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Isotoping Heegaard surfaces in neat positions with respect to critical distance Heegaard surfaces

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A B S T R A C T

Suppose a closed orientable 3-manifold M has a genus g Heegaard surface P with distance $d(P)=2g$. Let Q be another genus g Heegaard surface which is strongly irreducible. Then we show that there is a height function $f: M \to I$ induced from P such that by isotopy, Q is deformed into a position satisfying the following; (1) $f|_Q$ has $2g+2$ critical points $p_0, p_1, \dots, p_{2g+1}$ with $f(p_0) < f(p_1) < \cdots < f(p_{2g+1})$ where p_0 is a minimum and p_{2g+1} is a maximum, and $p_1, \dots, p_{2g}$ are saddles, (2) if we take regular values r_i $(i = 1, \dots, 2g+1)$ such that $f(p_{i-1}) < r_i < f(p_i)$, then $f^{-1}(r_i) \cap Q$ consists of a circle if i is odd, and $f^{-1}(r_i) \cap Q$ consists of two circles if i is even.

1. Introduction

Hempel [3] introduced the concept of *distance* of a Heegaard splitting, and it is shown by many authors that it well represents various complexities of 3-manifolds. For example, Scharlemann and Tomova [10] show that high distance Heegaard splittings are "rigid". More precisely:

Theorem 1.1. *(Corollary 4.5 of [10]) If a compact orientable* 3*-manifold M has a genus g Heegaard surface P with distance $d(P) > 2g$, then*

1. *any other Heegaard surface of the same genus is isotopic to P;*
2. *moreover, any Heegaard surface Q of M with $2g(Q) \leqslant d(P)$ is isotopic to a stabilization or boundary stabilization of P.*

Recently, Berge and Scharlemann [1] have shown the following fact:

Fact. *If a closed orientable* 3*-manifold M has a genus* 2 *Heegaard surface P with distance $d(P)=4$, then any other genus* 2 *Heegaard surfaces of M is isotopic to P. Moreover there exist examples of 3-manifolds each of which admits mutually non-isotopic genus* 2 *Heegaard surfaces with distance* 3.

Hence, it seems that it is natural to call P with distance $d(P)=2g(P)$ *a critical distance Heegaard surface*. Then we have the following question:

Question. If a closed orientable 3-manifold M has a genus g Heegaard surface P with distance $d(P)=2g$, then are any genus g Heegaard surfaces of M isotopic to P?

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In this paper, we tackle the above question. Unfortunately, we do not have the answer for the above yet. We show that any other genus g Heegaard surface Q can be isotoped to a neat position with respect to a height function induced from the genus g Heegaard surface P, if Q is not isotopic to P and $d(P) = 2g$. Precisely speaking:

Theorem 1.2. *Suppose that a closed orientable 3-manifold M has a genus g Heegaard surface P with distance $d(P) = 2g$. Let Q be another genus g Heegaard surface which is strongly irreducible. Then there is a height function $f : M \to I$ induced from P (that is, the level surfaces of f with height t ($\neq 0, 1$) are isotopic to P) such that by isotopy, Q is deformed into a position satisfying the following*:

1. *$f|_Q$ has $2g + 2$ critical points $p_0, p_1, \ldots, p_{2g+1}$ with $f(p_0) < f(p_1) < \cdots < f(p_{2g+1})$ where p_0 is a minimum and p_{2g+1} is a maximum, and $p_1, \ldots, p_{2g}$ are saddles,*
2. *if we take regular values r_i ($i = 1, \ldots, 2g + 1$) such that $f(p_{i-1}) < r_i < f(p_i)$, then $f^{-1}(r_i) \cap Q$ consists of a circle if i is odd, and $f^{-1}(r_i) \cap Q$ consists of two circles if i is even.*

The main tool of the proof of Theorem 1.2 is *Reeb graphs* derived from horizontal arcs of the *Rubinstein–Scharlemann graphic* (or *graphic* for short). Graphic is introduced by Rubinstein and Scharlemann [9] for studying Reidemeister–Singer distance of two strongly irreducible Heegaard splittings. Moreover, under certain technical conditions, Li [8] shows that there exist horizontal arcs disjoint from the union of the regions each of which is labeled X, x, Y or y (for the definitions, see Section 3) of the graphic. He gives an alternative proof of Theorem 1.1 by using such horizontal arcs.

In [4], to give a more detailed treatment for such horizontal arc, we introduce *Reeb graphs* derived from such horizontal arcs, and by introducing a method of assigning positive integer to each edge of Reeb graph, we give an estimation of Hempel distance (see Section 4). In this paper, by using Reeb graphs and the assigning method, we prove Theorem 1.2, and we hope that this could help giving the answer to the above question.

