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Dixon's ${}_3F_2(1)$-series and identities involving harmonic numbers and the Riemann zeta function

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ABSTRACT

Dixon's classical summation theorem on ${}_3F_2(1)$-series is reformulated as an equation of formal power series in an appropriate variable x. Then by extracting the coefficients of x^m, we establish a general formula involving harmonic numbers and the Riemann zeta function. Several interesting identities are exemplified as consequences, including one of the hardest challenging identities conjectured by Weideman (2003).

1. Introduction and Notation

Let $\mathbb{N}$ and $\mathbb{N}_0$ stand respectively for the sets of natural numbers and nonnegative integers. Define the generalized harmonic numbers by

$$H_0^{\langle m\rangle}=0 \quad\text{and}\quad H_n^{\langle m\rangle}=\sum_{k=1}^{n}\frac{1}{k^m}\quad\text{for } m,\ n\in\mathbb{N}.$$

The case $m=1$ reduces to the classical harmonic numbers: $H_n:=H_n^{\langle 1\rangle}$.

One of the hardest challenging harmonic number identities is the following identity conjectured by Weideman [16, Eq 20]. For all $n\in\mathbb{N}$, it holds that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left\{3(H_k-H_{n-k})^2+\left(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle}\right)\right\}=0. \tag{1}$$

It has been confirmed by Driver, Prodinger, Schneider and Weideman [12] through the computer algebra package *Sigma*. The partial fraction decomposition method has recently been utilized to prove this identity by Chu [5,7], where more identities involving binomial coefficients and harmonic numbers were found.

By expressing Dixon's classical formula as a formal power series identity in an appropriate variable x and then extracting the coefficients of x^m, we shall establish several finite and infinite series identities involving harmonic numbers and the Riemann zeta function. The main tools consist of the two expansion formulae of the Γ-function in terms of the exponential function as well as two expressions connecting binomial coefficients to Bell polynomials.

Throughout the paper, the shifted factorials are defined by

$$(x)_0\equiv 1 \quad\text{and}\quad (x)_n=x(x+1)\cdots(x+n-1)\quad\text{for } n\in\mathbb{N}$$

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which are connected to the Γ-function

$$\Gamma(x) = \int_0^\infty \tau^{x-1} e^{-\tau} d\tau \quad \text{with } \Re(x) > 0$$

via the following two relations

$$\frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n \quad \text{and} \quad \frac{\Gamma(x)}{\Gamma(x-n)} = (-1)^n (1-x)_n. \tag{2}$$

Then Dixon's theorem (cf. Bailey [1, Section 3.1]), which is well known in the theory of classical hypergeometric series, may be reproduced as

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{n!(1+a-b)_n(1+a-c)_n}$$
$$= \Gamma\left[\begin{matrix} 1+\frac{a}{2}, & 1+a-b, & 1+a-c, & 1+\frac{a}{2}-b-c \\ 1+a, & 1+\frac{a}{2}-b, & 1+\frac{a}{2}-c, & 1+a-b-c \end{matrix}\right]$$

where $\Re(1+\frac{a}{2}-b-c) > 0$ is assumed for convergence and the multivariate Γ-notation reads as

$$\Gamma\left[\begin{matrix} a, b, \ldots, c \\ A, B, \ldots, C \end{matrix}\right] = \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

Furthermore, we shall utilize the power series expansions of the Γ-function [4]

$$\Gamma(1-x) = \exp\left\{\sum_{k\ge 1} \frac{\sigma_k}{k} x^k\right\} \quad \text{and} \quad \Gamma\left(\frac{1}{2}-x\right) = \sqrt{\pi} \exp\left\{\sum_{k\ge 1} \frac{\tau_k}{k} x^k\right\}$$

where σ_k and τ_k are defined respectively by

$$\sigma_1 = \gamma \quad \text{and} \quad \sigma_m = \zeta(m) \quad \text{for } m \ge 2;$$
$$\tau_1 = \gamma + 2\ln 2 \quad \text{and} \quad \tau_m = (2^m - 1)\zeta(m) \quad \text{for } m \ge 2;$$

with γ being the Euler–Mascheroni constant defined by $\gamma = \lim_{n\to\infty}(H_n - \ln n)$ and $\zeta(x)$ the usual Riemann zeta function given by $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$.

