complex form

$$F(u)=\frac{\partial^2u}{\partial z\partial\bar z}+A(z,\bar z)\frac{\partial u}{\partial z}+B(z,\bar z)\frac{\partial u}{\partial\bar z}+C(z,\bar z)u=F(z,\bar z) \tag{8}$$

wherein

$$\begin{aligned}&A(z,\bar z)=\frac14\left[\alpha\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)+ib\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)\right]\\&B(z,\bar z)=\frac14\left[\alpha\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)-ib\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)\right]\\&C(z,\bar z)=\frac14c\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)\\&F(z,\bar z)=\frac14f\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)\end{aligned} \tag{9}$$

[1] We could have used the units M = mass, L = length, and T = time, instead.

Young et al. (2002) have previously proposed the Riemann Complex Boundary Element Method (RCBEM) based on the boundary element method (BEM) and the theory of Vekua and its modification, as well as complex Riemann function as the fundamental solution. The same authors verified the feasibility and accuracy of RCBEM by applying it to different case studies of potential flows, Helmholtz equation problem and advection–diffusion problem and results were compared with analytical solutions and other numerical models. The results have been found to be satisfactory and proved the applicability of RCBEM for various two-dimensional elliptic equation problems. Herein, Vekua's method is extended to consider diffusive-like flows around cracks.

As it is demonstrated in Appendix A the solution of Eq. (3) for the current pore pressure field attains the following final form

$$u(z,\bar{z})=\mathrm{Re}\left\{I_0(\lambda r^*)\varphi(z)+\frac{\lambda\sqrt{\bar{z}}}{2}\int_0^z\frac{I_1(\lambda\sqrt{\bar{z}(z-t)})}{\sqrt{z-t}}\left[\varphi(t)+\frac{4F(t,\bar{z})}{\lambda^2}\right]dt\right\} \tag{10}$$

where we have set $r^*=\sqrt{z\bar{z}}=|z|=\sqrt{x^2+y^2}$ and $\varphi(z)$ represents a potential that is found from boundary conditions (i.e. Appendix A). It is worth noting that for $\lambda\to 0$ the potential $\varphi(z)$ represents the steady-state pore pressure solution, since for this limit, the above Eq. (10) with $4F(t,\bar{z})/\lambda^2=-p^{[t]}(t)$ (from the last of relationships (9) and (4)) takes the form

$$u(z,0)|_{\lambda\to 0}=\varphi(z)|_{\lambda\to 0} \tag{11}$$

## 3. Solution of the Dirichlet initial-boundary value problem

In a first step we attack Dirichlet's problem, i.e. when along the entire contour of the crack $\Gamma$ of the rigid body (e.g. Fig. 1) the value of the pore pressure is prescribed that may also vary with time (i.e. the crack is pressurized by an injected fluid according to some manner along its lips through time)

$$u^+=g^{[\chi]}(x,0)=g^{[\chi]}(s),\quad s\in\Gamma,\ \chi\geqslant 0 \tag{12}$$

with $s$ to denote the abscissa running along the crack, the Greek lowercase letter $\chi$ appearing as superscript in brackets to denote hereafter the current time, and assuming that prior to pressurization of crack lips, there is zero pore pressure everywhere in the fluid saturated porous medium (or in the more realistic case of given reservoir or formation pore pressure, then this should be added to the transient solution of the pore pressure field that will be found from the above initial-boundary value problem). Lagging of the fluid front behind the crack tip (fracture front) could be also considered since it is of great interest in hydraulic fracturing processes in oil industry. Assuming that both $g^{[\chi]}(s)$ and the boundary value of the potential function $\varphi(t)$ satisfy the Hoelder condition[2] (Muskhelishvili, 1953) on boundary $\Gamma$, then the latter is a real function and may be expressed in the form of *modified simple layer potential* (Liolios and Exadaktylos, 2006)

$$\varphi(z)=\frac{1}{2\pi}\int_\Gamma\frac{\mu(t)dr}{r}=\frac{1}{2\pi}\int_\Gamma\mu(t)d\ln r=\frac{1}{2\pi}\int_0^\infty\frac{\mu(t)d|t-z|}{|t-z|} \tag{13}$$

where $\mu(t)$ is the real unknown value of the density of the Cauchy integral, $r=|t-z|$ denotes the distance between the points $t$ and $z$, and $|\cdot|$ denotes the modulus of the complex variable that it encloses. It should be noticed that the Cauchy integral is interpreted in its principal value sense. The physical meaning of Eq. (13) is that the pore pressure at every point $z$ of the plane, depends only on its inverse distance $r$ from every point $t$ of the crack contour $\Gamma$ multiplied by the appropriate weight, which is represented by the density function $\mu(t)$ defined along the crack lips.

Based on the representation (13) the fluid discharge vector q at a given point $z=x+iy$ of the plane in steady-state conditions, may be found from that is given by Eq. (1b) as follows

$$q(z)=q_s(z)+iq_n(z)=\frac{k}{2\pi}\int_\Gamma\frac{\mu(t)\cos\alpha(t,z)dr}{(t-z)^2}+i\frac{k}{2\pi}\int_\Gamma\frac{\mu(t)\sin\alpha(t,z)dr}{(t-z)^2} \tag{14}$$

where $q_s(z),\ q_n(z)$ denote the tangential and the normal fluid discharge, respectively, and $\alpha(t,z)$ is the angle enclosed by the vector $\overrightarrow{tz}$ and the positive direction of $Ox$-axis that is measured anticlockwise (Muskhelishvili, 1953).

Substituting the above value of $\varphi(z)$ as is given by Eq. (13) into Eq. (10) the following integral representation of $u$ is obtained

$$u(x,y)=\int_0^\infty\mu(t)M(z,t)dt+\mathrm{Re}\left\{\frac{2\sqrt{\bar{z}}}{\lambda}\int_0^z\frac{I_1(\lambda\sqrt{\bar{z}(z-\tau)})}{\sqrt{z-\tau}}F\left(\frac{\tau+\bar{z}}{2},\frac{\tau-\bar{z}}{2}\right)d\tau\right\} \tag{15}$$

in which

$$M(z,t)=\mathrm{Re}\left\{\frac{I_0(\lambda r^*)(d|t-z|/dt)}{2\pi|t-z|}+\frac{\lambda\sqrt{\bar{z}}/dt}{4\pi}\int_0^z\frac{I_1(\lambda\sqrt{\bar{z}(z-t_1)})d|t-t_1|}{\sqrt{z-t_1}|t-t_1|}dt_1\right\} \tag{16}$$