2. Preliminaries

2.1. Heegaard splittings

A genus $g(\geqslant 1)$ *handlebody* H is the boundary sum of g copies of a solid torus. Note that H is homeomorphic to the closure of a regular neighborhood of some finite graph Σ in $\mathbb{R}^3$. The image Σ of the graph is called a *spine* of H. By a technical reason, throughout this paper, we suppose that each vertex of spines of genus $g(> 1)$ handlebodies is of valency three (for a detailed discussion, see Section 2 of [7]). Let M be a closed orientable 3-manifold. We say that $A \cup_P B$ is a (genus g) Heegaard splitting of M if A, B are genus g handlebodies in M such that $M = A \cup B$ and $A \cap B = \partial A = \partial B = P$. Then P is called a (genus g) *Heegaard surface* of M. A disk D properly embedded in a handlebody H is called a *meridian disk* of H if ∂D is an essential simple closed curve in ∂H. If there are meridian disks D_A, D_B in A, B respectively so that $\partial D_A = \partial D_B$, $A \cup_P B$ is said to be *reducible*. If there are meridian disks D_A, D_B in A, B respectively so that $\partial D_A, \partial D_B$ are disjoint on P, $A \cup_P B$ is said to be *weakly reducible*. It is easy to see that if a Heegaard splitting $M = A \cup_P B$ is reducible, it is weakly reducible. If $A \cup_P B$ is not weakly reducible, it is said to be *strongly irreducible*.

2.2. Curve complexes

Let S be a closed connected orientable surface S of genus at least two, and $\mathcal{C}(S)$ the 1-skeleton of Harvey's complex of essential simple closed curves on S (see [2]), that is, $\mathcal{C}(S)$ denotes the graph whose 0-simplices are isotopy classes of essential simple closed curves and whose 1-simplices connect distinct 0-simplices with disjoint representatives. We remark that $\mathcal{C}(S)$ is connected. Let x, y be 0-simplices of $\mathcal{C}(S)$. Then we define the distance between x and y, denoted by $d_S(x, y)$, as the minimum of such d that there is a path in $\mathcal{C}(S)$ with d 1-simplices joining x and y. Let X, Y be subsets of the 0-simplices of $\mathcal{C}(S)$. Then we define

$$d_S(X, Y) = \min\{d_S(x, y) \mid x \in X, y \in Y\}.$$

Suppose that S is the boundary of a handlebody V. Then $\mathcal{M}(V)$ denotes the subset of $\mathcal{C}(S)$ consisting of the 0-simplices with representatives bounding meridian disks of V. For a genus $g(\geqslant 2)$ Heegaard splitting $A \cup_P B$, its Hempel distance, denoted by $d(P)$, is defined to be $d_P(\mathcal{M}(A), \mathcal{M}(B))$.

3. Rubinstein–Scharlemann graphics

Let M be a smooth closed orientable 3-manifold. A *sweep-out* is a smooth map $f : M \to I$ such that for each $x \in (0, 1)$, the level set $f^{-1}(x)$ is a closed surface, and $f^{-1}(0)$ (resp. $f^{-1}(1)$) is a connected, finite graph such that each vertex has valency three. Each of $f^{-1}(0)$ and $f^{-1}(1)$ is called a *spine* of the sweep-out. It is easy to see that each level surface of f is a Heegaard surface of M and the spines of the sweep-outs are spines of the two handlebodies in the Heegaard splitting. Conversely, given a Heegaard splitting $A \cup_P B$ of M, it is easy to see that there is a sweep-out f of M such that each level surface of f is isotopic to P, $f^{-1}(0)$ is a spine of A, and $f^{-1}(1)$ is a spine of B. We call it a *sweep-out obtained from $A \cup_P B$*.

Given two sweep-outs, f and g of M, we consider their product $f \times g$ (that is, $(f \times g)(x) = (f(x), g(x))$), which is a smooth map from M to $I \times I$. Kobayashi and Saeki [7] have shown that by arbitrarily small deformations of f and g, we can suppose that $f \times g$ is a stable map on the complement of the four spines. At each point in the complement of the spines, the differential of the map $f \times g$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$. This map have a one-dimensional kernel for a generic point in M. The *discriminant set for* $f \times g$ is the set of points where the differential has a higher-dimensional kernel. Mather's classification of stable maps [6] implies that at each point of the discriminant set, the dimension of the kernel of the differential is two, and the discriminant set is a one-dimensional smooth submanifold in the complement of the spines in M. Moreover the discriminant set consists of all such x that a level surface of f is tangent to a level surface of g at x (here, we note that the tangent point is either a "center" or "saddle").

Let f, g be as above with $f \times g$ stable. The image of the discriminant set is a graph in $I \times I$, which is called the *Rubinstein–Scharlemann graphic*. It is known that the Rubinstein–Scharlemann graphic is a finite 1-complex Γ with each vertex having valency four or two. Each valency four vertex is called a *crossing vertex*, and each valency two vertex is called a *birth-death vertex*. There are valency one or two vertices of the graphic on the boundary of $I \times I$. Each component of the complement of Γ in $I \times I$ is called a *region*. At each point of a region, the corresponding level surfaces of f and g are disjoint or intersect transversely. The stable map $f \times g$ is *generic* if each arc $\{s\} \times I$ or $I \times \{t\}$ contains at most one vertex of the graphic. By Proposition 6.14 of [7], by arbitrarily small deformation of f and g, we may suppose that $f \times g$ is generic.

3.1. Labeling regions of the graphic

Let f and g be sweep-outs obtained from Heegaard splittings $A \cup_P B$, $X \cup_Q Y$, respectively with $f \times g$ stable. Now we introduce how to label each region of the graphic with following the convention of [9]. For each $s \in I$, we put that $P_s = f^{-1}(s)$ (in particular, Σ_A denotes P_0 and Σ_B denotes P_1), $A_s = f^{-1}([0, s])$ and $B_s = f^{-1}([s, 1])$. Similarly, for $t \in I$, we put that $Q_t = g^{-1}(t)$ (in particular, Σ_X denotes Q_0 and Σ_Y denotes Q_1), $X_t = g^{-1}([0, t])$ and $Y_t = g^{-1}([t, 1])$. Let (s, t) be a point in a region of the graphic. Then either $P_s \cap Q_t = \phi$, or P_s and Q_t intersect transversely in a collection $C = \{c_1, \ldots, c_n\}$ of simple closed curves.