For the sequence of indeterminates $y := \{y_k\}_{k\in\mathbb{N}}$, the Bell polynomials $\Omega_m(y)$ (or the cyclic indicators of symmetric groups [10, Section 3.3]) are defined by the generating function

$$\sum_{m\ge 0} \Omega_m(y) x^m = \exp\left\{\sum_{k\ge 1} \frac{x^k}{k} y_k\right\}. \tag{3}$$

There is the following explicit expression

$$\Omega_m(y) := \Omega_m(y_1, y_2, \ldots, y_m) = \sum_{\omega(m)} \prod_{k=1}^{m} \frac{y_k^{\ell_k}}{\ell_k! k^{\ell_k}} \tag{4}$$

where the multiple sum runs over $\omega(m)$, the set of m-partitions represented by m-tuples of $(\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^m k\ell_k = m$.

Let $[x^m]f(x)$ stand for the coefficient of x^m in a formal power series $f(x)$. By means of the generating function method, it is not hard to show that these relations hold:

$$[x^m]\binom{n+\lambda x}{n} = \Omega_m(u), \qquad u_k := (-1)^{k-1}\lambda^k H_n^{\langle k\rangle}; \tag{5a}$$

$$[x^m]\binom{n-\lambda x}{n}^{-1} = \Omega_m(v), \qquad v_k := \lambda^k H_n^{\langle k\rangle}. \tag{5b}$$

In order to facilitate further applications, we display the first few Bell polynomials:

$$\Omega_0(y) \equiv 1, \tag{6a}$$
$$\Omega_1(y) = y_1, \tag{6b}$$
$$\Omega_2(y) = \frac{1}{2}(y_1^2 + y_2), \tag{6c}$$

$$\Omega_3(y) = \frac{1}{6}(y_1^3 + 3y_1y_2 + 2y_3), \tag{6d}$$

$$\Omega_4(y) = \frac{1}{24}(y_1^4 + 6y_1^2y_2 + 8y_1y_3 + 3y_2^2 + 6y_4), \tag{6e}$$

$$\Omega_5(y) = \frac{1}{120}(y_1^5 + 10y_1^3y_2 + 20y_1^2y_3 + 15y_1y_2^2 + 30y_1y_4 + 20y_2y_3 + 24y_5). \tag{6f}$$

2. Reformulation of Dixon's classical identity

Introducing an indeterminate x and the parameters $\lambda, \theta, \vartheta, \varepsilon, \delta$ subject to condition $\lambda = \theta + \vartheta = \varepsilon + \delta$, we may equivalently consider Dixon's formula as the following formal power series equation

$$ {}_3F_2\left[\begin{matrix}-\lambda x - n, -\theta x - n, -\varepsilon x - n\\ 1-\vartheta x, 1-\delta x\end{matrix}\middle| 1\right] = \Gamma\left[\begin{matrix}1-\vartheta x, \quad 1-\delta x\\ 1-\lambda x - n, 1-\lambda x + \theta x + \varepsilon x + n\end{matrix}\right] \tag{7a}$$

$$\times\ \Gamma\left[\begin{matrix}1-\frac{\lambda x+n}{2}, 1-\frac{\lambda x-3n}{2}+\theta x+\varepsilon x\\ 1-\frac{\lambda x-n}{2}+\theta x, 1-\frac{\lambda x-n}{2}+\varepsilon x\end{matrix}\right]. \tag{7b}$$

Let k be the summation index for the last ${}_3F_2$-series. Observing the two relations

$$(1-x)_k = \binom{k-x}{k}k! \quad \text{and} \quad (-y-n)_k = (-1)^k\binom{n+y}{k}k!$$

we may reformulate the general hypergeometric terms in (7a) as

$$\frac{(-1)^k\binom{\lambda x+n}{k}\binom{\theta x+n}{k}\binom{\varepsilon x+n}{k}}{\binom{k-\vartheta x}{k}\binom{k-\delta x}{k}}.$$

In view of (2), we can rewrite the Γ-function fraction in (7a) as

$$(-1)^n\frac{\binom{\lambda x+n-1}{n}}{\binom{\theta x+\varepsilon x-\lambda x+n}{n}}\Gamma\left[\begin{matrix}1-\vartheta x, \quad 1-\delta x\\ 1-\lambda x, 1-\lambda x+\theta x+\varepsilon x\end{matrix}\right]. \tag{8}$$

