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Some criteria for the existence of limit cycles for quadratic vector fields

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Abstract

In this paper we consider real quadratic systems. We present new criteria for the existence and uniqueness of limit cycles for such systems by using Darbouxian particular solutions. Some results are based on the study of such systems in $\mathbb{CP}^2$. We also generalize the well-know result of Bautin on the nonexistence of limit cycles for quadratic Lotka–Volterra systems.

1. Introduction

We consider here two-dimensional polynomial differential systems of the form

$$\dot{x} = \frac{dx}{dt} = P(x, y) = \sum_{i+j=0}^{2} a_{ij} x^i y^j, \qquad \dot{y} = \frac{dy}{dt} = Q(x, y) = \sum_{i+j=0}^{2} b_{ij} x^i y^j, \tag{1}$$

in which $P, Q \in \mathbb{R}[x, y]$ are relative primes polynomials where at least one of them has second degree. In what follows, system (1) will simply be called *quadratic system*. This class of systems have been studied intensively during this century and a lot of papers have been published on this subject, see the bibliographical survey [15].

A *critical point* of system (1) is a point $(x_i, y_i) \in \mathbb{C}^2$ such that $P(x_i, y_i) = Q(x_i, y_i) = 0$. On the other hand, limit cycles of planar vector fields were defined by Poincaré in the

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famous paper [13]. A *limit cycle* is a periodic orbit of system (1) which has an annulus-like neighborhood free of other periodic solutions. The investigation of the limit cycles for polynomial systems, from their inclusion in Hilbert's famous list of problems as part of 16th problem [12], remains the most important and difficult question in the subject. The nonexistence, existence, uniqueness and other properties of limit cycles have been studied extensively; see, for example, the book [16].

An important question is to discover when some trajectory of (1) can be described implicitly by the zero set of a polynomial $f(x,y)=0$. Regarding these algebraic solutions it is clear that the derivative of f respect to time (along the orbits of system (1)) should be annulled on the algebraic curve $f(x,y)=0$. Hence we are directly lead to the equation

$$\dot{f} := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

for some polynomial $K(x,y)$ of degree less than or equal to one, called the cofactor. We will denote this by $\deg K \leqslant 1$. The existence of algebraic trajectories has a strong influence on the behavior of polynomial systems; see, for instance, [8] and [9] where the Darboux integrability theory is exposed. On the other hand, it is well-known that a quadratic system with an invariant straight line can have at most one limit cycle, see [16]. Respect to invariant conics: a quadratic system can have no limit cycles in case of having an invariant hyperbola, see [6] or an invariant ellipse (except perhaps for the ellipse itself), see [14]. However, if the invariant conic is a parabola then a quadratic system can have limit cycles; see, for instance, [7]. Finally in [3], the class of real quadratic systems having a cubic irreducible invariant algebraic curve is examined by the authors, showing that no systems of this type has limit cycles except for two cases. In these cases, concrete examples are given with a limit cycle generated in a Hopf bifurcation.

The paper in organized as follows: In Section 2 we give a brief summary on classical and recent results that we shall use later. Section 3 is devoted to explain the main statements of this work.

2. Preliminary results on limit cycles

We shall need a number of well-known results and we briefly summarize them. We make use of the following Poincaré's method of tangential curves (see, for instance, [16]).

Theorem 1 (Poincaré). *A continuous autonomous system $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ has no periodic solutions in a region $D\subset\mathbb{R}^2$ if there exists a continuously differentiable function $F:D\to\mathbb{R}$ such that*

$$\dot{F} = P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y}$$

is of constant sign in D and the equality $\dot{F}=0$ cannot be satisfied on a whole orbit of the system.

The following theorem, see [16], is a classical method for proving the nonexistence of limit cycles in a simply connected region.