Definition 3.1. Let $C = \{c_1, \ldots, c_n\}$ be as above. Then C_P denotes the subset of C consisting of the elements which are essential on P_s. Furthermore the subset C_A of C_P is defined by:

$$C_A = \left\{c \in C_P \;\middle|\; c \text{ bounds a disk } D \text{ in } Q_t \setminus C_P \text{ such that } N(\partial D, D) \subset A_s\right\},$$

where $N(\partial D, D)$ is a regular neighborhood of ∂D in D. Analogously $C_B \subset C_P$ and $C_X, C_Y \subset C_Q$ are defined.

If C_A (resp. C_B, C_X, C_Y) is non-empty, the region is labeled A (resp. B, X, Y). If C_P and C_Q are both empty and A_s (resp. B_s) contains an essential curve of Q_t, then the region is labeled b (resp. a). If C_P and C_Q are both empty and X_t (resp. Y_t) contains an essential curve of P_s, then the region is labeled y (resp. x). R_A (resp. R_B, R_X, R_Y, R_a, R_b, R_x, R_y) denotes the closure of the union of the regions labeled A (resp. B, X, Y, a, b, x, y). R_ϕ denotes the closure of the union of the unlabeled regions. The next lemma is proved in the proof of Lemma 3.2 of [8] (see also [4], [5]).

Lemma 3.2. *Let M, P, Q be as above. Suppose P, Q are strongly irreducible. If Q is not isotopic to P, then there exists $t \in (0, 1)$ such that $I \times \{t\}$ is disjoint from $R_X \cup R_x$ and $R_Y \cup R_y$.*

For a proof of the next lemma, see Corollary 5.6 of [4].

Lemma 3.3. *Let t be as in Lemma* 3.2. *There is a subarc $[s_0, s_1] \times \{t\} \subset I \times \{t\}$ such that*:

- $(s_1, t) \in \{$*an edge of the graphic*$\}$, *and*
- *for any $s \in (s_0, s_1)$, $(s, t) \in R_\phi$, and for any small $\epsilon > 0$, $(s_0 - \epsilon, t) \in R_A$ and $(s_1 + \epsilon, t) \in R_B$.*

Then we apply the argument of the proof of Lemma 2.1 of [8] to obtain such Q' from Q_t by an ambient isotopy whose support is contained in $f^{-1}([s_0 + \epsilon, s_1 - \epsilon])$ that satisfies the following conditions.

Conditions 3.4. Let $Q^* = f^{-1}([s_0 + \epsilon, s_1 - \epsilon]) \cap Q'$. Then,

1. at each $s \in (s_0, s_1)$, Q^* is transverse to each P_s, except for finitely many critical levels $x_1, \ldots, x_n \in (s_0, s_1)$;
2. at each critical level x_i, Q^* is transverse to P_{x_i} except for a saddle or circle tangency, as shown in Fig. 2.1(a) of [8];
3. at each regular level $s \in (s_0, s_1)$, each component of $P_s \cap Q^*$ is a simple closed curve which is essential in both P_s and Q^*.

In the remainder of this paper, we abuse notion by denoting Q for Q'.

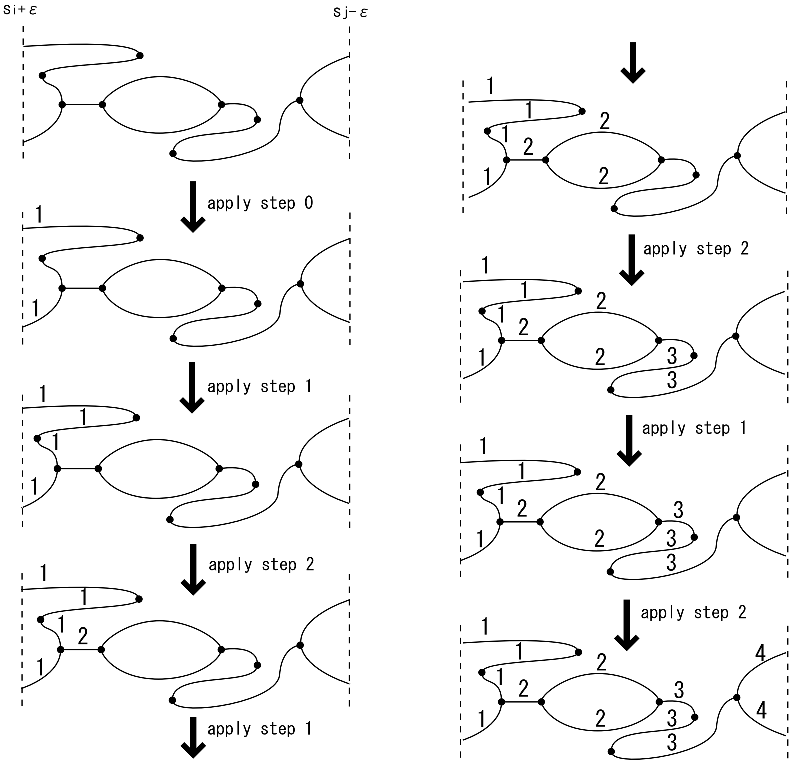

Fig. 1. Assigning an integer to each edge of G.