Now take the limit of Eq. (15) as the point $z=x+iy$ approaches some point $x_0$ of $\Gamma$ (i.e. along positive $Ox$-axis) from the domain $\Delta^+$ (Fig. 1) by using *Plemelj formulae or theorem* depicting the basic relationships among boundary values of Cauchy integrals (Muskhelishvili, 1953) and substituting the result into the boundary condition (12) we obtain the following singular integral equation of the 2nd kind that must be solved at each instant of time

$$\begin{aligned}&\frac{1}{2}\mu(x_0)+\int_0^\infty\mu(t)M(x_0,t)dt=g^{[\chi]}(x_0)-\\&-\frac{2\sqrt{x_0}}{\lambda}\int_0^{x_0}\frac{I_1(\lambda\sqrt{x_0(x_0-\tau)})}{\sqrt{x_0-\tau}}F\left(\frac{\tau+x_0}{2},\frac{\tau-x_0}{2}\right)d\tau\\&t_0\in[0,\infty)\\&M(x_0,t)=\mathrm{Re}\left\{\frac{I_0(\lambda x_0)(d|t-x_0|/dt)}{2\pi|t-x_0|}+\frac{\lambda\sqrt{x_0}/dt}{4\pi}\int_0^{x_0}\frac{I_1(\lambda\sqrt{x_0(x_0-t_1)})d|t-t_1|}{\sqrt{x_0-t_1}|t-t_1|}dt_1\right\}\end{aligned} \tag{17}$$

For easy reference we also recapitulate below the derived formulae

$$\begin{aligned}&\lambda^2=\frac{\beta}{k\Delta\chi},u(x,y)=p^{[\chi]}(x,y)\\&F\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)=\frac{1}{4}f\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)\\&f(x,y)=-\frac{\beta}{k\Delta\chi}p^{[\chi-\Delta\chi]}(x,y)=-\lambda^2p^{[\chi-\Delta\chi]}(x,y)\\&\varphi(z)=\frac{1}{2\pi}\int_0^\infty\frac{\mu(\tau)d|\tau-z|}{|\tau-z|}\end{aligned} \tag{18}$$

wherein the symbol $\Delta\chi$ appearing as superscript in brackets denotes the time increment. For the solution of the above problem an initial condition for the instant of time $\chi=0$ is also needed for the pore pressure along the crack. Furthermore, by taking into

[2] In mathematics, a real or complex-valued function $f$ on $d$-dimensional Euclidean space satisfies a *Hoelder condition*, or is *Hoelder continuous or H-continuous*, when there are nonnegative real constants $C$, $a$, such that

$$|f(x)-f(y)|\leqslant C|x-y|^a$$

for all $x$ and $y$ in the domain of $f$.

account that along the crack the potential assumes the following value

$$\varphi(x) = \frac{1}{2\pi}\int_0^\infty \frac{\mu(t)d|t-x|}{|t-x|} = \frac{1}{2\pi}\int_0^\infty \frac{\mu(t)dt}{t-x} \tag{19}$$

then Eq. (17) at the time instant $\chi$ simplifies as follows

$$\begin{aligned}&\frac{1}{2}\mu(x_0) + \frac{1}{2\pi}\int_0^\infty \left\{ I_0(\lambda x_0) + \frac{\lambda\sqrt{x_0}(x-x_0)}{2} \right.\\ &\times \left. \int_0^{x_0} \frac{1}{\sqrt{x_0-x_1}} \frac{I_1(\lambda\sqrt{x_0(x_0-x_1)})dx_1}{x-x_1} \right\} \frac{\mu(x)dx}{x-x_0}\\ &= g^{[\chi]}(x_0) + \frac{\lambda}{2}\sqrt{x_0}\int_0^{x_0} g^{[\chi-\Delta\chi]}(x)\frac{I_1(\lambda\sqrt{x_0(x_0-x)})}{\sqrt{x_0-x}}dx, \quad x_0\\ &\in [0,\infty),\ \chi > 0\end{aligned} \tag{20}$$

If the density function along the crack $\mu(x_0),\ x_0 \in [0,\infty)$ is found from the above singular integral equation of the 2nd kind by virtue of an appropriate quadrature rule, then by substituting this function into (Eq. (13)) the modified simple layer potential $\varphi(z)$ is also found. Subsequently, at every instant of time the solution for the pore pressure $u(z,\bar{z})$ is found from Eq. (10). Finally it is worth noting that if the prescribed pore pressure along the crack lips given by the function $g(x)$ remains constant through time, then in the above Eq. (20) one must set $g^{[\chi]}(x_0) = g^{[\chi-\Delta\chi]} = g(x_0),\ x_0 \in [0,\infty)$.

## 4. Solution of the singular integral equation of the second kind

### 4.1. Solution of the steady-state Dirichlet problem

Next, we investigate the behavior of the above singular integral Eq. (20) of the 2nd kind by taking the limit $\lambda \to 0$, that corresponds to either very long time after the application of the pressure on crack lips, or to a very large permeability $k$ of the porous medium, or for very low compressibility $\beta$ of the latter. This case corresponds obviously to the steady-state solution of the pressure field problem. It is therefore rather obvious from the preceding analysis, that the steady-state solution for the pore pressure (albeit not true for the fluid discharge) will be independent of the permeability k of the porous medium (e.g. see Eq. (21) below). Taking into account the limiting values of the involved Bessel functions one may easily end-up from Eq. (20) into a much simpler singular integral equation of the 2nd kind

$$\frac{1}{2}\mu(x_0) + \frac{1}{2\pi}\int_0^\infty \frac{\mu(x)dx}{x-x_0} = g(x_0), \quad x_0 \in [0,\infty),\ \lambda \to 0 \tag{21}$$

in which we assume that the known (input) function at the r.h.s. is Hoelder or H- continuous.

The endpoint of the crack at $x = 0$ is a point of geometric singularity. Physical arguments provide sufficient information about the behavior of the unknown density function $\mu(x_0)$. Invariably, these arguments simply account to stating that if the unknown function is a potential (i.e. pore pressure, temperature, displacement, etc.) it has to be bounded at the singular point $x = 0$. It can be shown that the fundamental function or weight function of Eq. (21) which characterizes the behavior of $\mu(x_0)$ at the singular point of the crack tip at (0,0) is given by (Muskhelishvili, 1953; Erdogan and Gupta, 1972; Erdogan et al., 1973; Liolios and Exadaktylos, 2006)

$$\hat{w}(x_0) = \sqrt{x_0} \tag{22}$$

Substituting the transformation $\mu(x) = \hat{w}(x)\hat{\mu}(x)$ into Eq. (21) it is found

$$\frac{1}{2}\hat{w}(x_0)\hat{\mu}(x_0) + \frac{1}{2\pi}\int_0^\infty \frac{\hat{w}(x)\hat{\mu}(x)dx}{x-x_0} = g(x_0) \tag{23}$$

The function $\hat{\mu}(x)$ along the semi-infinite crack is the new unknown density that is bounded at the crack tip and satisfies the Hoelder condition in order to ensure existence of the principal value of the Cauchy integral in Eq. (23).