Theorem 2 (Bendixon–Dulac's criterion). *A C^1 system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ defined in $\Omega \subset \mathbb{R}^2$ has no periodic solutions in a simply connected region $D \subset \Omega$ if there exists a continuously differentiate function $B : D \to \mathbb{R}$ such that the divergence $(BP)_x + (BQ)_y$ is of constant sign in D and is not identically zero in any open subset in D.*

Following Darboux [9], the polynomial vector field $\mathcal{X} = P(x, y)\,\partial/\partial x + Q(x, y)\,\partial/\partial y$ of $\mathbb{R}^2$ associated to system (1) can be thought in the *complex projective plane* $\mathbb{CP}^2$ by using the complex variables (X, Y, Z) of $\mathbb{CP}^2$ defined like $x = X/Z$, $y = Y/Z$; see also [5]. Hence, $\mathcal{X}$ is embedded in $\mathbb{CP}^2$ like the following particular homogeneous polynomial vector field $\widetilde{\mathcal{X}} = L(X, Y, Z)\,\partial/\partial X + M(X, Y, Z)\,\partial/\partial Y$, where $L(X, Y, Z) = Z^2 P(X/Z, Y/Z)$ and $M(X, Y, Z) = Z^2 Q(X/Z, Y/Z)$ are homogeneous polynomials of second degree. The vector field $\widetilde{\mathcal{X}}$ is called the *complex projectivization* of the vector field $\mathcal{X}$. The singular points of the complex projectivization $\widetilde{\mathcal{X}}$ of system (1) must satisfy the system of equations

$$ZL(X, Y, Z) = 0, \qquad ZM(X, Y, Z) = 0, \qquad XM(X, Y, Z) - YL(X, Y, Z) = 0.$$

It is easy to show that if $f(x, y) = 0$ is an invariant algebraic curve of degree n for system (1) with associated cofactor $K(x, y)$, i.e., $\mathcal{X}f = Kf$, then $\widetilde{F}(X, Y, Z) = Z^n f(X/Z, Y/Z) = 0$ is an invariant algebraic curve of $\widetilde{\mathcal{X}}$ with cofactor $\widetilde{K}(X, Y, Z) = ZK(X/Z, Y/Z)$. That is to say, $\widetilde{\mathcal{X}}\widetilde{F} = \widetilde{K}\widetilde{F}$. On the other hand, notice that every finite critical point $(x_0, y_0) \in \mathbb{C}^2$ of (1) verifies $L(x_0, y_0, 1) = M(x_0, y_0, 1) = 0$. Next useful theorem, proved in [2], provides sufficient conditions in order to have a quadratic system with all its limit cycles algebraic.

Theorem 3 (Chavarriga–Giacomini–Llibre). *Let $f(x, y) = 0$ be a real invariant algebraic curve of degree greater than or equal to two of a real quadratic system* (1) *with associate vector field $\mathcal{X} = P(x, y)\,\partial/\partial x + Q(x, y)\,\partial/\partial y$. Let K be the cofactor of $f = 0$. Suppose that there are two points $p_1, p_2 \in \mathbb{CP}^2$ such that $L(p_i) = M(p_i) = \widetilde{K}(p_i) = 0$ for $i = 1, 2$, where $L = Z^2 P(X/Z, Y/Z)$, $M = Z^2 Q(X/Z, Y/Z)$ and $\widetilde{K} = ZK(X/Z, Y/Z)$. Then every limit cycle of* (1) *must be algebraic and contained in $f(x, y) = 0$.*

3. Main statements

We will call a multi-valued function h of the form

$$h(x, y) = \exp\left[\frac{G(x, y)}{F(x, y)}\right] \prod_i f_i^{\lambda_i}(x, y),$$

with F, G and all f_i real polynomials and λ_i real constants, a *Darbouxian function*. We begin by giving a corollary that generalizes Theorem 3 and that we will use in the proof of the next theorem.

Corollary 4. *Consider the vector field $\mathcal{X} = P(x, y)\,\partial/\partial x + Q(x, y)\,\partial/\partial y$ associate to the real quadratic system* (1)*. Let $h(x, y)$ be a Darbouxian function such that $h(x, y) = 0$ is not a straight line and verifying $\mathcal{X}h = Kh$, where $K(x, y)$ is a polynomial with* $\deg K \leqslant 1$.