4. An estimation of Hempel distance by Reeb graph

We continue with Section 3. Particularly, let s_0, s_1 be as in Lemma 3.3, and Q^* as in Conditions 3.4. Define the equivalence relation $\sim$ on points on Q^* by $x \sim y$ whenever x, y are in the same component of a level set of $f|_{Q^*} : Q^* \to [s_0 + \epsilon, s_1 - \epsilon]$. The *Reeb graph* corresponding to $f|_{Q^*}$ is the quotient space of Q^* by the relation $\sim$. Then G denotes the Reeb graph corresponding to $f|_{Q^*}$. Note that G is a finite 1-complex such that the edges of G come from annuli in Q^* fibered by level loops, and that the valency two vertices correspond to circle tangencies, the valency three vertices correspond to saddle tangencies, and the valency one vertices correspond to components of ∂Q^*. In particular, if a valency one vertex corresponds to a component of $f^{-1}(s_0+\epsilon) \cap Q$ (resp. $f^{-1}(s_1 - \epsilon) \cap Q$), then it is called a ∂_-*-vertex* (resp. ∂_+*-vertex*). The union of ∂_--vertices (resp. ∂_+-vertices) is denoted by $\partial_- G$ (resp. $\partial_+ G$). Let $f^* : G \to [s_0 + \epsilon, s_1 - \epsilon]$ be the function induced from $f|_{Q^*}$. Note that for each $s \in (s_0 + \epsilon, s_1 - \epsilon)$, $f^{*-1}(s) (\subset G)$ consists of a finite number of points corresponding to the components of $P_s \cap Q^*$.

In [4], a method of estimating Hempel distance by using Reeb graph is given. We note that the Reeb graphs in [4] are slightly different from the above Reeb graphs, since circle tangencies did not come to appear in [4]. However it is easy to see that the arguments in [4] work in the setting of this paper. We introduce it here.

Let G, $\partial_\pm G$, f^* be as above. We assign a positive integer to each edge of G according to the following steps. Let $w_1, \ldots, w_k$ be the vertices of G which are not ∂-vertices. We suppose that $w_1, \ldots, w_k$ are positioned in this order from the left, i.e., $f^*(w_1) < f^*(w_2) < \cdots < f^*(w_k)$.

Now we define Steps 0, 1 and 2 inductively for assigning positive integers to the edges of G (see Fig. 1).

Step 0. We assign 1 to every edge adjacent to $\partial_- G$.

Step 1. Suppose that there is a valency two vertex w_i adjacent to edges e_l, $e_{l'}$ such that e_l has already been assigned and $e_{l'}$ has not been assigned yet. Then we assign the same integer as that of e_l to $e_{l'}$. We apply this assignment as much as possible.

In our assigning process, we will repeat applications of Steps 1 and 2. Before describing Step 2, we will give a general condition that the assignments have in the process. Suppose we finish Step 1 in repeated applications of Steps 1 and 2. At this stage, either every edge of G is assigned exactly one integer, or there is a unique vertex w_i such that there is an unassigned edge adjacent to w_i, and that each edge of G containing a point x with $f^*(x) < f^*(w_i)$ has already been assigned exactly one integer. Then we suppose that the assigned integers satisfy the following condition $(*)$. (Note that the conditions are clearly satisfied after Steps 0 and 1.)

$(*)$ For a small $\epsilon > 0$, let L_i be the set of the edges of G each containing a point x with $f^*(x) = f^*(w_i) - \epsilon$. Then it satisfies one of the following conditions:
   (1) All of the elements of L_i are assigned with the same integer, say n.
   (2) The set of the integers assigned to the elements of L_i consists of consecutive two integers, say $n-1$ and n.

Step 2.

1. Suppose that the vertex w_i satisfies the condition (1). Then we assign $n+1$ to the unassigned edge(s) adjacent to w_i.
2. Suppose that the vertex w_i satisfies the condition (2). Then we assign n to the unassigned edge(s) adjacent to w_i.

After finishing Step 2, we apply Step 1. Here we note that there are no multiple assignments. If all of the edges are assigned integers, then we are done. Suppose there is an unassigned edge. Then there is a unique vertex w_j such that there is an unassigned edge adjacent to w_j, and that each edge of G that contains a point y such that $f^*(y) < f^*(w_j)$ has already been assigned. Then we can show that the assignments at the current stage, also satisfies $(*)$ (for the proof, see Lemma 7.1 of [4]). And by repeating the processes, we finally assign an integer to each edge of G.

The next theorem is proved as in the proof of Theorem 7.3 of [4].

Theorem 4.1. *Let P, Q and G be as above. Let n be the minimum of the integers assigned to the edges adjacent to $\partial_+ G$. Then the distance $d(P)$ is at most $n+1$.*

5. Proof of main theorem

Let P, Q, Q^*, G, $\partial_\pm G$, f, f^* be as in Section 4. In this section, we suppose that P and Q are genus g Heegaard surfaces and $d(P) = 2g$, and then we consider the Reeb graph G derived from Q^*.

Since Q is connected, there is a component of Q^*, say $\hat{Q}^*$, whose Reeb graph contains a ∂_--vertex and a ∂_+-vertex. Let $\hat{G}^*$ be the Reeb graph corresponding to $\hat{Q}^*$.