For $n = 2m$, the Γ-function fraction in (7b) can be restated as

$$\frac{(-1)^m\frac{(3m)!}{(m!)^3}\binom{3m+\theta x+\varepsilon x-\frac{\lambda x}{2}}{3m}}{\binom{m-1+\frac{\lambda x}{2}}{m}\binom{m+\theta x-\frac{\lambda x}{2}}{m}\binom{m+\varepsilon x-\frac{\lambda x}{2}}{m}}\Gamma\left[\begin{matrix}1-\frac{\lambda x}{2}, 1-\frac{\lambda x}{2}+\theta x+\varepsilon x\\ 1-\frac{\lambda x}{2}+\theta x, 1-\frac{\lambda x}{2}+\varepsilon x\end{matrix}\right]. \tag{9}$$

When $n = 2m + 1$, we can similarly reformulate the Γ-function fraction in (7b) as

$$\frac{(-1)^m\left(\frac{1-\lambda x}{2}+\theta x+\varepsilon x\right)_{3m+2}}{\left(\frac{\lambda x+1}{2}\right)_m\left(\frac{1-\lambda x}{2}+\theta x\right)_{m+1}\left(\frac{1-\lambda x}{2}+\varepsilon x\right)_{m+1}}\Gamma\left[\begin{matrix}\frac{1-\lambda x}{2}, \frac{1-\lambda x}{2}+\theta x+\varepsilon x\\ \frac{1-\lambda x}{2}+\theta x, \frac{1-\lambda x}{2}+\varepsilon x\end{matrix}\right]$$

which can further be expressed in terms of binomial coefficients as

$$\frac{(-1)^m}{\left(\frac{\lambda x+1}{2}\right)_m}\frac{\binom{4+6m-\lambda x+2\theta x+2\varepsilon x}{4+6m}\binom{1+m-\frac{\lambda x}{2}+\theta x}{1+m}\binom{1+m-\frac{\lambda x}{2}+\varepsilon x}{1+m}}{\binom{2+3m-\frac{\lambda x}{2}+\theta x+\varepsilon x}{2+3m}\binom{2+2m-\lambda x+2\theta x}{2+2m}\binom{2+2m-\lambda x+2\varepsilon x}{2+2m}} \tag{10a}$$

$$\times\frac{(m!)^2(3+6m)!}{2^{1+2m}\{(1+2m)!\}^2(1+3m)!}\Gamma\left[\begin{matrix}\frac{1-\lambda x}{2}, \frac{1-\lambda x}{2}+\theta x+\varepsilon x\\ \frac{1-\lambda x}{2}+\theta x, \frac{1-\lambda x}{2}+\varepsilon x\end{matrix}\right]. \tag{10b}$$

Dividing both sides of the equation (7a) and (7b) by $\binom{\lambda x+n}{n}\binom{\theta x+n}{n}\binom{\varepsilon x+n}{n}$ and keeping in mind the two relations

$$\frac{\binom{\lambda x+2m-1}{2m}}{\binom{\frac{\lambda x}{2}+m-1}{m}}=\frac{\binom{\lambda x+2m}{2m}}{\binom{\frac{\lambda x}{2}+m}{m}},$$

$$\frac{\binom{\lambda x+2m}{1+2m}}{(\frac{\lambda x+1}{2})_m}=\frac{m!2^{2m}\lambda x}{(1+2m)!}\binom{\frac{\lambda x}{2}+m}{m};$$

we may finally express the resulting equation as follows

$$x^3W(x)+\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3T_k(x)=\begin{cases}(-1)^m\dfrac{(3m)!}{(m!)^3}U(x), & n=2m;\\ \dfrac{(-1)^{m+1}}{2}\dfrac{\lambda x(3+6m)!(m!)^3V(x)}{\{(1+2m)!\}^3(1+3m)!}, & n=2m+1;\end{cases}\tag{11}$$

where $T_k(x)$, $U(x)$, $V(x)$ and $W(x)$ are the formal power series explicitly given by

$$T_k(x)=\left\{\binom{k-\vartheta x}{k}\binom{k-\delta x}{k}\binom{\lambda x+n-k}{n-k}\binom{\theta x+n-k}{n-k}\binom{\varepsilon x+n-k}{n-k}\right\}^{-1},\tag{12a}$$