If there are two points $p_1, p_2 \in \mathbb{CP}^2$ such that $L(p_i) = M(p_i) = \widetilde{K}(p_i) = 0$ for $i = 1, 2$, where $\widetilde{K}(X, Y, Z) = Z^{\deg K} K(X/Z, Y/Z)$, then every limit cycle of $\mathcal{X}$ must be contained in $h(x, y) = 0$.

Proof. This corollary is a direct consequence of the proof of Theorem 3. In fact, these proof only uses the algebraicity of K but in any moment apply that h needs to be algebraic. $\square$

There are examples of polynomial vector fields $\mathcal{X}$ with nonalgebraic particular solutions $h(x, y) = 0$ and verifying $\mathcal{X}h = Kh$ for some polynomial K; see, for instance, [8] where appear the exponential factors $h(x, y) = \exp(G(x, y)/F(x, y))$ with $F = 0$ invariant algebraic curve of $\mathcal{X}$. There are also different examples, which are not Darbouxian functions. For instance, in [4] appears that

$$h(x, y) = bx^2 + (c - a)xy - 1 + (a + c)(x^2 + y^2)\arcsin\left(x/\sqrt{x^2 + y^2}\right)$$

is a particular solution of the cubic vector field with linear part of center type and degenerate infinity $\mathcal{X} = [-y + x\Lambda(x, y)]\,\partial/\partial x + [x + y\Lambda(x, y)]\,\partial/\partial y$, where $\Lambda(x, y) = ax^2 + bxy + cy^2$. Obviously, if $a + c \neq 0$ then $h(x, y)$ is nonalgebraic but it verifies $\mathcal{X}h = Kh$ for $K(x, y) = \Lambda(x, y)$.

In this work the sense of *independent* is the given in the next definition.

Definition 5. The set of real algebraic curves $f_i(x, y) = 0$ for $i = 1, \ldots, n$ will be called independent if do not exist real numbers α_i such that the Darbouxian function $\prod_i f_i^{\alpha_i}(x, y) = 0$ is a straight line.

Let $F = 0$ and $G = 0$ be two projective algebraic curves and p a point of $\mathbb{CP}^2$. We will call *intersection point* a point p such that it belongs to both curves $F = 0$ and $G = 0$. We denote by $I(p, F \cap G)$ the *intersection number* of the algebraic curves $F = 0$ and $G = 0$ at a point p of $\mathbb{CP}^2$ and we think of it as the number of times that these two curves intersect at p. A more formal definition can be found in [10].

Theorem 6. *Consider the vector field $\mathcal{X} = P(x, y)\,\partial/\partial x + Q(x, y)\,\partial/\partial y$ associate to the real quadratic system* (1) *and possessing two independent real invariant algebraic curves $f_i(x, y) = 0$ with cofactors $K_i(x, y)$ for $i = 1, 2$. Let $\widetilde{\mathcal{X}} = L(X, Y, Z)\,\partial/\partial X + M(X, Y, Z)\,\partial/\partial Y$ be the complex projectivization of $\mathcal{X}$. If a point (X_0, Y_0, Z_0) of $\mathbb{CP}^2$ exists such that*

$$L(X_0, Y_0, Z_0) = M(X_0, Y_0, Z_0) = \widetilde{K}_1(X_0, Y_0, Z_0) = \widetilde{K}_2(X_0, Y_0, Z_0) = 0,$$

where $\widetilde{K}_i = ZK_i(X/Z, Y/Z)$ for $i = 1, 2$, then any limit cycle γ of system (1) *must be algebraic and $\gamma \subset \Sigma := \{(x, y) \in \mathbb{R}^2\colon\ f_1(x, y) f_2(x, y) = 0\}$.*

Proof. Since $\deg L = \deg M = 2$, from Bezout's theorem it is known that there are four points (counted with their multiplicity) $p_i = (X_i, Y_i, Z_i) \in \mathbb{CP}^2$ such that $L(X_i, Y_i, Z_i) = M(X_i, Y_i, Z_i) = 0$ with $i = 0, 1, 2, 3$. By hypothesis $L(p_0) = M(p_0) = 0$ and $\widetilde{K}_1(p_0) = \widetilde{K}_2(p_0) = 0$.