Claim 5.1. *The Reeb graph $\hat{G}^*$ contains exactly $2g-2$ valency three vertices, and each component of $cl(Q \setminus \hat{Q}^*)$ is an annulus.*

Proof. By 3 of Condition 3.4, each component of $\partial\hat{Q}^*$ is essential on Q. Hence by Euler characteristic argument, we see that $\hat{Q}^*$ contains at most $2g-2$ saddles. By the rule in the assigning process, for each valency three vertex v, the difference in the integers assigned to the edges adjacent to v is at most one. Hence the integer assigned to the edges of $\hat{G}^*$ containing ∂_+-vertex is at most $2g-1$. This fact together with Theorem 4.1 and the assumption $d(P) = 2g$ show that each edge containing ∂_+-vertex is assigned $2g-1$. Moreover by the rule in the assigning process, we see that $\hat{Q}^*$ contains exactly $2g-2$ saddles. Hence $\chi(Q \setminus \hat{Q}^*) = \chi(Q) - \chi(\hat{Q}^*) = 0$, and this implies that each component of $cl(Q \setminus \hat{Q}^*)$ is an annulus. □

Let $\mathcal{G}$ be any path in the Reeb graph $\hat{G}^*$ joining a ∂_--vertex and a ∂_+-vertex.

Claim 5.2. *The path $\mathcal{G}$ contains all of the valency three vertices in $\hat{G}^*$.*

Proof. Assume that $\mathcal{G}$ does not contain all of the valency three vertices. Then by Claim 5.1, $\mathcal{G}$ contains at most $2g-3$ valency three vertices. This fact together with the rule in the assigning process show that the edge of $\mathcal{G}$ adjacent to an ∂_+-vertex is assigned an integer less than $2g-1$. Then Theorem 4.1 shows that $d(P) < 2g$, a contradiction. Hence this claim holds. □

Let $v_1, \ldots, v_{2g-2}$ be the valency three vertices contained in $\mathcal{G}$. By the proof of Claim 5.2, we see that by changing subscripts if necessary, we may suppose for each i, the set of the integers assigned to the edges adjacent to v_i consists of consecutive two integers $\{i, i+1\}$.

Claim 5.3. $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_{2g-2})$.

Proof. Suppose that there exists i such that $f^*(v_i) > f^*(v_{i+1})$, then there is a point x in the path $\mathcal{G}$, which joins v_{i+1} and a ∂_+-vertex such that $f^*(v_{i+1}) < f^*(x) < f^*(v_i)$. Note that by Claim 5.2, the edge containing x is assigned at least $i+2$. Then there is a point x' in the path $\mathcal{G}$, which joins a ∂_--vertex and v_i such that $f^*(x) = f^*(x')$. By Claim 5.2, the integer assigned to the edge containing x' is at most i, contradicting condition $(*)$. □

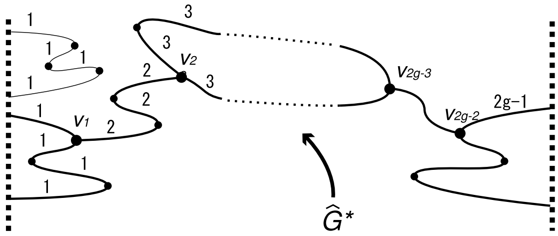

Fig. 2. The Reeb graph $\hat{G}^*$.

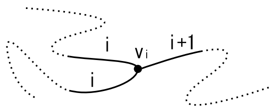

Fig. 3. i is odd.

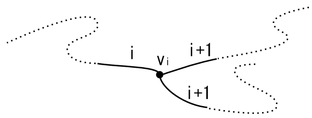

Fig. 4. i is even.

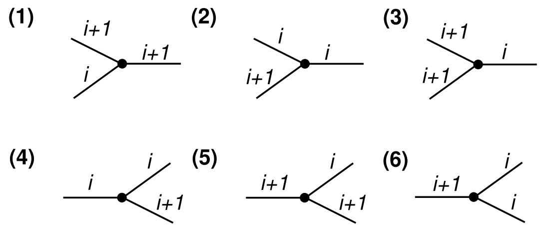

Fig. 5. The possible patterns of assigned edges adjacent to v_i.

By Claim 5.1 and the assumption $d(P) = 2g$, we see that $\hat{Q}^*$ is the unique component whose Reeb graph contains both a ∂_--vertex and a ∂_+-vertex (Fig. 2).

Claim 5.4. *For v_i $(i = 1, \dots, 2g-2)$, if i is odd, then the number of edges assigned i adjacent to v_i is exactly two. If i is even, then the number of edges assigned i adjacent to v_i is exactly one.*

Proof. By the fact that each component of $cl(Q \setminus \hat{Q}^*)$ is an annulus (Claim 5.1) and 3 of Conditions 3.4, we see that each component of Q^* other than $\hat{Q}^*$ is an annulus. This fact together with Theorem 4.1 and the assumption $d(P) = 2g$ show that both of the boundary components of each annulus are contained in $f^{-1}(s_0 + \epsilon)$ or $f^{-1}(s_1 - \epsilon)$. This shows that the number of edges containing $\partial_- G$ is even, and the number of ∂_--vertices contained in $\hat{G}^*$ is even. This together with the fact that v_1 is the only vertex which is adjacent to edges assigned 1, show that the number of edges assigned 1 in $\hat{G}^*$ adjacent to $\partial_- G$ is exactly two. This implies that the number of edges assigned 1 in $\hat{G}^*$ adjacent to v_1 is two and the number of edges assigned 2 adjacent to v_1 is one. This implies that the number of edges assigned 2 adjacent to v_2 is exactly one. By this fact and the fact that for each i, the set of the integers assigned to the edges adjacent to v_i consists of consecutive two integers $\{i, i+1\}$, we see that the remaining two edges are assigned 3. By repeating the arguments, we have this claim (Figs. 3 and 4). □

We say that v_i is *in a normal position*, if v_i satisfies the following:

For each edge e adjacent to v_i, if e is assigned i (resp. $i+1$), then for each $x \in e$, $f^*(x) < f^*(v_i)$ (resp. $f^*(x) > f^*(v_i)$).