$$U(x)=\Gamma\begin{bmatrix}1-\lambda x+\theta x,\ 1-\lambda x+\varepsilon x\\ 1-\lambda x,\ 1-\lambda x+\theta x+\varepsilon x\end{bmatrix}\Gamma\begin{bmatrix}1-\frac{\lambda x}{2},\ 1-\frac{\lambda x}{2}+\theta x+\varepsilon x\\ 1-\frac{\lambda x}{2}+\theta x,\ 1-\frac{\lambda x}{2}+\varepsilon x\end{bmatrix}\tag{12b}$$

$$\times\frac{\binom{3m+\theta x+\varepsilon x-\frac{\lambda x}{2}}{3m}}{\binom{m+\frac{\lambda x}{2}}{m}\binom{m+\theta x-\frac{\lambda x}{2}}{m}\binom{m+\varepsilon x-\frac{\lambda x}{2}}{m}\binom{\theta x+2m}{2m}\binom{\varepsilon x+2m}{2m}\binom{\theta x+\varepsilon x-\lambda x+2m}{2m}},\tag{12c}$$

$$V(x)=\frac{\binom{4+6m-\lambda x+2\theta x+2\varepsilon x}{4+6m}}{\binom{2+3m-\frac{\lambda x}{2}+\theta x+\varepsilon x}{2+3m}}\binom{m+\frac{\lambda x}{2}}{m}\binom{m-\frac{\lambda x}{2}+\theta x}{m}\binom{m-\frac{\lambda x}{2}+\varepsilon x}{m}\tag{12d}$$

$$\times\frac{\Gamma\begin{bmatrix}1-\lambda x+\theta x,\ 1-\lambda x+\varepsilon x\\ 1-\lambda x,\ 1-\lambda x+\theta x+\varepsilon x\end{bmatrix}\Gamma\begin{bmatrix}\frac{1-\lambda x}{2},\ \frac{1-\lambda x}{2}+\theta x+\varepsilon x\\ \frac{1-\lambda x}{2}+\theta x,\ \frac{1-\lambda x}{2}+\varepsilon x\end{bmatrix}}{\binom{1+2m+\lambda x}{1+2m}\binom{1+2m+\theta x}{1+2m}\binom{1+2m+\varepsilon x}{1+2m}\binom{1+2m-\lambda x+2\theta x}{1+2m}\binom{1+2m-\lambda x+2\varepsilon x}{1+2m}\binom{1+2m-\lambda x+\theta x+\varepsilon x}{1+2m}},\tag{12e}$$

$$W(x)=\lambda\theta\varepsilon\frac{(-1)^{n+1}}{(n+1)^3}\sum_{k=0}^{\infty}\frac{(k!)^3}{(n+2)_k^3}\frac{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{k-\varepsilon x}{k}}{\binom{n+k+1-\vartheta x}{n+k+1}\binom{n+k+1-\delta x}{n+k+1}}.\tag{12f}$$

By means of (5a) and (5b), we can compute without difficulty the coefficients

$$[x^m]T_k(x)=\Omega_m(t_k)\quad\text{with } t_k^{\langle i\rangle}=(\vartheta^i+\delta^i)H_k^{\langle i\rangle}+(-1)^i(\lambda^i+\theta^i+\varepsilon^i)H_{n-k}^{\langle i\rangle};$$

and

$$[x^m]W(x)=(-1)^{n+1}\lambda\theta\varepsilon\sum_{k=0}^{\infty}\frac{(k!)^3}{(n+1)_{k+1}^3}\Omega_m(w_k)$$

where

$$w_k^{\langle i\rangle}=(\vartheta^i+\delta^i)H_{n+k+1}^{\langle i\rangle}-(\lambda^i+\theta^i+\varepsilon^i)H_k^{\langle i\rangle}.$$

Recalling the two expansion formulae of the Γ-function mentioned before, we have the following two further coefficients

$$[x^m]U(x)=\Omega_m(u)$$

where

$$u^{\langle i\rangle}=\left\{(\lambda-\theta)^i+(\lambda-\varepsilon)^i+\left(\frac{\lambda}{2}\right)^i+\left(\frac{\lambda}{2}-\theta-\varepsilon\right)^i\right\}\sigma_i-\left\{\left(\frac{\lambda}{2}-\theta\right)^i+\left(\frac{\lambda}{2}-\varepsilon\right)^i+\lambda^i+(\lambda-\theta-\varepsilon)^i\right\}\sigma_i$$
$$+(-1)^i\left\{\left(\frac{\lambda}{2}\right)^i+\left(\theta-\frac{\lambda}{2}\right)^i+\left(\varepsilon-\frac{\lambda}{2}\right)^i\right\}H_m^{\langle i\rangle}+(-1)^i\left\{\theta^i+\varepsilon^i+(\theta+\varepsilon-\lambda)^i\right\}H_{2m}^{\langle i\rangle}-\left(\frac{\lambda}{2}-\theta-\varepsilon\right)^iH_{3m}^{\langle i\rangle}$$