Since the coefficients of the polynomials L, M, $\widetilde{K}_1$ and $\widetilde{K}_2$ are real, if the point p_0 has some nonreal coordinate, then its complex conjugate point $p_1 = \bar{p}_0 = (\overline{X}_0, \overline{Y}_0, \overline{Z}_0)$ also verifies $L(\bar{p}_0) = M(\bar{p}_0) = 0$ and $\widetilde{K}_1(\bar{p}_0) = \widetilde{K}_2(\bar{p}_0) = 0$. In this case we can apply Corollary 4 to the Darbouxian function $h(x, y) = f_1 f_2$ and the theorem is proved.

Now let us assume that the three coordinates of p_0 are real. Therefore we can have either $p_1 \neq p_0$ with the three coordinates of p_1 reals or $I(p_0, L \cap M) \geqslant 2$ and the rest of points p_i with $p_i \neq p_0$ have complex coordinates. In this last case, we can assume that p_0 is a finite point because otherwise $\mathcal{X}$ does not have any finite real critical point and therefore it cannot have limit cycles. Hence, p_0 is the only real finite critical point of $\widetilde{\mathcal{X}}$ which we can suppose $p_0 = (0, 0, 1)$ without loss of generality. In consequence, since $I(p_0, L \cap M) \geqslant 2$, if we consider the polynomials P and Q it is clear that either some of them do not have linear part or their linear parts are proportional. But in this situation the origin of $\mathbb{R}^2$ is the only finite real critical point of $\mathcal{X}$ which moreover has at least one null associate eigenvalue. Therefore, since it is well known that if a quadratic system has a limit cycle then in its interior planar region it should have only one critical point of the system and moreover it must be a focus, the quadratic system (1) has either nilpotent linear part or it does not have linear part, i.e., $P(x, y) = y + P_2(x, y)$ and $Q(x, y) = Q_2(x, y)$ or $P(x, y) = P_2(x, y)$ and $Q(x, y) = Q_2(x, y)$, respectively, where P_2 and Q_2 are real homogeneous polynomials of second degree. This last case corresponds to an homogeneous quadratic system and it is well known that these systems cannot have limit cycles.

Regarding the above nilpotent case, we see that $Q_2(x, y)$ cannot be defined because if $\dot{y}$ is defined then the differential system does not have any limit cycle. Hence, since Q_2 is homogeneous, it can be written as the product of two different linear factors, $Q_2(x, y) = L_1(x, y)L_2(x, y)$. Obviously, $L_i(x, y) \neq y$ because otherwise system (1) possesses the invariant straight line $y = 0$ which passes through the focus, what is a contradiction.

On the other hand, in order to have (0,0) as the unique finite real critical point of the nilpotent system (1), Q_2 must divide to P_2. Moreover, since $I(p_0, L \cap M) \geqslant 2$, system (1) should be written in the form $\dot{x} = y + P_2(x, y)$, $\dot{y} = \lambda P_2(x, y)$, with $\lambda \in \mathbb{R}$ different from zero. In this case, making the linear change of coordinates $z = \lambda x - y$ and after a rescaling of the time given by $d\tau = \lambda\, dt$, the above system becomes

$$z' = y, \qquad y' = \widetilde{P}_2(z, y), \tag{2}$$

where $' = d/d\tau$ and $\widetilde{P}_2(z, y) = P_2((z + y)/\lambda, y)$. This system, by hypothesis has two invariant curves $F_i(z, y) = 0$ with $i = 1, 2$. We can suppose without lost of generality that their associated cofactors C_1 and C_2 are linearly independent because otherwise there exists a real constant α different from zero such that $C_1(z, y) = \alpha C_2(z, y)$ and therefore, from Darboux's integrability theory, $F_1 F_2^{-\alpha}$ is an analytic first integral for the quadratic system in $\mathbb{R}^2 \backslash \Sigma$ and the system cannot have limit cycles, except perhaps in Σ.

