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# Uniqueness theorems for entire functions concerning fixed points ☆

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ABSTRACT

This paper is devoted to studying the uniqueness problem of entire functions sharing one value or fixed points. We improve some results given by Fang and extend some results given by Fang and Qiu and by Lin and Yi.



## 1. Introduction

Let $f(z)$ be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We shall use the standard notations in the Nevanlinna value distribution theory of meromorphic functions such as $T(r,f)$, $N(r,f)$ and $m(r,f)$ (see, e.g., [7,14,15]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o\,(T(r,f))$ as $r \to \infty$ possibly outside a set $r$ of finite linear measure. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$.

Let $p$ be a positive integer and $a \in \mathbf{C}\bigcup\{\infty\}$. We denote by $N_p\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ where an $m$-fold zero is counted $m$ times if $m \le p$ and $p$ times if $m > p$.

We say that two meromorphic functions $f$ and $g$ share a small function $a$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share $a$ CM (counting multiplicities).

Hayman [8] and Clunie [3] proved the following result.

**Theorem A.** *Let $f$ be a transcendental entire function, $n \ge 1$ be a positive integer, then $f^n f' = 1$ has infinitely many zeros.*

Fang and Hua [6] and, Yang and Hua [13] obtained a unicity theorem corresponding to the above result.

**Theorem B.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n \ge 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share* 1 CM, *then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_1$ and $c$ are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f = tg$ for a constant $t$ such that $t^{n+1} = 1$.*

Hennekemper [9], Chen [2], and Wang [12] extended Theorem A by proving the following theorem.

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**Theorem C.** *Let $f$ be a transcendental entire function, and $n, k$ be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many zeros.*

Fang [4] proved the following result corresponding to Theorem C.

**Theorem D.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_1$ and $c$ are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant $t$ such that $t^n = 1$.*

A similar result for meromorphic functions appears in [1, Theorem 2]. Unfortunately, the proof there contains an incorrect detail. (See the final Section.)

In [4], Fang also obtained the following results.

**Theorem E.** *Let $f$ be a transcendental entire function, $n, k$ be two positive integers with $n \geq k+2$. Then $(f^n(f-1))^{(k)} = 1$ has infinitely many zeros.*

**Theorem F.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n \geq 2k+8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

For the case $k = 1$ in Theorem F, Lin and Yi [10] obtained the following result.

**Theorem G.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n \geq 7$ be a integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f = g$.*

Theorem G means that $n \geq 2k+6$ in Theorem F when $k = 1$, so a natural question is that can $n \geq 2k+8$ in Theorem F be replaced by $n \geq 2k+6$? In this paper, we give an affirmative answer to the above question and get the following results improving Theorems E and F.

**Theorem 1.** *Let $f$ be a transcendental meromorphic function, and $n, k$ be two integers with $n \geq 4$ and $k \geq 1$. Then $(f^n(f-1))^{(k)} = 1$ has infinitely many zeros.*

**Remark 1.** From the proof of Theorem 1, we get that if $f$ is an entire function and $k = 1$, we only need $n \geq 3$.

**Theorem 2.** *Suppose that $f$ is a transcendental meromorphic function with finite number of poles, $g$ is a transcendental entire function, and let $n, k$ be two positive integers with $n \geq 2k+6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Obviously, Theorem 2 is a generalization of Theorem G.

We say that a finite value $z_0$ is a fixed point of $f$ if $f(z_0) = z_0$. Define

$$E_f = \{z \in \mathbb{C} : f(z) = z, \text{ counting multiplicities}\}. \tag{1}$$

It is easy to see that a polynomial $P$ with degree $n \geq 2$ has $n$ fixed points (counting multiplicities). A transcendental function may not have a fixed point. For example, $f = e^z + z$.

Fang and Qiu [5] obtained the following result.

**Theorem H.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share $z$ CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$ or $f = tg$ for a constant $t$ such that $t^{n+1} = 1$.*

Lin and Yi [11] obtained:

**Theorem I.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n \geq 7$ is an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, then $f = g$.*

In this paper, we prove the following results concerning fixed points.