For numerical reasons it is advantageous, although not imperative, to eliminate the infinite boundary in the integral of Eq. (23). This can be achieved by suitable coordinate transformation. The following variable transformation proposed by Ioakimidis and Theocaris (1980) is adopted here

$$x = \frac{1}{c}\ln\frac{2}{1-t} \tag{24}$$

or

$$t = 1 - 2\exp(-cx) \tag{25}$$

with

$$dx = \frac{1}{c}\frac{dt}{1-t} = \frac{\exp(cx)}{2c}dt \tag{26}$$

where $c$ is a scaling constant. In view of the transformation (25) the semi-infinite integration interval $[0,\infty)$ of the variable $x$ changes into the integration interval $[-1,1]$ of the corresponding variable $t$. Having surpassed this difficulty, the unknown density $\hat{\mu}(x_0)$ may be calculated by recourse to the *Gauss–Chebyshev* numerical integration method for singular integral equations (Erdogan and Gupta, 1972; Erdogan et al., 1973) exploiting the properties of orthogonal Chebyshev polynomials of the first and second kind. By applying numerical integration rule to Eq. (23), and considering Eqs. (24)–(26) there results the following system of linear algebraic equations with unknowns the *densities* $\hat{\mu}(x_r)$ at the collocation points $x_r$ along the crack,

$$\begin{aligned}&\frac{1}{2}\sqrt{x_r}\hat{\mu}(x_r) + \frac{1}{2}\sum_{j=1}^{n}\frac{B_j(x_j)}{x_j-x_r}\hat{\mu}(x_j) = g(x_r), \quad r = 1,2,\ldots,n+1\\ &B_j(x_j) = A_j(t_j)\frac{\sqrt{x_j}}{w^*(t_j)}\frac{dx}{dt} = A_j(t_j)\frac{\sqrt{x_j}}{\sqrt{1-(1-2\exp(-cx_j))^2}}\frac{\exp(cx_j)}{2c},\\ &w^*(t_j) = (1-t_j^2)^{1/2}, \quad A_j(t_j) \approx \frac{1-t_j^2}{n+1}, \quad x_j = \frac{1}{c}\ln\left(\frac{2}{1-t_j}\right),\\ &\quad x_r = \frac{1}{c}\ln\left(\frac{2}{1-t_r}\right)\end{aligned} \tag{27}$$

wherein $n$ is the number of integration points and $r$ is the number of collocation points. According to Erdogan and Gupta (1972) the integration points $t_j$ in the interval $[-1,1]$ are the roots of the Chebyshev polynomials of the second kind and of order $n$ denoted as usual as $U_n(t_j) = 0$

$$t_j = \cos\left(\frac{j\pi}{n+1}\right), \quad j = 1,2,\ldots,n \tag{28}$$

while the collocation points $t_r$ also in the interval $[-1,1]$ are the roots of the Chebyshev polynomials of the first kind and of order $n+1$ denoted as usual as $T_{n+1}(t_r) = 0$, that is

$$t_r = \cos\left(\frac{\pi(2r-1)}{2(n+1)}\right), \quad r = 1,2,\ldots,n+1 \tag{29}$$

It may be noticed that the new abscissas $x_j$ and weights $B_j$ in the semi-infinite interval $[0,\infty)$ are related to the initial abscissas $t_j$ and the corresponding weights $A_j$ in the interval $[-1,1]$ according the transformation of Eq. (26) and the second of relations (27), respectively. In addition we may note that there are $n + 1$ collocation points to determine the $n$ unknown densities $\hat{\mu}(x_1),\ \hat{\mu}(x_2),\ldots,\hat{\mu}(x_n)$. It is also worth noticing that there is no need to satisfy in this case the single-valuedness of the pore pressure since the body is simply connected. Thus, it is sufficient to choose only $n$ of these points to determine the densities $\hat{\mu}(x_k)$. Herein the most harmless point to neglect is that one located farther from the crack tip, i.e. that one corresponding to $r = 1$.

Having found the density of the Cauchy integral along the crack, then the pore pressure and flow discharge may be found at any point of the unbounded medium. Indeed, from Eq. (13), and the integration scheme of Eq. (27), the potential function at any point $z = x + iy$ of the plane may be derived by employing the related Gauss–Chebyshev integration formula for non-singular integrals (Abramowitz and Stegun, 1965)

$$\varphi(z) = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{\tau}\hat{\mu}(\tau)d|\tau - z|}{|\tau - z|} = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{\tau}\hat{\mu}(\tau)(\tau - x)d\tau}{|\tau - z|^2} \approx \frac{1}{2}\sum_{j=1}^{n} B_j(x_j)\frac{(x_j - x)}{|x_j - z|^2}\hat{\mu}(x_j) \quad (30)$$

The above solution represents the steady-state pore pressure field around the crack tip $p(z)$. Further, it is not difficult to derive the components of the fluid discharge vector $q(z) = q_s(z) + iq_n(z)$ from Eq. (14)

$$\begin{aligned} q(z) &= q_s(z) + iq_n(z) \\ &= \frac{k}{2\pi}\int_0^\infty \frac{\sqrt{t}\hat{\mu}(t)\cos\alpha(t,z)dr}{|t - z|^2} + i\frac{k}{2\pi}\int_0^\infty \\ &\quad \times \frac{\sqrt{t}\hat{\mu}(t)\sin\alpha(t,z)dr}{|t - z|^2} \\ &\approx \frac{k}{2}\sum_{j=1}^{n} B_j(x_j)\cos\alpha(t,z)\frac{(x_j - x)}{|x_j - z|^3}\hat{\mu}(x_j) + i\frac{k}{2}\sum_{j=1}^{n} B_j(x_j) \\ &\quad \times \sin\alpha(t,z)\frac{(x_j - x)}{|x_j - z|^3}\hat{\mu}(x_j) \end{aligned} \quad (31)$$

In the following the convergence of the proposed numerical solution of the pressure and flow discharge fields around the crack tip is studied. In order to explore the convergence of the solution, we consider a straight semi-infinite crack lying along the positive $Ox$-axis and with the origin of the coordinate system placed at its tip. The crack lips are subjected to uniform dimensionless unit pressure $g(x) = 1$ (using the sign convention of positive compressive stresses and negative tensile stresses), whereas the scaling factor is taken as $c = 0.1$ in order to produce collocation points sufficiently far from the crack tip. Also, the pressure is made dimensionless by dividing it with 1 Pa, whereas the permeability coefficient of the porous medium is taken equal to 1 $m^2/(\text{Pa s})$ and the fluid discharge is made dimensionless by dividing it with the permeability coefficient. Figs. 2 and 3 display the dimensionless pore pressure p distribution along $Ox$ and $Oy$-axes, respectively for 12, 18, 24, and 30 integration points, respectively. Also, Figs. 4 and 5 illustrate the dimensionless specific flow discharges $q_x$ and $q_y$ along $Ox$ and $Oy$-axes, respectively. Finally, Fig. 6 illustrates the distribution of the steady-state dimensionless pore pressure field around the crack tip region. These figures illustrate that the Gauss–Chebyshev integration scheme of the derived singular integral equation of the second kind converges rapidly to the analytical pore pressure solution. Hence, this quadrature scheme will be followed also for the solution of the transient problem in Section 4.2.