Notice that system (2) has the exponential factor $\exp(x)$. Since by hypothesis system (2) has a real focus at (0,0) and the cofactors $C_i(z, y)$ are real independent polynomials of degree less than or equal to one verifying $C_1(0, 0) = C_2(0, 0) = 0$, we can take real constants λ_1 and λ_2 not both zero such that $H(z, y) = \exp(x) F_1^{\lambda_1}(z, y) F_2^{\lambda_2}(z, y)$ is an analytic first integral of system (2) in $\mathbb{R}^2 \backslash \{F_1 F_2 = 0\}$. This is due to the fact that

$$H' = \frac{\partial H}{\partial z} y + \frac{\partial H}{\partial y} \widetilde{P}_2(z, y) = \big[y + \lambda_1 C_1(z, y) = \lambda_2 C_2(z, y)\big] H,$$

and we can choose the real constants λ_1 and λ_2 not both zero such that previous expression $y + \lambda_1 C_1(z, y) + \lambda_2 C_2(z, y)$ is annulled. Undoing the change of variables we have that the initial quadratic system admits the first integral $H(x, y) = \exp(x) f_1^{\lambda_1}(x, y) f_2^{\lambda_2}(x, y)$ which is analytic in $\mathbb{R}^2 \setminus \Sigma$ and therefore it cannot have limit cycles outside of Σ.

In short, we continue the proof assuming that there exists a point $p_1 \in \mathbb{CP}^2$ with real coordinates such that $p_0 \neq p_1$ and $L(p_1) = M(p_1) = 0$.

We can suppose without loss of generality that the cofactors K_1 and K_2 are linearly independent, see the above argument of Darboux's integrability.

Since the cofactors $\widetilde{K}_i$ are independent polynomials of degree less than or equal to one verifying $\widetilde{K}_1(p_0) = \widetilde{K}_2(p_0) = 0$ and in addition $\widetilde{K}_i(p_1)$ is real for $i = 1, 2$, we can always choose real constants λ_1 and λ_2 not both zero such that $\widetilde{K}(X, Y, Z) := \lambda_1 \widetilde{K}_1(X, Y, Z) + \lambda_2 \widetilde{K}_2(X, Y, Z)$ satisfies $\widetilde{K}(p_1) = 0$. Finally, we have that the Darbouxian function $h(x, y) = f_1^{\lambda_1}(x, y) f_2^{\lambda_2}(x, y)$ verifies $\mathcal{X} h = K h$, where $\widetilde{K} = Z K(X/Z, Y/Z)$, and therefore applying Corollary 4 we prove the theorem. $\square$

Let $f(x, y) = 0$ be a planar algebraic curve of degree n defined in the affine plane $\mathbb{R}^2$. Consider a point $p = (x_0, y_0) \in \mathbb{R}^2$ such that $f(x_0, y_0) = 0$. This point is called a *multiple point of order* r (with $r \geqslant 2$) of the curve if it is such that every line through p meets the curve r times at p. If the origin $p = (0, 0)$ is a multiple point of order r then $f(x, y) = \sum_{k=r}^{n} f_k(x, y)$, $r \geqslant 2$, where $f_k(x, y)$ are homogeneous polynomials of degree k and $f_r(x, y) = \prod_{i=1}^{s} L_i^{r_i}(x, y)$ with $\sum_{i=1}^{s} r_i = r$. L_i are different straight lines called *tangents* to $f = 0$ at p. In this situation p is an *isolated* multiple point if every tangent L_i is complex. In the particular case of double points, i.e., $r = 2$, we have $f(x, y) = ax^2 + bxy + cy^2 + \sum_{k=3}^{n} f_k(x, y)$, and $p = (0, 0)$ is isolated if the discriminant $\Delta = b^2 - 4ac < 0$.