**Theorem 3.** *Let $f$ be a transcendental entire function, and $n, k$ be two positive integers with $n \geq k+2$, then $(f^n)^{(k)}$ has infinitely many fixed points.*

**Theorem 4.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n > 2k+4$. If $E_{(f^n)^{(k)}} = E_{(g^n)^{(k)}}$, then either*

(1) *$k = 1$, $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1$, $c_1$ and $c$ are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$ or*
(2) *$f = tg$ for a constant $t$ such that $t^n = 1$.*

**Theorem 5.** *Let $f$ be a transcendental entire function, and $n, k$ be two positive integers with $n \geq k+2$. Then $(f^n(f-1))^{(k)}$ has infinitely many fixed points.*

**Theorem 6.** *Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n \geq 2k+6$. If $E_{(f^n(f-1))^{(k)}} = E_{(g^n(g-1))^{(k)}}$, then $f = g$.*

**Remark 2.** Noting that $f^n f' = \frac{1}{n+1}(f^{n+1})'$, we can get Theorem H from Theorem 4 when $k = 1$. Thus Theorem 4 extends Theorem H. Similarly, Theorem 6 extends Theorem I.

## 2. Some lemmas

For the proof of our results, we need the following lemmas.

**Lemma 1** (*Milloux Inequality [7]*). *Suppose that $f$ is a nonconstant meromorphic function and $k$ is a positive integer. Then*

$$T(r,f) \leq \overline{N}(r,f) + N(r,1/f) + N\left(r, \frac{1}{f^{(k)}-1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f).$$

**Lemma 2** (*[12], Lemma 1*). *Suppose that $f$ is a transcendental meromorphic function, $k \geq 3$ is an integer and $\varepsilon > 0$. Then*

$$(k-2)\overline{N}(r,f) + N(r,1/f) \leq 2\overline{N}(r,1/f) + N(r,1/f^{(k)}) + \varepsilon T(r,f) + S(r,f).$$

**Lemma 3** (*[10], Lemma 2*). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2}$$

*where $F$ and $G$ are two nonconstant meromorphic functions. If $F$ and $G$ share 1 CM and $H \not\equiv 0$, then*

$$T(r,F) + T(r,G) \leq 2\left(N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G)\right) + S(r,F) + S(r,G). \tag{3}$$

**Lemma 4** (*[16], Lemma 2.4*). *Let $f$ be a nonconstant meromorphic function, $p, k$ be positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f), \tag{4}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f). \tag{5}$$

**Lemma 5** (*[7], Theorem 3.10*). *Suppose that $f$ is a nonconstant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N(r,1/f) + N(r,1/f^{(k)}) = S(r,f'/f),$$

*then $f = e^{az+b}$, where $a \neq 0$, $b$ are constants.*

**Lemma 6.** *Suppose that $f$ and $g$ are two nonconstant entire functions, and $n, k$ are two positive integers, and denote $F = (f^n(f-1))^{(k)}$, $G = (g^n(g-1))^{(k)}$. If there exist two non-zero constants $c_1$ and $c_2$ such that $\overline{N}(r, \frac{1}{F-c_1}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G-c_2}) = \overline{N}(r, \frac{1}{F})$, then $n \leq 2k+3$.*

**Proof.** By the second fundamental theorem (see, e.g., [14], Theorem 1.8), we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,1/F) + \overline{N}\left(r, \frac{1}{F-c_1}\right) + S(r,F) \\ &\leq \overline{N}(r,1/F) + \overline{N}(r,1/G) + S(r,F). \end{aligned}$$

By Lemma 4 and the above inequality, and using the standard Valiron–Mohon'ko theorem (see, e.g., [14], Theorem 1.13), we get

$$\begin{aligned} (n+1)T(r,f) &\leq N_{k+1}\left(r, \frac{1}{f^n(f-1)}\right) + N_{k+1}\left(r, \frac{1}{g^n(g-1)}\right) + S(r,f) \\ &\leq (k+1)\left(\overline{N}(r,1/f) + \overline{N}(r,1/g)\right) + T(r,f) + T(r,g) + S(r,f). \end{aligned}$$

Similarly,

$$(n+1)T(r,g) \leq (k+1)\left(\overline{N}(r,1/f) + \overline{N}(r,1/g)\right) + T(r,f) + T(r,g) + S(r,g).$$

The above two inequalities yield

$$(n-2k-3)\left(T(r,f)+T(r,g)\right) \le S(r,f)+S(r,g),$$

resulting in $n \le 2k+3$, completing the proof.