### 4.2. Solution of the transient Dirichlet problem

Having solved the steady-state Dirichlet problem, the solution of the corresponding transient problem may be performed without any difficulty by following a similar Gauss–Chebyshev integration scheme of the singular integral equation (20) at each time step. That is to say, for an arbitrary time-invariant pore pressure distribution suddenly applied at instant of time $\chi = 0^+$ on the crack lips, and considering the numerical integration scheme given by Eqs. (27)–(29), then Eq. (20) becomes

$$\begin{aligned} &\frac{1}{2}\sqrt{x_r}\hat{\mu}(x_r) + \frac{I_0(\lambda x_r)}{2}\sum_{j=1}^{n}\frac{B_j(x_j)}{x_j - x_r}\hat{\mu}(x_j) \\ &\quad - \frac{\lambda\sqrt{x_r}(x_j - x_r)}{4}\sum_{j=1}^{n} B_j(x_j)\hat{\mu}(x_j)\int_0^{x_r}\frac{1}{\sqrt{x_r - x_1}}\frac{I_1(\lambda\sqrt{x_r(x_r - x_1)})dx_1}{x_1 - x_j} \\ &\quad = g^{[\chi]}(x_r) + \frac{\lambda\sqrt{x_r}}{2}\int_0^{x_r} g^{[\chi-\Delta\chi]}(x)\frac{I_1(\lambda\sqrt{x_r(x_r - x)})}{\sqrt{x_r - x}}dx, \\ &r = 1, 2, \ldots, n+1 \\ &g^{[\chi]}(x) = g^{[\chi-\Delta\chi]}(x) = \cdots = g^{[0]}(x) = g(x) \\ &0 \leqslant x < \infty, \quad y = 0 \end{aligned} \quad (32)$$

The definite integral in the l.h.s. of the above Eq. (32) may be also approximated as a finite sum with appropriate quadrature abscissas and weights based on properties of orthogonal polynomials, that is to say

$$\begin{aligned} &\int_0^{x_r}\frac{1}{\sqrt{x_r - x_1}}\frac{I_1(\lambda\sqrt{x_r(x_r - x_1)})dx_1}{x_1 - x_j} \\ &\quad = \int_0^{x_r}\frac{1}{\sqrt{(x_r - x_1)x_1}}\frac{\sqrt{x_1}I_1(\lambda\sqrt{x_r(x_r - x_1)})dx_1}{x_1 - x_j} \\ &\quad \approx \frac{\pi}{n_1}\sum_{k=1}^{n_1}\frac{\sqrt{x_k}I_1(\lambda\sqrt{x_r(x_r - x_k)})}{x_k - \hat{x}_j} \end{aligned} \quad (33)$$

wherein $n_1$ is the number of integration points, and $x_r, x_j$ are the collocation and integration points, respectively, given by relationships (29) and (28), through the transformation (24). In the above quadrature rule the integration points in the interval $[0, x_r]$ may be found by the linear transformation of the integration points $x'_k$ in the interval $[-1, 1]$ as follows

$$x_k = \frac{x_r}{2}x'_k + \frac{x_r}{2}, \quad k = 1, 2, \ldots, n_1 \quad (34)$$

According to Erdogan and Gupta (1972) the integration points $x'_k$ in the interval $[-1, 1]$ are the roots of the Chebyshev polynomials of the first kind and of order $n$ denoted as usual as $T_{n_1}(t_j) = 0$

$$x'_k = \cos\left(\frac{(2k - 1)\pi}{2n_1}\right), \quad k = 1, 2, \ldots, n_1 \quad (35)$$

For the evaluation of the above integral we may distinguish two cases, namely one with $x_j > x_r$ and another with $0 < x_j < x_r$. In the first case the above integral is non-singular and the integration is performed with Eq. (33) by simply setting $\hat{x}_j = x_j$. However, in the second case the integral in (33) is singular and should be evaluated accordingly. In this case the abscissas $x_j$ are chosen in such a manner so as: (1) to be roots of Chebyshev polynomials of the second kind $U_{n_1-1}(x'_j) = 0$, and (2) to be located as close as possible to $x_j$, that is

$$\begin{aligned} &r \approx \left\lceil \frac{n_1}{\pi}\cos^{-1}\left(\frac{2x_j}{x_r} - 1\right)\right\rceil \\ &x'_j = \cos\left(\frac{\pi r}{n_1}\right) \\ &\hat{x}_j = \frac{x_r}{2}(x'_j + 1) \end{aligned} \quad (36)$$

Finally, the quadrature scheme of the definite integral in the r.h.s. of Eq. (32) is also based on the above relations with Chebyshev polynomials of the first kind in the following manner

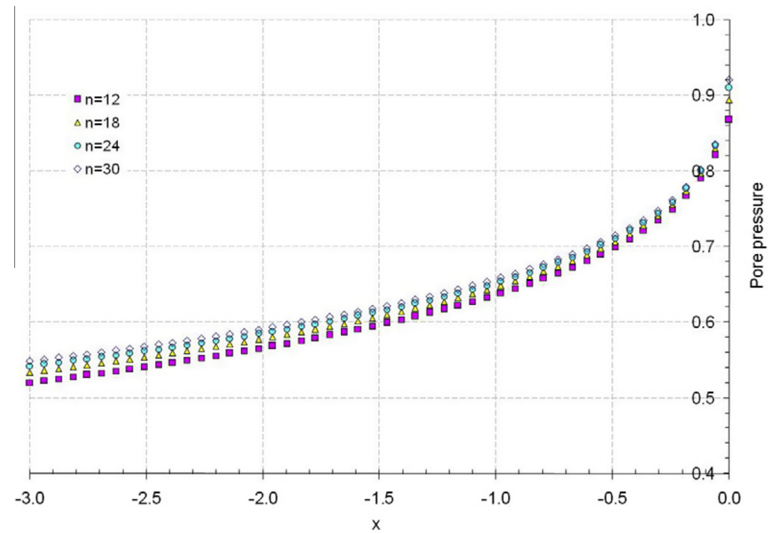


**Fig. 2.** Dimensionless pore pressure distribution in front of the crack tip along *Ox* axis.

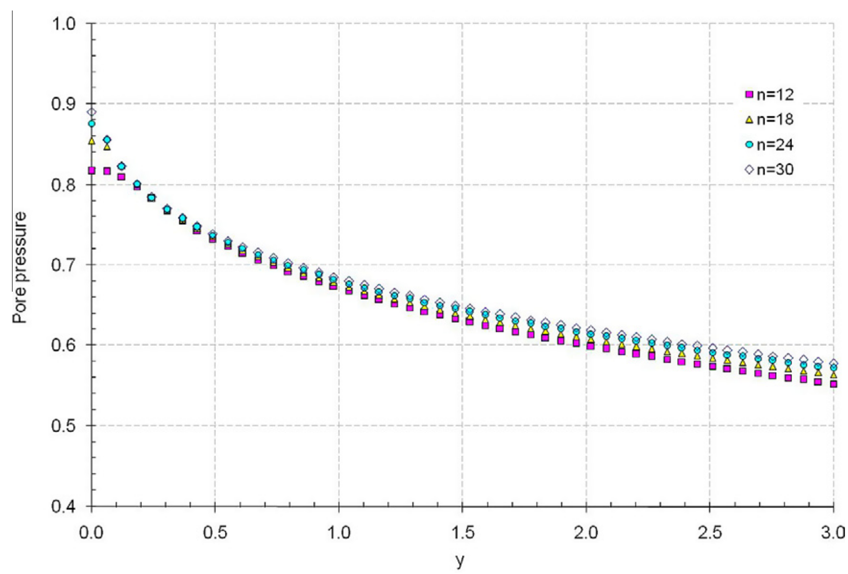


**Fig. 3.** Dimensionless pore pressure distribution along *Oy* axis.

$$\int_0^{x_r} g(x)\frac{I_1(\lambda\sqrt{x_r(x_r-x)})}{\sqrt{x_r-x}}dx = \int_0^{x_r} g(x)\frac{\sqrt{x}I_1(\lambda\sqrt{x_r(x_r-x)})}{\sqrt{(x_r-x)x}}dx \approx \frac{\pi}{n_1}\sum_{k=1}^{n_1} g(x_k)\sqrt{x_k}I_1(\lambda\sqrt{x_r(x_r-x_k)}) \tag{37}$$