and

$$[x^m]V(x) = \Omega_m(v)$$

where

$$\begin{aligned} v^{\langle i\rangle} &= \left\{(\lambda-\theta)^i + (\lambda-\varepsilon)^i - \lambda^i - (\lambda-\theta-\varepsilon)^i\right\}\sigma_i - \left\{\left(\frac{\lambda}{2}-\theta\right)^i + \left(\frac{\lambda}{2}-\varepsilon\right)^i - \left(\frac{\lambda}{2}\right)^i - \left(\frac{\lambda}{2}-\theta-\varepsilon\right)^i\right\}\tau_i \\ &\quad - (-1)^i\left\{\left(\frac{\lambda}{2}\right)^i + \left(\theta-\frac{\lambda}{2}\right)^i + \left(\varepsilon-\frac{\lambda}{2}\right)^i\right\}H_m^{\langle i\rangle} + (-1)^i\left\{\lambda^i+\theta^i+\varepsilon^i+(2\theta-\lambda)^i+(2\varepsilon-\lambda)^i\right. \\ &\quad \left. +\ (\theta+\varepsilon-\lambda)^i\right\}H_{1+2m}^{\langle i\rangle} + \left(\frac{\lambda}{2}-\theta-\varepsilon\right)^i H_{2+3m}^{\langle i\rangle} - (\lambda-2\theta-2\varepsilon)^i H_{4+6m}^{\langle i\rangle}. \end{aligned}$$

Equating the coefficients of x^m on both sides of (11), we establish the following formula.

Theorem 1 (*Harmonic Number Identity*). *Let $\{t_k, w_k, u, v\}$ be the four sequences just defined above. Then for $m, \ell \in \mathbb{N}$, there holds the algebraic identity:*

$$\begin{aligned} &(-1)^{n+1}\lambda\theta\varepsilon\sum_{k=0}^{\infty}\frac{(k!)^3}{(n+1)_{k+1}^3}\Omega_{\ell-3}(w_k) + \sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Omega_\ell(t_k) \\ &= \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3}\Omega_\ell(u), & n=2m; \\ \dfrac{(-1)^{m+1}}{2}\dfrac{\lambda(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\Omega_{\ell-1}(v), & n=2m+1. \end{cases} \end{aligned}$$

When one of the parameters λ, ε and θ equals zero, this theorem gives numerous finite sum formulae on harmonic numbers. Instead for $\lambda\varepsilon\theta \neq 0$, Theorem 1 will lead to several infinite series identities involving both harmonic numbers and the Riemann zeta function. For further identities of finite and infinite series of similar type, the reader can refer to [5–8,12–14] and [4,9,11,17], respectively. In addition, it is pointed out by an anonymous referee that the expansion of hypergeometric sum expressions, as carried out in this paper, has applications in particle physics (cf. Vermaseren [15]).

3. Examples: Harmonic number identities

As applications of Theorem 1, this section will display several interesting finite and infinite series identities involving harmonic numbers and the Riemann zeta function.

3.1. $\ell = 0$

In this case, we recover Dixon's classical identity on alternating sums of cubic binomial coefficients

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3}, & n=2m; \\ 0, & n=2m+1. \end{cases}$$

3.2. $\ell = 1$

The corresponding identity displayed in Theorem 1 reads as

$$\begin{aligned} &\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\{(\vartheta+\delta)H_k - (\lambda+\theta+\varepsilon)H_{n-k}\} \\ &= \begin{cases} (-1)^{m+1}\dfrac{(3m)!}{(m!)^3}\left(\theta+\varepsilon-\dfrac{\lambda}{2}\right)(H_m+2H_{2m}-H_{3m}), & n=2m; \\ \dfrac{(-1)^{m+1}\lambda}{2}\dfrac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}, & n=2m+1. \end{cases} \end{aligned}$$

Under the involution $k \to n-k$, this identity can be simplified as [12, Eq 2]