**Lemma 7.** *Suppose that $F$ and $G$ are given by Lemma 6. If $n > 2k+1$ and $F = G$, then $f = g$.*

**Proof.** From $F = G$, we have

$$(f^n(f-1))^{(k)} = (g^n(g-1))^{(k)}.$$

By integration, we get

$$(f^n(f-1))^{(k-1)} = (g^n(g-1))^{(k-1)} + c_{k-1},$$

where $c_{k-1}$ is a constant. If $c_{k-1} \neq 0$, we get from Lemma 6 that $n \le 2k+1$, which is a contradiction. Hence $c_{k-1} = 0$. Repeating the same reasoning, we obtain

$$(f^n(f-1))^{(k-2)} = (g^n(g-1))^{(k-2)}.$$

Continuing inductively, we arrive at

$$f^n(f-1) = g^n(g-1).$$

Let $h = f/g$. If $h \neq 1$, then by the above equation we have

$$g = \frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^n}.$$

Thus, we deduce by Picard's theorem that $h$ has two Picard exceptional values at most if $h$ is a nonconstant function. Noting that $n \ge 4$, there exists a $\omega$ such that $h-\omega$ has zeros, where $\omega^{n+1} = 1$. Thus $g$ must have poles from the last equation, which is impossible. Hence $h$ is a constant, $g$ is a constant too, which is a contradiction. Thus, $f(z) \equiv g(z)$.

**Lemma 8.** *Suppose that $f$ is a transcendental meromorphic function with finite number of poles, $g$ is a transcendental entire function, and $n, k$ are two positive integers. Denote $F = (f^n(f-1))^{(k)}$, $G = (g^n(g-1))^{(k)}$. If $F \cdot G = \alpha$, where $\alpha = 1$ or $\alpha = z^2$, then $n \le k+2$.*

**Proof.** Suppose that $n > k+2$. From $F \cdot G = \alpha$, we have

$$(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = \alpha. \tag{6}$$

If $z_0$ is a zero of $f$ with the order $p$, then $z_0$ is a zero of $(f^n(f-1))^{(k)}$ with the order $np-k$. Noting that $g$ is an entire function and $n > k+2$, then $z_0$ is a zero of $\alpha$ with the order at least 3 from (6), which is impossible. Thus $f$ has no zeros. Let

$$f(z) = \frac{e^{\beta}}{h},$$

where $\beta$ is a nonconstant entire function and $h$ is a polynomial. Thus, by induction we get

$$(f^{n+1})^{(k)} = \left(\frac{e^{(n+1)\beta}}{h^{n+1}}\right)^{(k)} = P_1(\beta', \beta'', \ldots, \beta^{(k)}, h)e^{(n+1)\beta}, \tag{7}$$

$$(f^{n})^{(k)} = \left(\frac{e^{(n)\beta}}{h^{n}}\right)^{(k)} = P_2(\beta', \beta'', \ldots, \beta^{(k)}, h)e^{n\beta}, \tag{8}$$

where $P_1$ and $P_2$ are differential polynomials in $\beta', \beta'', \ldots, \beta^{(k)}$, with coefficients which are rational functions in $h$ or its derivatives. Obviously, $P_1 \not\equiv 0$, $P_2 \not\equiv 0$, $T(r,P_1) = S(r,f)$, $T(r,P_2) = S(r,f)$. From (7), (8) and (6) we have

$$N\left(r, \frac{1}{P_1e^{\beta}-P_2}\right) = S(r,f).$$

By the second fundamental theorem for small functions (see, e.g., [14], Theorem 1.36), we have

$$\begin{aligned} T(r,f) &\le T(r,P_1e^{\beta}) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{P_1e^{\beta}-P_2}\right) + \overline{N}\left(r, \frac{1}{P_1e^{\beta}}\right) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which is a contradiction. Hence $n \le k+2$. The proof of Lemma 8 is completed. $\square$

The following lemmas can be proved similarly as Lemmas 6 and 7, respectively.