where the integration points are given by Eqs. (34) and (35).

Regarding the previously examined case of crack lips subjected to a constant uniform unit pressure $g(x) = 1$, with the scaling factor $c = 0.1$, the solutions for the distribution of the density $\hat{\mu}(x_r)$ of the Cauchy integral at the collocation points $x_r$ along the crack for various values of $\lambda$ ranging from 0 to 0.3, and for a fixed number of integration points $n_1 = 15$, are shown in Fig. 7.

From the density numerical data such as those displayed in Fig. 7 it was found that the following approximate albeit accurate formula, that considerably simplifies the transient problem since it links the density of the latter problem with the corresponding density of the steady-state problem at some abscissa along the crack, holds true

$$\hat{\mu}(x) \approx \left(1 - \frac{\lambda}{0.4}\right)\hat{\mu}_0(x) = \left(1 - \frac{1}{0.4\sqrt{D\cdot t}}\right)\hat{\mu}_0(x) \quad 0 \leqslant x < \infty,\ y = 0 \tag{38}$$

wherein $\hat{\mu}_0(x)$ denotes the density of the steady-state problem that was solved in the preceding paragraph. This means that the upper limit for the parameter $\lambda$ is the value of 0.4. Eq. (38) clearly represents a *similarity transformation* in the sense described by Barenblatt (1996). According to Barenblatt a time-developing phenomenon is called self-similar if the spatial distribution of its properties (in this case the pore pressure) at various different moments of time can be obtained from one another by a similarity transformation such as relationship (38). The comparison of the predictions of the above formula (38) with the numerical solution of Eq. (32) is shown in Fig. 7 for three values of parameter $\lambda$.

In turn, the potential function for any $\lambda$ ranging from 0 to 0.4 is the following

$$\varphi(z) \approx \frac{(1-\frac{\lambda}{0.4})}{2\pi}\int_0^{\infty}\frac{\sqrt{t}\hat{\mu}_0(t)d|t-z|}{|t-z|} \tag{39}$$

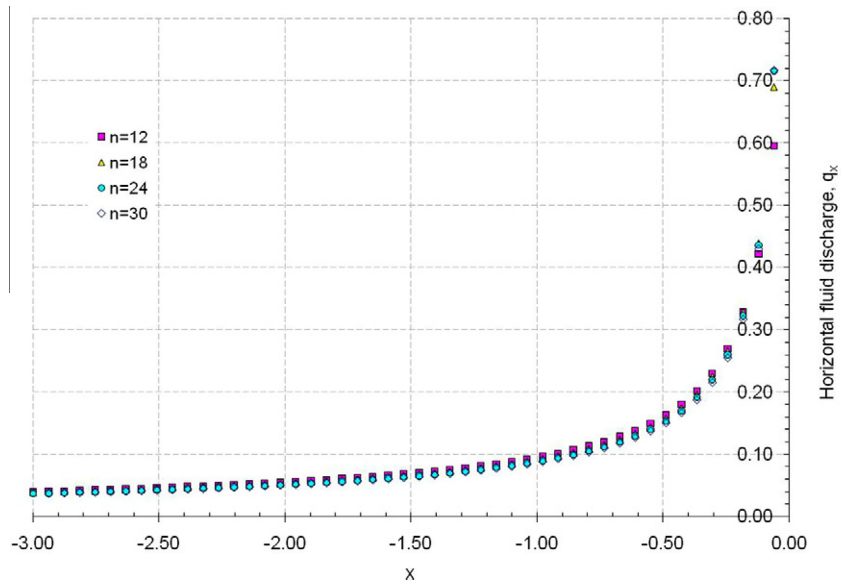


**Fig. 4.** Distribution of the component $q_x$ of the fluid discharge vector in front of the crack tip along *Ox* axis.

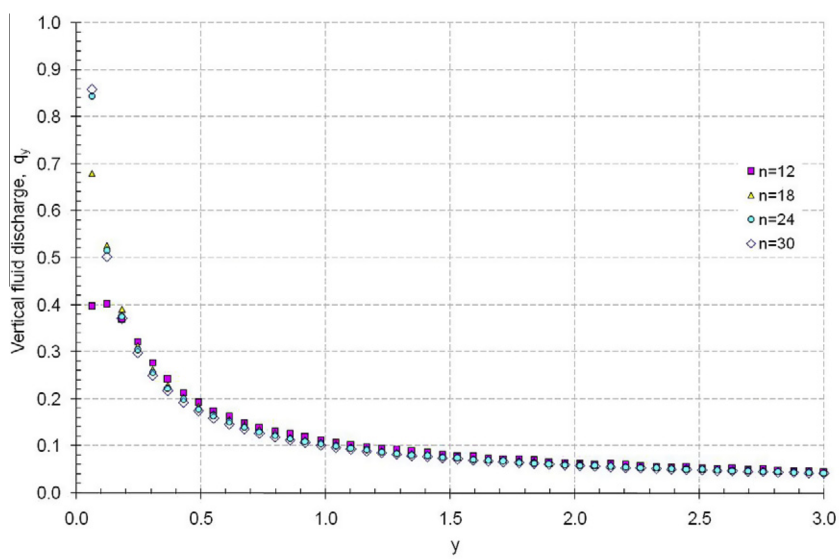


**Fig. 5.** Distribution of the component $q_y$ of the fluid discharge vector along *Oy* axis.

Having found the density of the Cauchy integral at the collocation points along the crack either by solving Eq. (32) or by using approximate formula (38), then the pore pressure and flow discharge may be found at any point of the unbounded medium outside the crack. First, the potential function $\varphi(z)$ at any point $z = x + iy$ of the plane may be derived by virtue of Eq. (30) as was done for the steady-state problem attacked previously. The only task is the computation of the integral appearing in Eq. (10), at the current time $t$ i.e.

$$I'(z) = \mathrm{Re}\left\{\frac{\lambda\sqrt{\bar{z}}}{2}\int_0^z \frac{I_1(\lambda\sqrt{\bar{z}(z-t)})}{\sqrt{z-t}}[\varphi(t) - p^{[t-\Delta t]}(t)]dt\right\} \tag{40}$$