Claim 5.5. *For each v_i, there is an ambient isotopy $\varphi_t^{(i)}$ of M whose support is contained in $f^{-1}([f^*(v_i) - \epsilon, f^*(v_i) + \epsilon])$ such that $\varphi_1^{(i)}(Q^*)$ satisfies Conditions 3.4, hence, the Reeb graph of $\varphi_1^{(i)}(Q^*)$ is defined, where the Reeb graph contains exactly one valency three vertex between the levels $f^*(v_i) - \epsilon$ and $f^*(v_i) + \epsilon$, which is in a normal position.*

Proof. Suppose that v_i is not in a normal position. Then the possible patterns of assigned edges adjacent to v_i are shown in Fig. 5 (1)–(6).

We consider the pattern (1). The component of $f^{-1}([f^*(v_i) - \epsilon, f^*(v_i) + \epsilon]) \cap Q^*$ containing the critical point of $f|_{Q^*}$ is a wedge of two circles c_1, c_2 on Q^*. Here, c_1 corresponds to the edge assigned $i+1$. Now we take a narrow annulus $\mathcal{A}$ such that $\mathcal{A} \cap Q^* = \partial\mathcal{A} \cap Q^* = c_1$, $f|_{\mathcal{A}}$ has no critical point, and $f(\partial\mathcal{A} \setminus c_1) = f^*(v_i) + \epsilon/2$. Then we push c_1 along $\mathcal{A}$ to deform Q^* as in Fig. 6. By this figure, we see that this ambient isotopy gives $\varphi_t^{(i)}$.

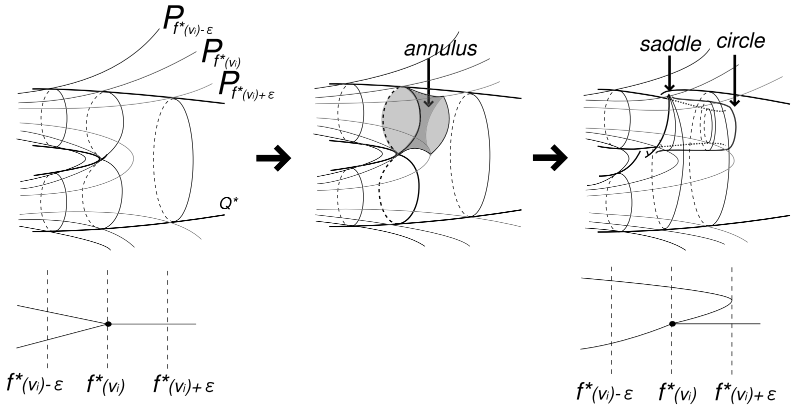

Fig. 6. The pattern (1).

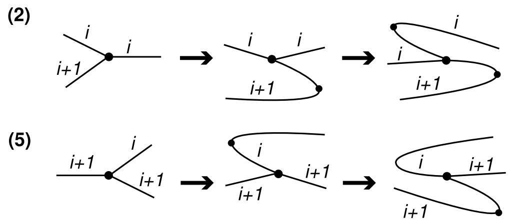

Fig. 7. The patterns (2) and (5).

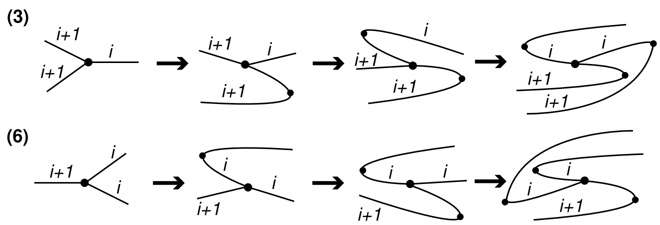

Fig. 8. The patterns (3) and (6).

The pattern (4) can be treated in a similar argument as in the pattern (1).

The patterns (2), (5) can be treated by two successive applications of the above arguments as described in Fig. 7. Details the left to the reader.

The patterns (3), (6) can be treated by three successive applications of the above arguments as described in Fig. 8. Details the left to the reader. $\square$

Here, for simplicity, we use Q^* for $\varphi_1^1(\varphi_1^2(\cdots(\varphi_1^{2g-2}(Q^*))\cdots))$.