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 H_k = \begin{cases} \dfrac{(-1)^m}{2}\dfrac{(3m)!}{(m!)^3}(H_m+2H_{2m}-H_{3m}), & n=2m; \\ \dfrac{(-1)^{m+1}}{6}\dfrac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}, & n=2m+1. \end{cases} \tag{13}$$

3.3. $\ell = 2$

When $n = 2m$, we have the following general identity.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left\{(\vartheta^2+\delta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2+\varepsilon^2)H_{n-k}^{\langle 2\rangle}+\{(\vartheta+\delta)H_k-(\lambda+\theta+\varepsilon)H_{n-k}\}^2\right\} \tag{14a}$$

$$=(-1)^m\frac{(3m)!}{(m!)^3}\left[\left\{\frac{\lambda^2}{4}+\left(\theta-\frac{\lambda}{2}\right)^2+\left(\varepsilon-\frac{\lambda}{2}\right)^2\right\}H_m^{\langle 2\rangle}+\left\{\theta^2+\varepsilon^2+(\theta+\varepsilon-\lambda)^2\right\}H_{2m}^{\langle 2\rangle}\right.$$

$$\left.+\left(\theta+\varepsilon-\frac{\lambda}{2}\right)^2(H_m+2H_{2m}-H_{3m})^2-\left(\theta+\varepsilon-\frac{\lambda}{2}\right)^2H_{3m}^{\langle 2\rangle}\right]. \tag{14b}$$

It is trivial to check that for $\lambda = 4$ and $\varepsilon, \theta = 1 \pm \sqrt{-3}$, the last identity becomes Weideman's (1). Furthermore, we can deduce, by specifying the parameters λ, ε and θ, the following harmonic number identities.

Example 1 *($\lambda = 0$ and $\varepsilon, \theta = \pm 1$: [12, Eq 3] and [14, Eq 1.12]).*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\{H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle}\}=(-1)^m\frac{(3m)!}{(m!)^3}\left\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\right\}.$$

Example 2 *($\lambda = 4$ and $\varepsilon, \theta = 1 \pm 3\sqrt{-1}$: [6, Example 8]).*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3(H_k-H_{2m-k})^2=\frac{(-1)^{m+1}}{3}\frac{(3m)!}{(m!)^3}\left\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\right\}.$$

Example 3 *($\lambda = 0$, $\varepsilon = 1$ and $\theta = \sqrt{-1}$: [6, Example 6] and [12, Eq 23]).*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3(H_k+H_{2m-k})^2=(-1)^m\frac{(3m)!}{(m!)^3}\left\{(H_m+2H_{2m}-H_{3m})^2+H_{2m}^{\langle 2\rangle}-H_{3m}^{\langle 2\rangle}\right\}.$$

According to linear combinations, the last two examples are equivalent to the following two identities (cf. [12, Eqs. 22 and 23] and [14, Eqs. 1.10 and 1.11]).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3H_k^2=\frac{(-1)^m}{12}\frac{(3m)!}{(m!)^3}\left[3(H_m+2H_{2m}-H_{3m})^2-H_m^{\langle 2\rangle}+2H_{2m}^{\langle 2\rangle}-3H_{3m}^{\langle 2\rangle}\right], \tag{15a}$$

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3H_kH_{2m-k}=\frac{(-1)^m}{12}\frac{(3m)!}{(m!)^3}\left[3(H_m+2H_{2m}-H_{3m})^2+H_m^{\langle 2\rangle}+4H_{2m}^{\langle 2\rangle}-3H_{3m}^{\langle 2\rangle}\right]. \tag{15b}$$

Instead, for $\ell = 2$ and $n = 1 + 2m$, the corresponding identity reduces to

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left\{(\vartheta^2+\delta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2+\varepsilon^2)H_{n-k}^{\langle 2\rangle}+\{(\vartheta+\delta)H_k-(\lambda+\theta+\varepsilon)H_{n-k}\}^2\right\}$$

$$=(-1)^m\frac{\lambda(3+6m)!(m!)^3(\frac{\lambda}{2}-\theta-\varepsilon)}{\{(1+2m)!\}^3(1+3m)!}\left\{H_m-4H_{1+2m}-H_{2+3m}+2H_{4+6m}\right\}.$$

Under the replacement $k \to n - k$, this identity simplifies to the following one.