**Lemma 9.** *Suppose that $f$ and $g$ are two nonconstant entire functions, and $n$, $k$ are two positive integers, and denote $F=(f^n)^{(k)}$, $G=(g^n)^{(k)}$. If there exist two non-zero constants $c_1$ and $c_2$ such that $\overline{N}(r,\frac{1}{F-c_1})=\overline{N}(r,\frac{1}{G})$ and $\overline{N}(r,\frac{1}{G-c_2})=\overline{N}(r,\frac{1}{F})$, then $n\le 2k+4$.*

**Lemma 10.** *Suppose that $F$ and $G$ are given as in Lemma 9. If $n>2k$ and $F=G$, then $f=tg$ for a constant $t$ such that $t^n=1$.*

## 3. Proofs of results

**Proof of Theorem 1.** Denote $F=f^n(f-1)$. From Lemmas 1 and 2 and the standard Valiron–Mohon'ko theorem, we have

$$\begin{aligned}(n+1)T(r,f) &= T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+N(r,1/F)+N\left(r,\frac{1}{F^{(k)}-1}\right)-N\left(r,\frac{1}{F^{(k+1)}}\right)+S(r,f)\\ &= \overline{N}(r,f)+N(r,1/F)+N\left(r,\frac{1}{F^{(k)}-1}\right)-N\left(r,\frac{1}{F^{(k+1)}}\right)+S(r,f),\end{aligned}$$

$$(k-1)\overline{N}(r,f)+N(r,1/F)\le 2\overline{N}(r,1/F)+N(r,1/F^{(k+1)})+\varepsilon T(r,f)+S(r,F).$$

The two inequalities above yield

$$\begin{aligned}(n+1-\varepsilon)T(r,f) &\le 2\overline{N}(r,1/F)+N\left(r,\frac{1}{F^{(k)}-1}\right)+S(r,f)\\ &\le 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{F^{(k)}-1}\right)+S(r,f)\\ &\le 4T(r,f)+N\left(r,\frac{1}{F^{(k)}-1}\right)+S(r,f).\end{aligned}$$

Noting that $n\ge 4$, we conclude that $(f^n(f-1))^{(k)}-1$ has infinitely many zeros.

Next, we suppose that $k=1$. From the second fundamental theorem, we have

$$T(r,F')\le \overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}\left(r,\frac{1}{F'-1}\right)+\overline{N}(r,f)+S(r,f).$$

By the above inequality and Lemma 4 applied to $F$, we have

$$\begin{aligned}(n+1)T(r,f) &\le N_2\left(r,\frac{1}{f^n(f-1)}\right)+\overline{N}\left(r,\frac{1}{F'-1}\right)+T(r,f)+S(r,f)\\ &\le 2\overline{N}\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{F'-1}\right)+T(r,f)+S(r,f)\\ &\le 4T(r,f)+\overline{N}\left(r,\frac{1}{F'-1}\right)+S(r,f),\end{aligned}$$

that is

$$(n-3)T(r,f)\le \overline{N}\left(r,\frac{1}{(f^n(f-1))'-1}\right)+S(r,f).$$

Noting that $n\ge 4$, $(f^n(f-1))'-1$ has infinitely many zeros from the above inequality. □

**Proof of Theorem 2.** Denote

$$F=(f^n(f-1))^{(k)},\qquad G=(g^n(g-1))^{(k)}. \tag{9}$$

Then $F$ and $G$ share 1 CM. Let $H$ be given by (2). If $H\not\equiv 0$, by Lemma 3 we know that (3) holds. From Lemma 4, we have