Since the above integral is path-independent, a straight line integration from 0 to $z$ is adequate, i.e.

$$t(s) = x(s) + iy(s) = s + i\frac{y}{x}s, \quad 0 \leqslant s \leqslant x, \qquad dt = \left(1 + i\frac{y}{x}\right)ds \tag{41}$$

Substituting relationships (41) into Eq. (40) and after some algebraic manipulations it is found

$$I_1'(z) = \mathrm{Re}\left\{\frac{\lambda\sqrt{|z|}}{2\sqrt{x}}\int_0^x \frac{\sqrt{s}I_1(\lambda\sqrt{(x-s)(x^2+y^2)/x})}{\sqrt{(x-s)s}}\left[\varphi(s+i\frac{y}{x}) - p^{[t-\Delta t]}\left(s+i\frac{y}{x}\right)\right]ds\right\} \tag{42}$$

Due to Eq. (42) the final form of the current pore pressure $p^{[t]}(x,y)$ from Eqs. (10) and (30) takes the form

$$\begin{aligned} p^{[t]}(x,y) \approx{}& \frac{I_0(\lambda\sqrt{x^2+y^2})}{2}\sum_{j=1}^{n} B_j(x_j)\frac{(x_j - x)}{|x_j - (x+iy)|^2}\hat{\mu}(x_j) \\ &+ \mathrm{Re}\left\{\frac{\lambda\sqrt{x^2+y^2}}{2\sqrt{x}}\int_0^x \frac{\sqrt{s}I_1(\lambda\sqrt{(x-s)(x^2+y^2)/x})}{\sqrt{(x-s)s}}\right. \\ &\left.\times\left[\frac{1}{2}\sum_{j=1}^{n}B_j(x_j)\frac{(x_j - s)}{|x_j - (s+i\frac{y}{x}s)|^2}\hat{\mu}(x_j) - p^{[t-\Delta t]}(s+iy/x)\right]ds\right\} \end{aligned} \tag{43}$$

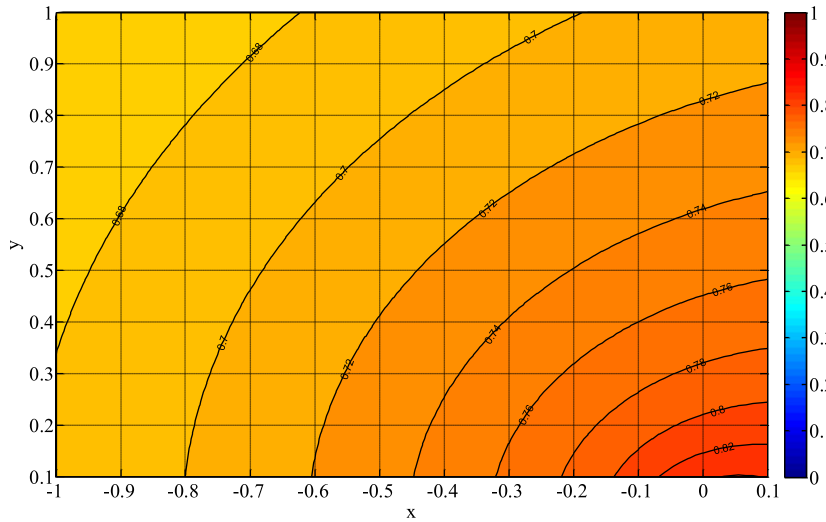


**Fig. 6.** Contour plot of the dimensionless pore pressure around the crack tip located on $x = 0$, $y = 0$ and extending along the positive $Ox$-axis.

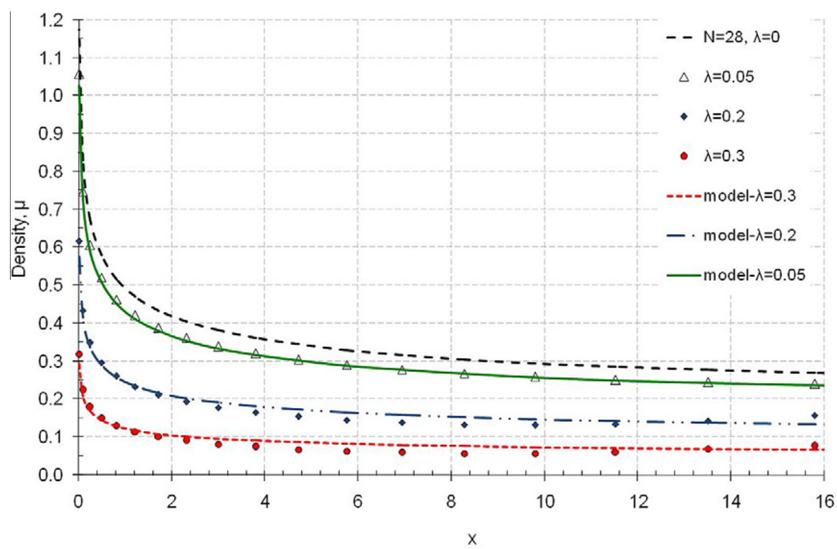


**Fig. 7.** Distribution of the density of the Cauchy integral along the crack for various values of $\lambda$ at hand.

Now, taking the limit $\sqrt{x^2+y^2} \to 0$ and considering the asymptotic formula $I_0(\lambda r) \approx 1 + (\lambda r)^2/4 + o(r^3)$, then the pore pressure field around the crack tip has as follows

$$p^{[t]}(x,y) \approx \sum_{j=1}^{n} B_j(x_j) \frac{(x_j - x)}{|x_j - (x+iy)|^2} \hat{\mu}(x_j) \tag{44}$$

Finally, considering the simple approximation (38) the transient pore pressure field may be easily found without solving the cumbersome system (32) but rather from the steady-state solution $\hat{\mu}_0(x)$ as follows

$$\begin{aligned} p^{[t]}(x,y) &\approx \left(1 - \frac{\lambda}{0.4}\right) \sum_{j=1}^{n} B_j(x_j) \frac{(x_j - x)}{|x_j - (x+iy)|^2} \hat{\mu}_0(x) \\ &= \left(1 - \frac{1}{0.4\sqrt{D\Delta t}}\right) \sum_{j=1}^{n} B_j(x_j) \frac{(x_j - x)}{|x_j - (x+iy)|^2} \hat{\mu}_0(x) \end{aligned} \tag{45}$$

For example, Figs. 8 and 9 illustrate the distribution of the pore pressure along $Ox$ and $Oy$-axes for the values of $\lambda = 1/\sqrt{D\Delta t}$ at hand.

This is a significant theoretical result since it greatly simplifies the study of the transient pore pressure problem around the crack tip by knowing only the solution of the much simpler steady-state problem.