Claim 5.6. *There is an ambient isotopy $\psi_t^{(1)}$ of M whose support is contained in $f^{-1}([s_0+\epsilon, f^*(v_2)])$ such that $\psi_1^{(1)}(Q^*)$ satisfies Conditions 3.4, hence, the Reeb graph of $\psi_1^{(1)}(Q^*)$ is defined, and it satisfies the following. There is exactly one valency three vertex in the levels $(s_0+\epsilon, f^*(v_2))$, which has the level $f^*(v_1)$ and is in a normal position, and every edge which is assigned 1 is contained in between the levels $s_0+\epsilon$ and $f^*(v_1)$.*

Proof. Let $\mathcal{P}_1, \mathcal{P}_2$, be the path in $\hat{G}^*$ each of which joins a ∂_--vertex and v_1. We consider the subset $\mathcal{Q}_i$ of $\mathcal{P}_i$ consisting of the point x with $f^*(x) \geqslant f^*(v_1)$. Since v_1 is in a normal position (Claim 5.5), $\mathcal{Q}_i$ is a union of subarcs contained in $Int(\mathcal{P}_i)$. By condition $(*)$ in Section 4, we see that $\mathcal{Q}_i$ is contained between the levels $f^*(v_1)$ and $f^*(v_2)$. Note that the preimage of each arc is an annulus properly embedded in the product region $f^{-1}([f^*(v_1), f^*(v_2)])$, whose boundary is contained in the boundary component of $f^{-1}(f^*(v_1))$. This shows that there is an ambient isotopy of M whose support is contained in a small neighborhood of $f^{-1}([f^*(v_1), f^*(v_2)])$, which pushes the annuli corresponding to $\mathcal{Q}_1 \cup \mathcal{Q}_2$ out of $f^{-1}([f^*(v_1), f^*(v_2)])$. Note that this ambient isotopy does not affect the vertex v_1. Hence we obtained a desired ambient isotopy (Fig. 9). $\square$

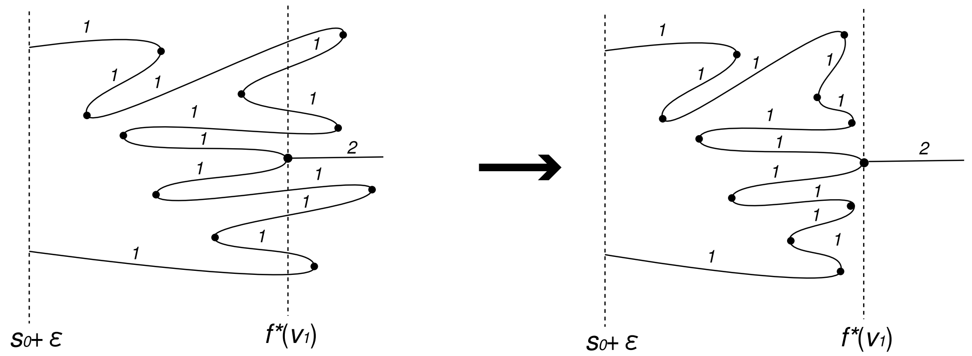

Fig. 9. Ambient isotopy $\psi_t^{(1)}$.

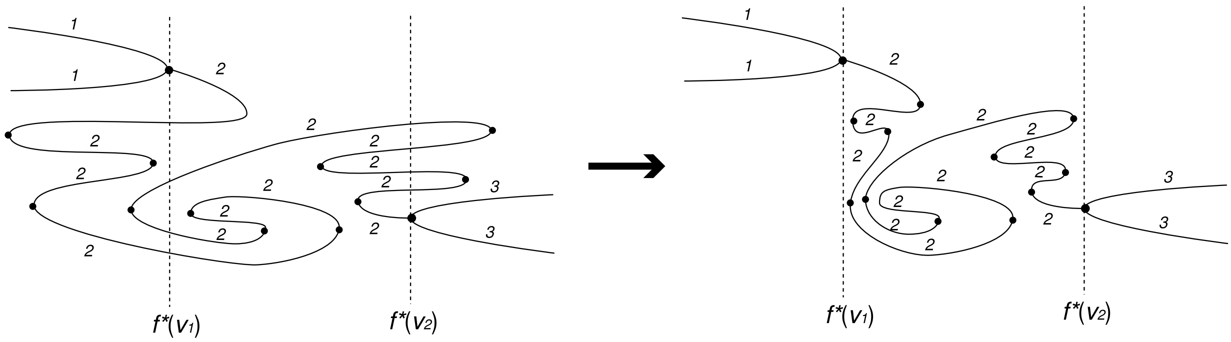

Fig. 10. Ambient isotopy $\psi_t^{(2)}$.

Here, for simplicity, we use Q^* for $\psi_1^{(1)}(Q^*)$. By using similar arguments as in the proof of Claim 5.6, we have the following.

Claim 5.7. *There is an ambient isotopy $\psi_t^{(2)}$ of M whose support is contained in $f^{-1}([f^*(v_1), f^*(v_3)])$ which satisfies Conditions* 3.4, *hence, the Reeb graph of $\psi_1^{(2)}(Q^*)$ is defined, and it satisfies the following. There is exactly one valency three vertex in the levels $(f^*(v_1), f^*(v_2))$, which has the level $f^*(v_2)$ and is in a normal position, and every edge which is assigned* 2 *is contained in between the levels $f^*(v_1)$ and $f^*(v_2)$.*

Then we successively apply similar argument to $v_3, v_4, \dots, v_{2g-2}$, and we finally obtain the next claim (see Fig. 10).