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right) &\le T(r,F)-(n+1)T(r,f)+N_{k+2}\left(r,\frac{1}{f^n(f-1)}\right)+S(r,f),\\ N_2\left(r,\frac{1}{G}\right) &\le T(r,G)-(n+1)T(r,g)+N_{k+2}\left(r,\frac{1}{g^n(g-1)}\right)+S(r,g),\\ N_2\left(r,\frac{1}{F}\right) &\le N_{k+2}\left(r,\frac{1}{f^n(f-1)}\right)+S(r,f),\end{aligned}$$

$$N_2\left(r,\frac{1}{G}\right)\le N_{k+2}\left(r,\frac{1}{g^n(g-1)}\right)+S(r,g).$$

Noting that $f$ has only finite number of poles, from the last four inequalities and (3), we get

$$\begin{aligned}(n+1)(T(r,f)+T(r,g)) &\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_{k+2}\left(r,\frac{1}{f^n(f-1)}\right)+N_{k+2}\left(r,\frac{1}{g^n(g-1)}\right)+S(r)\\ &\le 2N_{k+2}\left(r,\frac{1}{f^n(f-1)}\right)+2N_{k+2}\left(r,\frac{1}{g^n(g-1)}\right)+S(r)\\ &\le (2k+4)\left(\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right)+2N_{k+2}\left(r,\frac{1}{f-1}\right)+2N_{k+2}\left(r,\frac{1}{g-1}\right)+S(r),\end{aligned}$$

where $S(r)=S(r,f)+S(r,g)$. That is

$$(n-2k-5)\left(T(r,f)+T(r,g)\right)\le S(r),$$

which contradicts the assumption $n\ge 2k+6$. Hence $H=0$. Integrating twice, we get from (2) that

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{10}$$

where $A$ ($\neq 0$) and B are constants. From (10) we have

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)},\qquad G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{11}$$

We discuss the following three cases.

**Case 1**. Suppose that $B\neq 0,-1$. From (11) we have $\overline{N}\left(r,1/\left(F-\frac{B+1}{B}\right)\right)=\overline{N}(r,G)$. From the second fundamental theorem, we have

$$\begin{aligned}T(r,F) &\le \overline{N}(r,1/F)+\overline{N}\left(r,\frac{1}{\left(F-\frac{B+1}{B}\right)}\right)+S(r,F)\\ &=\overline{N}(r,1/F)+\overline{N}(r,G)+S(r,F)\\ &=\overline{N}(r,1/F)+S(r,F).\end{aligned}$$

From the above inequality and (4) applied to $F$, we have

$$\begin{aligned}T(r,F) &\le N_1(r,1/F)+S(r,f)\\ &\le T(r,F)-T\left(r,f^n(f-1)\right)+N_{k+1}\left(r,\frac{1}{f^n(f-1)}\right)+S(r,f),\end{aligned}$$

namely,

$$\begin{aligned}(n+1)T(r,f) &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{f-1}\right)+S(r,f)\\ &\le (k+2)T(r,f)+S(r,f),\end{aligned}$$

which contradicts the assumption $n\ge 2k+6$.

**Case 2**. Suppose that $B=0$. From (11) we have

$$F=\frac{G+(A-1)}{A},\qquad G=AF-(A-1). \tag{12}$$

If $A\neq 1$, from (12) we obtain $\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)=\overline{N}(r,1/G)$ and $\overline{N}\left(r,\frac{1}{G+(A-1)}\right)=\overline{N}(r,1/F)$. By Lemma 6, $n\le 2k+3$ contradicting the assumption $n\ge 2k+6$. Thus $A=1$ and consequently $F=G$, so that $f$ does not have any pole and $f(z)\equiv g(z)$ by Lemma 7.

**Case 3**. Suppose that $B=-1$. From (11) we have

$$F=\frac{A}{-G+(A+1)},\qquad G=\frac{(A+1)F-A}{F}. \tag{13}$$

If $A\neq -1$, we obtain from (13) that $\overline{N}\left(r,1/\left(F-\frac{A}{A+1}\right)\right)=\overline{N}(r,1/G)$, $\overline{N}\left(r,1/\left(G-A-1\right)\right)=\overline{N}(r,F)$. By the same reasoning in Case 1 and Case 2, we get a contradiction. Hence $A=-1$. From (13), we have $F\cdot G=1$, which is not possible by Lemma 8. This completes the proof of Theorem 2. $\square$

Since the proof of Theorem 3 is similar to the proof of Theorem 5, we only prove Theorem 5 here.