### 4.3. Physical significance of the solution

This work may constitute the first step of attacking the transient poroelastic problem of the pressurized Griffith crack which is related to the hydraulic fracturing process in petroleum engineering. Hydraulic fracturing involves the propagation of a fracture in a porous brittle material, such as rock, due to the pressure exerted on the fracture surfaces by a viscous fluid that is pumped into the fracture. During the process of a hydraulic fracturing, the pumping rate is maintained at a higher rate than the fluid leak-off rate trough the interconnected pores, and the newly created fracture will continue to propagate and grow in the formation until shut-in. The analysis of the problem of a fluid-driven fracture propagating through a poroelastic medium involves the formulation of such model of an hydraulic fracture that is at the cross-road of four classical disciplines of engineering

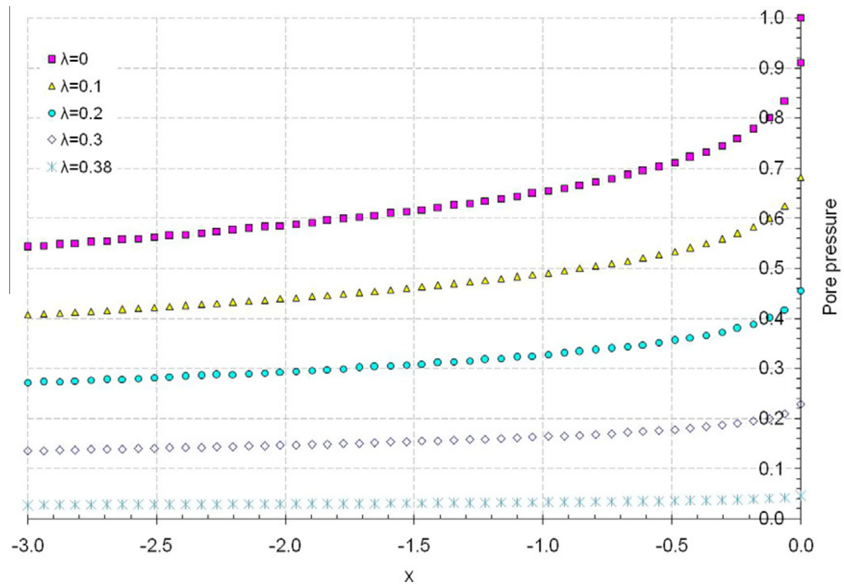


**Fig. 8.** Dimensionless pore pressure distribution in front of the crack tip along *Ox* axis for various values of $\lambda$-parameter.

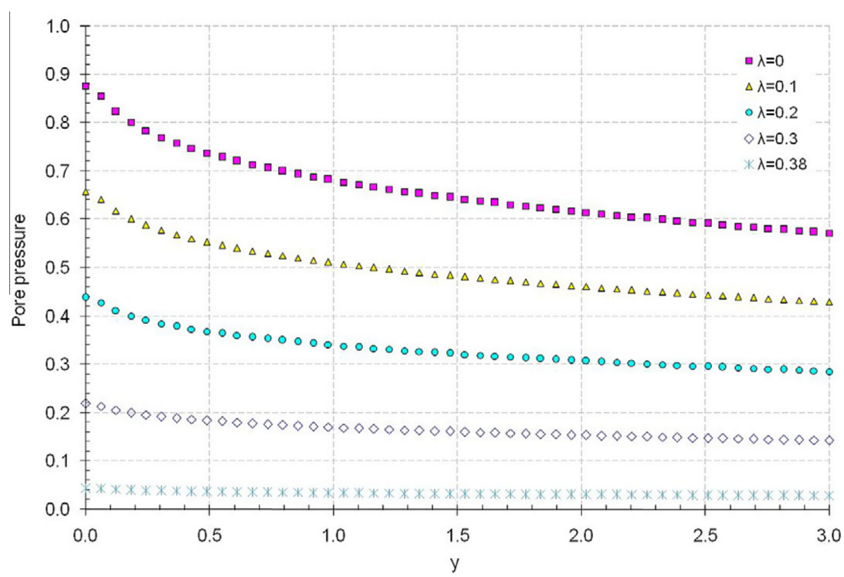


**Fig. 9.** Dimensionless pore pressure distribution along *Oy* axis for various values of $\lambda$-parameter.

mechanics: lubrication theory consisting from the continuity equation and Poiseuille's equation along the crack, filtration theory, fracture mechanics, and poroelasticity, with the latter including both elasticity and diffusion. Such a resulting mathematical model consists of a set of linear or non-linear integrodifferential history-dependent equations with singular behavior at the moving fracture front (Kovalyshen, 2010). It is noted here that most papers on hydraulic fracturing assume one-dimensional diffusion around the crack, thus neglecting the two-dimensional character of the diffusion problem presented here. Detournay and Garagash (2003) have presented a comprehensive literature review of the poroelastic problem of propagating cracks in permeable formations.

## 5. Concluding remarks

In this paper the parabolic partial differential equation of fluid mass balance in a porous medium is first transformed into the modified Helmholtz's equation by virtue of an implicit (backward) finite difference approximation of the time derivative. Subsequently by following Vekua's method of solving elliptical pde's we find the corresponding Riemann function that is used to construct the general solution. By assuming that the pore pressure field is described by a modified simple layer potential in the form of Cauchy integral the Dirichlet boundary value problem of prescribed pore pressure along a semi-infinite crack was formulated. The resulting singular integral equation of the second kind is first solved for the steady-state problem by virtue of Gauss–Chebyshev quadrature scheme. It was demonstrated that the applied quadrature method converges rapidly. Subsequently, the more elaborate transient problem is solved by following a similar procedure. From the numerical solution of the resulting equation for various values of the time parameter it was found that the density of the Cauchy integral for the transient problem may be found from the corresponding density along the crack by means of a similarity relationship.

Another possible application of the analytical solution presented herein, apart from being used as benchmark solution for studying the accuracy of numerical methods, is to be used for developing some new types of transient tip elements, so that the near-tip pore-fluid flow systems can be accurately simulated in numerical models.

Accordingly, the Neumann boundary value problem could be solved albeit following another integration method as is described by Liolios and Exadaktylos (2006). This analysis will be presented in a following paper since it is also of significant technological interest. Also, the present theory can be extended to anisotropic permeability as has been done in (Barzokas and Exadaktylos, 1995).

## Appendix A. Derivation of the explicit solution of the elliptic pde

Herein for easy reference the method proposed by Vekua (1967) is first outlined. Subsequently, the Riemann function of the non-homogeneous modified Helmholtz's equation (3) in the main text is found, and finally the general solution of the problem at hand is given.