Claim 5.8. *We may suppose that Q^* satisfies the following*:

1. *each v_i in G^* is in a normal position*;
2. *for each i $(1 \leqslant i \leqslant 2g-3)$, each edge assigned $i+1$ is contained in between the levels $f^*(v_i)$ and $f^*(v_{i+1})$. The edge assigned* 1 *is contained in the levels $s_0+\epsilon$ and $f^*(v_1)$, and the edge assigned $2g-1$ is contained in the levels $f^*(v_{2g-2})$ and $s_1-\epsilon$.*

Claim 5.9. *There is an ambient isotopy Φ_t of M such that $\Phi_1(Q^*)$ satisfies Condition* 3.4, *hence, the Reeb graph of $\Phi_1(Q^*)$ is defined, and it satisfies the following. For each i $(1 \leqslant i \leqslant 2g-3)$, each edge assigned $i+1$ joins v_i and v_{i+1}, each edge assigned* 1 *joins a ∂_--vertex and v_1, and each edge assigned $2g-1$ joins v_{2g-2} and ∂_+-vertex.*

Proof. By Claim 5.8, we see that for each i $(1 \leqslant i \leqslant 2g-3)$, each component of the subset of Q^* between the levels $f^*(v_i)$ and $f^*(v_{i+1})$, or between the levels $s_0+\epsilon$ and $f^*(v_1)$, or between the levels $f^*(v_1)$ and $s_1+\epsilon$, is an annulus (with pinches introduces by saddles) properly embedded in the corresponding product regions $f^{-1}([f^*(v_i), f^*(v_{i+1})])$, or $f^{-1}([s_0+\epsilon, f^*(v_1)])$, or $f^{-1}([f^*(v_{2g-2}), s_1-\epsilon])$. Hence by using ambient isotopy, we can straighten each annulus with the boundary components fixed so that no critical point exists in the interior of the annulus. The composition of the above ambient isotopes gives Φ_t (Fig. 11). □

Here, for simplicity, we use Q^* for $\Phi_1(Q^*)$.

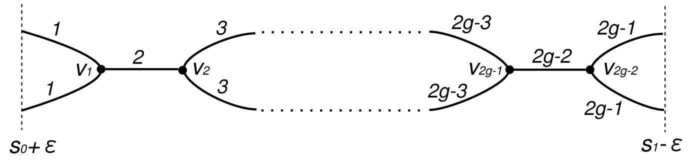

Fig. 11. The Reeb graph of $\hat{Q}^*$.

Completion of the proof. We have obtained Q with Q^* satisfies the conditions in Claim 5.9. Recall that $\hat{Q}^*$ is the component of Q^* joining a ∂_--vertex and a ∂_+-vertex. By Claim 5.1 and 3 of Conditions 3.4, we see that each component of Q^* other than $\hat{Q}^*$ is an annulus properly embedded in the product region $f^{-1}([s_0+\epsilon, s_1-\epsilon])$. Then by the proof of Claim 5.6, we can push such annulus out of the product region. Hence we may suppose that $\hat{Q}^* = Q^*$, and this implies that $cl(Q \setminus \hat{Q}^*)$ consists of two annuli, say $\mathcal{A}_-, \mathcal{A}_+$ such that $\mathcal{A}_-$ is properly embedded in the handlebody $f^{-1}([0, s_0+\epsilon])$, and $\mathcal{A}_+$ is properly embedded in the handlebody $f^{-1}([s_1-\epsilon, 1])$. Assume that $\mathcal{A}_-$ is compressible in the handlebody $f^{-1}([0, s_0+\epsilon])$. Then by compressing $\mathcal{A}_-$, we see that each component of $\partial\mathcal{A}_-$ bounds a disk in the handlebody, and hence has distance 0 in the handlebody. This together with the saddles in Q^* show that $d(P)$ is at most $2g-1$, a contradiction. Hence $\mathcal{A}_-$ is incompressible in the handlebody. Then we have the following two cases.

Case 1. $\mathcal{A}_-$ is not essential in the handlebody (that is, $\mathcal{A}_-$ is ∂-parallel in the handlebody).

In this case, we can isotope $\mathcal{A}_-$ with fixing $\partial\mathcal{A}_-$ so that $\mathcal{A}_-$ is contained in a small collar neighborhood of the boundary of the handlebody, which is the union of level surfaces. Then it is easy to see that we can further isotope $\mathcal{A}_-$ in the collar neighborhood so that the critical points of $f|_{\mathcal{A}_-}$ consists of one minimal point, and one saddle point.

Case 2. $\mathcal{A}_-$ is essential in the handlebody.

In this case, we can ∂-compress $\mathcal{A}_-$ to obtain a meridian disk, say D_-, in the handlebody. Then we can take a spine Σ_- of the handlebody such that Σ_- intersects D_- transversely in a single point. Then we retake f with respect to Σ_- so that $\Sigma_- \cap D_-$ is the unique critical point (which is obviously minimal) of $f|_{D_-}$. Note that $\mathcal{A}_-$ can be recovered from D_- by adding a band. We may suppose that the band is contained in a collar neighborhood as in Case 1. Hence we may suppose that this band contains exactly one critical point which is a saddle. Hence the critical points of $f|_{\mathcal{A}_-}$ consists of one minimal point and one saddle point.

In either case we have shown that the critical points of $f|_{\mathcal{A}_-}$ consists of one minimal point and one saddle point. The annulus $\mathcal{A}_+$ properly embedded in the handlebody $f^{-1}([s_1-\epsilon, 1])$ is treated in same manner. These fact together with Claim 5.9 completes the proof of Theorem 1.2.

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