Corollary 7. *Consider a real quadratic system* (1) *possessing two independent real invariant algebraic curves* $f_1(x, y) = 0$ *and* $f_2(x, y) = 0$. *Let* $(x_0, y_0) \in \mathbb{R}^2$ *be the focus inside the region enclosed by a limit cycle* γ *of the system. If* (x_0, y_0) *is not an isolated multiple point of the algebraic curve* $f_1(x, y) f_2(x, y) = 0$ *then the limit cycle* γ *must be algebraic and* $\gamma \subset \Sigma = \{(x, y) \in \mathbb{R}^2\colon\ f_1(x, y) f_2(x, y) = 0\}$.

Proof. It is well known (see, for instance, [16]) that a limit cycle of a quadratic system surrounds an unique critical point (x_0, y_0) of the system. Moreover, this critical point must be of focus type. Therefore if the system admits two real invariant algebraic curves $f_i(x, y) = 0$ for $i = 1, 2$, and (x_0, y_0) is not an isolated multiple point of these curves, then $f_i(x_0, y_0) \neq 0$ for $i = 1, 2$. So, from equations $\dot{f}_i = K_i f_i$ with $i = 1, 2$ it is deduced that $K_1(x_0, y_0) = K_2(x_0, y_0) = 0$, where K_1 and K_2 are the cofactors of f_1 and f_2, respectively. Finally, we obtain the claim by applying Theorem 6. $\square$

Ye [16] classified quadratic systems that can have limit cycles in the following three families:

$$\dot{x} = \delta x - y + \ell x^2 + mxy + ny^2, \qquad \dot{y} = x(1 + ax + by);$$

$a = b = 0$ (family I), $a \neq 0$ and $b = 0$ (family II), and $b \neq 0$ (family III). It is well known, see [16] for instance, that a quadratic system of the family I has at most one limit cycle. Next theorem tackles to this type of systems.

Definition 8. Let $C \subset \mathbb{R}^2$ be a connected component of the real point set of a real algebraic curve different of an isolated multiple point. We say that C is a *circuit* if it is not homeomorphic neither to a straight line neither to a circle.

Theorem 9. *Consider a quadratic system of the form*

$$\dot{x} = y, \qquad \dot{y} = Q(x, y). \tag{3}$$

If system (3) *admits a real invariant algebraic curve* $f(x, y) = 0$ *without isolated finite multiple points neither circuits then any limit cycle* γ *of* (3) *must be algebraic and contained in* $f(x, y) = 0$.

Proof. For the proof of this theorem we make use Bendixon–Dulac's criterion. Since $f(x, y) = 0$ is an invariant algebraic curve for quadratic system (3) it verifies $\dot{f} = K(x, y) f(x, y)$ for some polynomial $K(x, y) = ax + by + c$. Taking the Dulac's function $B(x, y) = f^{\lambda}(x, y) \exp(\alpha x)$ with λ and α real constants and computing the divergence of the vector field (yB, QB) we obtain

$$(yB)_x + (QB)_y = BQ_y + \dot{B} = B[\lambda K + \alpha y + Q_y].$$

(a) If $a \neq 0$ then we can always choose constants λ and α such that $\lambda K + \alpha y + Q_y$ is a constant. On the other hand, if the curve $f(x, y) = 0$ does not have any oval (neither any isolated multiple point and circuits by hypothesis) then it makes a partition of the plane into simply connected regions D_i of the form $\mathbb{R}^2 = \bigcup_i D_i \cup \Sigma$, where $\Sigma = \{(x, y) \in \mathbb{R}^2\colon\ f(x, y) = 0\}$. Since $f(x, y) = 0$ is invariant for the flow of the system we know that either $\gamma \subset D_i$ or $\gamma \subset \Sigma$. But applying Bendixon–Dulac's criterion to each one of the regions D_i the first option above is impossible. Moreover, since $f(x, y) = 0$ does not have ovals then $\gamma \not\subset \Sigma$ and we conclude that the system does not have any limit cycle.