**Proof of Theorem 5.** Denote $F = f^n(f-1)$. From the second fundamental theorem for small functions, we have

$$T(r, F^{(k)}) \le \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, F).$$

By the above inequality and Lemma 4 with $p = 1$ applied to $F$, we have

$$\begin{aligned}(n+1)T(r,f) &\le N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r,f)\\ &\le (k+2)T(r,f) + \overline{N}\left(r, \frac{1}{(f^n(f-1))^{(k)} - z}\right) + S(r,f).\end{aligned}$$

Noting that $n \ge k+2$, $(f^n(f-1))^{(k)}$ has infinitely many fixed points from the above inequality. □

**Proof of Theorem 6.** By the assumption and Theorem 5 we know that either both $f$ and $g$ are transcendental entire functions or both $f$ and $g$ are polynomials.

First, we consider the case when $f$ and $g$ are transcendental entire functions. Let

$$F = \frac{(f^n(f-1))^{(k)}}{z}, \qquad G = \frac{(g^n(g-1))^{(k)}}{z}.$$

Then $F$ and $G$ share 1 CM. By the same argument in the proof of Theorem 2, we get $F = G$ or $F \cdot G = 1$.

If $F = G$, then $(f^n(f-1))^{(k)} = (g^n(g-1))^{(k)}$. From Lemma 7, we obtain $f = g$.

If $F \cdot G = 1$, then $(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = z^2$. From Lemma 8, we get a contradiction.

Now we consider the case where both $f$ and $g$ are polynomials. Then there exists a non-zero constant $c$ such that

$$(f^n(f-1))^{(k)} - z = c\left((g^n(g-1))^{(k)} - z\right). \tag{14}$$

Taking the derivative of (14) gives

$$(f^n(f-1))^{(k+1)} = c(g^n(g-1))^{(k+1)} + 1 - c. \tag{15}$$

If $c \ne 1$, we deduce by Lemma 6 that $n \le 2k+5$ from (15), which is a contradiction. Thus $c = 1$. From (14), we have

$$(f^n(f-1))^{(k)} = (g^n(g-1))^{(k)}. \tag{16}$$

By Lemma 7, we obtain $f = g$. This completes the proof of Theorem 6. □

**Proof of Theorem 4.** From Theorem H, we need to consider the case $k \ge 2$ only. By the assumption and Theorem 3 we know that either both $f$ and $g$ are transcendental entire functions or both $f$ and $g$ are polynomials.

First, we consider the case when $f$ and $g$ are transcendental entire functions. Let

$$F = \frac{(f^n)^{(k)}}{z}, \qquad G = \frac{(g^n)^{(k)}}{z}.$$

Then $F$ and $G$ share 1 CM. By the same argument as in the proof of Theorem 2, we can get $F = G$ or $F \cdot G = 1$.

If $F = G$, then $(f^n)^{(k)} = (g^n)^{(k)}$. From Lemma 10, we obtain $f = tg$, where $t$ is a constant satisfying $t^n = 1$.

Therefore, we may now assume that $F \cdot G = 1$, hence

$$(f^n)^{(k)}(g^n)^{(k)} = z^2. \tag{17}$$

Since $f$ and $g$ are entire functions and $n > 2k+4$, we can deduce from (17) that $f$ and $g$ have no zeros.

In fact, suppose that $z_0$ is a zero of $f$ with the multiplicity $p \ge 1$. Then $z_0$ is a zero of $(f^n)^{(k)}$ with the multiplicity $np - k$. Since $g$ is an entire function, $z_0$ is a zero of $z^2$ with the multiplicity $np - k$, which is a contradiction. Thus, we may now write

$$f = e^{\alpha}, \qquad g = e^{\beta}, \tag{18}$$

where $\alpha, \beta$ are two nonconstant entire functions. Then $T(r, \frac{(f^n)'}{f^n}) = T(r, n\alpha')$. From (17), we know that either both $\alpha$ and $\beta$ are transcendental functions or both $\alpha$ and $\beta$ are polynomials. From (17), we conclude that

$$N\left(r, 1/(f^n)^{(k)}\right) = N\left(r, 1/z^2\right) = O(\log r).$$

From this and (18), we have

$$N(r, f^n) + N\left(r, 1/f^n\right) + N\left(r, 1/(f^n)^{(k)}\right) = O(\log r).$$

Suppose that $\alpha$ is a transcendental entire function. We deduce from Lemma 5 that $\alpha$ is a polynomial, which is a contradiction.