Example 4 *(Schneider [14, Section 3.3]).*

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\{3H_k^2+H_k^{\langle 2\rangle}\}=\frac{(-1)^m}{2}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{H_m-4H_{1+2m}-H_{2+3m}+2H_{4+6m}\right\}.$$

However, when the factor $3H_k^2 + H_k^{\langle 2\rangle}$ is split into two terms, the corresponding sums do not have closed expressions. Schneider [14, Eq 1.15] has verified this fact by showing the following interesting identity

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3H_k^{\langle 2\rangle}=\frac{(-1)^m}{12}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{-2+\sum_{i=1}^{m}\frac{(5+36i+72i^2)\{(2i)!\}^3\{(3i)!\}^2}{2i(1+2i)(1+6i)!(i!)^6}\right\}.$$

In order to reduce lengthy expressions, the following two abbreviated notations will be utilized in the rest of the paper:

$$\Phi_m:=H_m+2H_{2m}-H_{3m}\quad\text{and}\quad\Psi_m=H_m-4H_{1+2m}-H_{2+3m}+2H_{4+6m}.$$

3.4. $\ell=3$

By specifying the parameters λ, ε and θ concretely in Theorem 1, we can derive the following seven harmonic number identities.

Example 5 ($\lambda=4$ *and* $\varepsilon,\theta=1\pm\sqrt{-3}$).

$$\sum_{k=0}^{\infty}\frac{(k!)^3}{(1+2m)_{k+1}^3}+\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3H_k^{\langle 3\rangle}=\frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\left\{2\zeta(3)-H_{2m}^{\langle 3\rangle}+H_m^{\langle 3\rangle}\right\}.$$

Example 6 ($\lambda=4$ *and* $\varepsilon,\theta=1\pm\sqrt{-3}$).

$$\sum_{k=0}^{\infty}\frac{2(k!)^3}{(2+2m)_{k+1}^3}+\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{9(H_k^3+H_kH_k^{\langle 2\rangle})-9(H_k^{\langle 2\rangle}+3H_k^2)H_{1+2m-k}+2H_k^{\langle 3\rangle}\right\}$$
$$=2(-1)^{m+1}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\zeta(2).$$

Example 7 ($\lambda=4$ *and* $\varepsilon,\theta=1\pm3\sqrt{-1}$).

$$\sum_{k=0}^{\infty}\frac{5(k!)^3}{(2+2m)_{k+1}^3}+\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{9H_k^3-4H_k^{\langle 3\rangle}-27H_k^2H_{1+2m-k}\right\}$$
$$=\frac{(-1)^{m+1}}{2}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{10\zeta(2)+3H_m^{\langle 2\rangle}-15H_{1+2m}^{\langle 2\rangle}\right\}.$$

Example 8 ($\lambda=0$, $\varepsilon=1$ *and* $\theta=\sqrt{-1}$).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\left\{H_k^3-H_k^{\langle 3\rangle}+3H_kH_{2m-k}^2\right\}=\frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\left\{\Phi_m^3-3\Phi_m(H_{3m}^{\langle 2\rangle}-H_{2m}^{\langle 2\rangle})-2H_{3m}^{\langle 3\rangle}+H_{2m}^{\langle 3\rangle}-H_m^{\langle 3\rangle}\right\}.$$

Example 9 ($\lambda=2$, $\varepsilon=1+\sqrt{-3}$ *and* $\theta=0$).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\left\{(3H_{2m-k}-2H_k)H_k^2+(H_{2m-k}-2H_k)H_k^{\langle 2\rangle}\right\}$$
$$=\frac{(-1)^m}{8}\frac{(3m)!}{(m!)^3}\left\{\Phi_m^3+\Phi_m(H_m^{\langle 2\rangle}+4H_{2m}^{\langle 2\rangle}-3H_{3m}^{\langle 2\rangle})-2H_{3m}^{\langle 3\rangle}+2H_m^{\langle 3\rangle}\right\}.$$

Example 10 ($\lambda=2$, $\varepsilon=1+\sqrt{-3}$ *and* $\theta=0$).

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{H_kH_{1+2m-k}^{\langle 2\rangle}+3H_kH_{1+2m-k}^2\right\}$$
$$=\frac{(-1)^m}{24}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{3\Psi_m^2+3H_{2+3m}^{\langle 2\rangle}+8H_{1+2m}^{\langle 2\rangle}-H_m^{\langle 2\rangle}-12H_{4+6m}^{\langle 2\rangle}\right\}.$$

Example 11 ($\lambda=2$, $\varepsilon=1$ *and* $\theta=0$).