By integrating the adjoint form of the *homogeneous* pde given by Eq. (8) of the main text, we obtain the so-called 'Riemann function' that satisfies the following Volterra equation of the 2nd kind (Vekua, 1967)

$$R(z,\bar{z};t,\tau)-\int_t^z B(\xi,\bar{z})R(\xi,\bar{z};t,\tau)d\xi - \int_\tau^{\bar{z}} A(z,\eta)R(z,\eta;t,\tau)d\eta + \int_t^z d\xi\int_\tau^{\bar{z}} C(\xi,\eta)R(\xi,\eta;t,\tau)d\eta = 1 \quad (A.1)$$

as well as equations

$$\begin{aligned} &R(t,\bar{z};t,\tau) = \exp\int_\tau^{\bar{z}} A(t,\eta)d\eta, \quad (t\in\Delta^+,\bar{z},\tau\in\Delta^-),\\ &R(z,\tau;t,\tau) = \exp\int_t^{z} B(\xi,\tau)d\xi, \quad (z,t\in\Delta^+,\tau\in\Delta^-),\\ &R(t,\tau;t,\tau)=1, \quad (t\in\Delta^+,\tau\in\Delta^-). \end{aligned} \quad (A.2)$$

The last of the above equations represents the normalization condition that is satisfied by the Riemann function. Also, the functions $A(z,\bar{z})$, $B(z,\bar{z})$, $C(z,\bar{z})$ are analytic functions of the two complex variables enclosed in parentheses for $z\in\Delta^+$, $\bar{z}\in\Delta^-$ (e.g. Fig. 1).

Then the solution of Eq. (8) in the simply connected domain $\Delta^+$ is given in terms of the Riemann function as follows (Vekua, 1967)

$$\begin{aligned} u(z,\bar{z}) &= u(z_0,\bar{z}_0)R(z,\bar{z};z_0,\bar{z}_0)+\int_{z_0}^z \Phi_1(t)R(z,\bar{z};t,\bar{z}_0)dt\\ &\quad+\int_{\bar{z}_0}^{\bar{z}}\Phi_2(\tau)R(z,\bar{z};z_0,\tau)d\tau+\int_{z_0}^z dt\\ &\quad\times\int_{\bar{z}_0}^{\bar{z}} F(t,\tau)R(z,\bar{z};t,\tau)d\tau \end{aligned} \quad (A.3)$$

where we have set

$$\begin{aligned} \Phi_1(z) &= \frac{\partial u(z,\bar{z}_0)}{\partial z}+B(z,\bar{z}_0)u(z,\bar{z}_0)\\ \Phi_2(\bar{z}) &= \frac{\partial u(z_0,\bar{z})}{\partial \bar{z}}+A(z_0,\bar{z})u(z_0,\bar{z}) \end{aligned} \quad (A.4)$$

Since the coefficients of the pde (5) of the main text are real functions, then integrating by parts Eq. (A.3) we obtain the result

$$u(z,\bar{z}) = \mathrm{Re}\left\{H_0(z)\varphi(z)+\int_{z_0}^z H(z,t)\varphi(t)dt+\int_{z_0}^z dt\int_{\bar{z}_0}^{\bar{z}}F(t,\tau)R(z,\bar{z};t,\tau)d\tau\right\} \quad (A.5)$$

where $\mathrm{Re}(\cdot)$ denotes the real part of what encloses, $\varphi(z)$ is an arbitrary holomorphic function in the region D determined by the boundary condition, and

$$\begin{aligned} H_0(z) &= R(z,\bar{z};z_0,\bar{z}_0)\wedge H(z,t)\\ &= -\frac{\partial}{\partial t}R(z,\bar{z};t,\bar{z}_0)+B(t,\bar{z}_0)R(z,\bar{z};t,z_0) \end{aligned} \quad (A.6)$$

Also, the holomorphic function $\varphi(z)$ admits the representation

$$\varphi(z) = 2u(z,\bar{z}_0)-u(z_0,\bar{z}_0)R(z,\bar{z}_0;z_0,\bar{z}_0) \quad (A.7)$$

It will not affect the generality of the solution if we assume that the point $z_0=\bar{z}_0=0$ belongs to the domain D. Furthermore our problem is characterized by $A(z,\bar{z})=B(z,\bar{z})=0$, hence Eqs. (A.6) and (A.7) take the form

$$\begin{aligned} &H_0(z) = R(z,\bar{z};0,0)\wedge H(z,t) = -\frac{\partial}{\partial t}R(z,\bar{z};t,0),\\ &\varphi(z) = 2u(z,0)-u(0,0)R;(z,0;0,0) \end{aligned} \quad (A.8)$$

The Riemann function of our problem at hand is derived directly from Eq. (A.1). In this case $C(z,\bar{z})=-\lambda^2/4$, hence Eq. (A.1) takes the form[3]

$$R(z,\bar{z};t,\tau)-\frac{\lambda^2}{4}\int_t^z d\xi\int_\tau^{\bar{z}}R(\xi,\eta;t,\tau)d\eta = 1 \quad (A.9)$$

The solution of the above equation by employing the method of successive approximations (Tricomi, 1985) turns out to be

$$R(z,\bar{z};t,\tau) = I_0(\lambda\sqrt{(z-t)(\bar{z}-\tau)}) \quad (A.10)$$

where $I_0$ is the modified Bessel function of the first kind and of order zero. The same expression was also presented by Young et al. (2002). Substituting this value in Eq. (A.8) we find

$$\begin{aligned} &H_0(z) = I_0(\lambda r^*), \quad r^*=\sqrt{z\bar{z}}=\sqrt{x^2+y^2},\\ &H(z,t) = \frac{1}{2}\lambda\sqrt{\bar{z}}\frac{I_1(\lambda\sqrt{\bar{z}(z-t)})}{\sqrt{z-t}},\\ &\varphi(z) = 2u(z,0)-u(0,0) \quad \text{or} \quad \varphi(z) = 2u\left(\frac{z}{2},\frac{z}{2i}\right)-u(0,0) \end{aligned} \quad (A.11)$$

where $I_1$ is the modified Bessel function of the first kind and of first order. Finally, taking into account the above formal results, the solution for the current pore pressure attains the form

$$u(z,\bar{z}) = \mathrm{Re}\left\{I_0(\lambda r^*)\varphi(z)+\frac{\lambda\sqrt{\bar{z}}}{2}\int_0^z\frac{I_1(\lambda\sqrt{\bar{z}(z-t)})}{\sqrt{z-t}}\varphi(t)dt+\int_0^z dt\int_0^{\bar{z}}I_0(\lambda\sqrt{(z-t)(\bar{z}-\tau)})F(t,\tau)d\tau\right\} \quad (A.12)$$

By virtue of the identity (Abramowitz and Stegun, 1965)

$$\int_0^{\bar{z}} I_0(\lambda\sqrt{(z-t)(\bar{z}-\tau)})d\tau = \frac{2\sqrt{\bar{z}}I_1(\lambda\sqrt{\bar{z}(z-t)})}{\lambda\sqrt{z-t}} \quad (A.13)$$

as well as the last of relations (9) in the main text, then Eq. (A.12) simplifies as follows

$$u(z,\bar{z}) = \mathrm{Re}\left\{I_0(\lambda r^*)\varphi(z)+\frac{\lambda\sqrt{\bar{z}}}{2}\int_0^z\frac{I_1(\lambda\sqrt{\bar{z}(z-t)})}{\sqrt{z-t}}\left[\varphi(t)+\frac{4F(t,\bar{z})}{\lambda^2}\right]dt\right\} \quad (A.14)$$

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[3] For Laplace's eqn it is obvious that $R(z,\bar{z};t,\tau)=1$ and from (A.5) $u(z,\bar{z})=\mathrm{Re}\{\varphi(z)\}$.

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