We may assume that $\alpha$ is a polynomial of degree $p$ and $\beta$ is a polynomial of degree $q$. If $p = q = 1$, we write

$$f = \mathrm{e}^{Az+B}, \qquad g = \mathrm{e}^{Cz+D},$$

where $A, B, C$ and $D$ are constants such that $AC \neq 0$. Substituting above $f$ and $g$ into (17), we get

$$A^k C^k n^{2k} \mathrm{e}^{n(A+C)z+n(B+D)} = z^2,$$

which is impossible. Then $\max\{p, q\} > 1$. Without loss of generality, we suppose that $p > 1$. Then $(f^n)^{(k)} = P\mathrm{e}^{n\alpha}$, where $P$ is a polynomial of degree $kp - k \geq k \geq 2$. From (17), we have $p = k = 2$, and then $q = 1$ again from (17). Suppose that

$$f^n = \mathrm{e}^{n(A_1 z^2 + B_1 z + C_1)}, \qquad g^n = \mathrm{e}^{n(D_1 z + E_1)}$$

where $A_1$ ($\neq 0$), $B_1$, $C_1$, $D_1$ ($\neq 0$) and $E_1$ are constants. Then

$$(f^n)'' = n(4nA_1^2 z^2 + 4nA_1 B_1 z + nB_1^2 + 2A_1)\mathrm{e}^{n(A_1 z^2 + B_1 z + C_1)},$$
$$(g^n)'' = n^2 D_1^2 \mathrm{e}^{n(D_1 z + E_1)}.$$

Substituting the last two equations into (17), we have

$$Q(z)\mathrm{e}^{n\left(A_1 z^2 + (B_1 + D_1)z + C_1 + E_1\right)} = z^2, \tag{19}$$

where $Q(z)$ is a polynomial of degree 2. Since $A_1 \neq 0$, we get a contradiction from (19).

Next, we consider the case where both $f$ and $g$ are polynomials. Then there exists a non-zero constant $K$ such that

$$(f^n)^{(k)} - z = K((g^n)^{(k)} - z). \tag{20}$$

If $K \neq 1$, taking the derivative of (20) gives

$$(f^n)^{(k+1)} = K(g^n)^{(k+1)} + 1 - K. \tag{21}$$

By Lemma 9 and (21), we obtain $n \leq 2k + 4$, which is a contradiction. Hence, $K = 1$ and (20) yields

$$(f^n)^{(k)} = (g^n)^{(k)}. \tag{22}$$

From Lemma 10 and (22), we obtain $f = tg$, where $t$ is a constant satisfying $t^n = 1$. This completes the proof of Theorem 4. □

## 4. Annex remarks

In this section, we point out an incorrect detail in the proof of Theorem 2 in [1] as follows.

In [1, p. 1200], on the third line above formula (4.10), the authors say:

"Similarly,

$\infty$ is a Picard exceptional value of $f$ and $g$".

In fact, by the reasoning suggested, one can conclude that the poles of $f$ can occur at zeros of $(g^n)^{(k)}$, and not at the zeros of $g$ itself. (In fact, $g$ has no zeros.)

For example. Suppose that $k = 2$. From (4.8), we obtain

$$n^2 f^{n-2} g^{n-2} \left((n-1)f'^2 + ff''\right)\left((n-1)g'^2 + gg''\right) = 1.$$

If $z_0$ is a pole of $f$ with multiplicity $p$, then $z_0$ is a pole of $f^{n-2}\left((n-1)f'^2 + ff''\right)$ with multiplicity $np + 2$. If $z_0$ is a zero of $\left((n-1)g'^2 + gg''\right)$ with multiplicity $np + 2$ but not a zero of $g$, we cannot get the contradiction stated in [1].

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