It remains only to study the case for which $f(x, y) = 0$ has one oval. This oval can be either a limit cycle of the system or an orbit belonging to the continuous band of periodic orbits of a center. But in the last option the quadratic system cannot have limit cycles because it is well known, see [16], that a quadratic system cannot have a center and a limit cycle simultaneously.

(b) If $a = 0$ then we can apply the Poincaré method of tangential curves. To do this, consider the function $F(x, y) = f(x, y) \exp(-bx)$. The rate of change of this function along orbits of (3) is

$$\dot{F} = \exp(-bx)[-b\dot{x} f + \dot{f}] = \exp(-bx)\bigl[-byf + (by + c) f\bigr] = \exp(-bx) c f(x, y).$$

If $c = 0$ then F is an analytic first integral of system (3) defined in the whole plane and consequently the system does not have limit cycles. Otherwise $\dot{F}$ has constant sign outside of the set $\Sigma = \{(x, y) \in \mathbb{R}^2\colon\ f(x, y) = 0\}$. So Poincaré's method assures the nonexistence of limit cycles except perhaps in Σ and this completes the proof of the theorem. $\square$

Theorem 10. *Let $f_i(x, y) = 0$ with $i = 1, 2$ be two invariant algebraic curves of a quadratic system* (1) *such that the complement set of $\Sigma = \{(x, y) \in \mathbb{R}^2\colon\ f_1 f_2 = 0\}$ is the union of simply connected domains. Then the system does not have any limit cycle.*

Proof. For the proof of this theorem we make use the Bendixon–Dulac's criterion. Since $f_i(x, y) = 0$ with $i = 1, 2$ are two invariant algebraic curves of a quadratic system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, they verify $\dot{f} = K_i(x, y) f_i(x, y)$ for some polynomials $K_i(x, y)$ of degree less than or equal to one. We can assume without lost of generality that the cofactors K_1 and K_2 are linearly independent because otherwise, see the proof of Theorem 6, we have the first integral $f_1(x, y) f_2(x, y)^\alpha$ with $\alpha \in \mathbb{R}$ and the possible limit cycles of the system are in Σ. But, since the complement set of Σ is the union of simply connected domains, Σ cannot contain ovals and therefore the system does not have any limit cycle.

Taking the Dulac's function $B(x, y) = \prod_{i=1}^{2} f_i^{\lambda_i}(x, y)$ with λ_1 and λ_2 real constants and computing the divergence of the vector field (PB, QB) we obtain

$$(BP)_x + (BQ)_y = B(P_x + Q_y) + \dot{B} = B\left[P_x + Q_y = \sum_{i=1}^{2} \lambda_i K_i\right].$$

Now, taking into account that $P_x + Q_y$, K_1 and K_2 are polynomials of at most first degree and the independence of the cofactors, we can always to choose the constants λ_1 and λ_2 such that $P_x + Q_y + \sum_{i=1}^{2} \lambda_i K_i = c$, c being a real constant. If $c = 0$ then B is an integrating factor for the quadratic system and hence, see [11], the possible limit cycles of the quadratic system are contained in $B^{-1}(x, y) = 0$. But this is impossible because the algebraic curve $f_1 f_2 = 0$ does not have ovals. We can continue the proof supposing that $c \neq 0$. In that case, since by hypothesis the algebraic curve $f_1 f_2 = 0$ does not have ovals neither isolated multiple points it is clear that the interior of the possible limit cycle is always a simply connected region $D \subset \mathbb{R}^2$ such that the function $B \in \mathcal{C}^1(D)$. Then applying Bendixon–Dulac's criterion we prove the theorem. □

Remark. Theorem 10 generalizes the well known result that the existence of two invariant straight lines for a quadratic system precludes the presence of limit cycles as it has been shown by Bautin [1]. As far as we know, in that paper the existence of invariant curves was used for the first time to prove the nonexistence of limit cycles.

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