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{3H_k^3+5H_kH_k^{\langle 2\rangle}+2H_k^{\langle 3\rangle}-(9H_k^2+5H_k^{\langle 2\rangle})H_{1+2m-k}\right\}$$
$$=\frac{(-1)^m}{3}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{H_m^{\langle 2\rangle}-5H_{1+2m}^{\langle 2\rangle}\right\}.$$

We remark that the last identity can also be obtained by combining Example 6 with Example 7 and canceling the parts containing $\zeta(2)$.

3.5. $\ell = 4$

For $\ell \geq 4$, the identities specialized from Theorem 1 are generally quite complicated. We limit ourselves to present only four summation formulae for exemplification.

Example 12 ($\lambda = 0$ *and* $\varepsilon, \theta = \pm 1$).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\left\{H_k^{\langle 4\rangle}+H_k^{\langle 2\rangle}H_k^{\langle 2\rangle}+H_k^{\langle 2\rangle}H_{2m-k}^{\langle 2\rangle}\right\}=\frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\left\{H_m^{\langle 4\rangle}+H_{2m}^{\langle 4\rangle}+\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\}^2\right\}.$$

Example 13 ($\lambda = 1$, $\varepsilon = 0$ *and* $\theta = -1$).

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{10H_k^{\langle 4\rangle}+30H_k^2H_k^{\langle 2\rangle}+7H_k^{\langle 2\rangle}H_k^{\langle 2\rangle}+24H_kH_k^{\langle 3\rangle}+12H_k^2H_{1+2m-k}^{\langle 2\rangle}+9H_k^4\right\}$$
$$=\frac{(-1)^m(3+6m)!(m!)^3}{4\{(1+2m)!\}^3(1+3m)!}\Big[3\Psi_m^3+6H_m^{\langle 3\rangle}-64H_{1+2m}^{\langle 3\rangle}-6H_{2+3m}^{\langle 3\rangle}+48H_{4+6m}^{\langle 3\rangle}$$
$$-\Psi_m\left\{11H_m^{\langle 2\rangle}-64H_{1+2m}^{\langle 2\rangle}-9H_{2+3m}^{\langle 2\rangle}+36H_{4+6m}^{\langle 2\rangle}\right\}\Big].$$

Example 14 ($\lambda = 4$ *and* $\varepsilon, \theta = 1 \pm \sqrt{-3}$).

$$\sum_{k=0}^{\infty}\frac{(k!)^3(H_k-H_{2+2m+k})}{(2+2m)_{k+1}^3}+\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\left\{H_k^{\langle 4\rangle}+(H_k-H_{1+2m-k})H_k^{\langle 3\rangle}\right\}$$
$$=\frac{(-1)^m}{6}\frac{(3+6m)!(m!)^3}{\{(1+2m)!\}^3(1+3m)!}\left\{4\zeta(3)+H_m^{\langle 3\rangle}-7H_{1+2m}^{\langle 3\rangle}\right\}.$$

Finally, by combining the two special cases of Theorem 1 specified with $\lambda = \pm 4$, $\varepsilon, \theta = 4 \pm (1+\sqrt{-1})\sqrt{6+6\sqrt{-1}}$ and $2 \pm (1+\sqrt{-1})\sqrt{6+12\sqrt{-1}}$, we find the following strange looking identity.

Example 15 (*Harmonic Number Identity*).

$$\sum_{k=0}^{\infty}\frac{(k!)^3\{H_k+H_{1+2m+k}\}}{(1+2m)_{k+1}^3}+\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\left\{H_k^{\langle 4\rangle}+(H_k+H_{2m-k})H_k^{\langle 3\rangle}\right\}$$
$$=\frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\left\{3\zeta(4)+2\Phi_m\zeta(3)+H_m^{\langle 4\rangle}-2H_{2m}^{\langle 4\rangle}+\Phi_m(H_m^{\langle 3\rangle}-H_{2m}^{\langle 3\rangle})\right\}.$$

When $m = 0$, it recovers the identity (cf. [2, Eq 4], [4, Eq 1.3], [9, B7a] and [11, Eq 7])

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4)=\frac{\pi^4}{72}.$$

Further infinite series identities of this type have been derived by Zheng [17]. Moreover, analogous multiple Euler sums are heavily used in physics [3,15], where highly non-trivial computations have been accomplished.

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