Cyclic vectors of diagonal operators on the space of functions analytic on a disk

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Abstract

The purpose of this paper is to study cyclic vectors and invariant subspaces of operators on the space of functions analytic on an open disk in the complex plane having as eigenvectors the monomials $z^n$.

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1. Introduction

A vector $x$ in a complete metrizable topological vector space $\mathcal{X}$ is said to be cyclic for a continuous linear operator $T : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$ if the closed linear span of the orbit $\{T^n x: n \geq 0\}$ of $x$ under $T$ is all of $\mathcal{X}$. Operators which have a cyclic vector are said to be cyclic. Cyclicity results yield interesting approximation results. For instance, the Weierstrass Approximation Theorem asserts that the function $f(x) \equiv 1$ on $[0, 1]$ is cyclic for the operator $T : g(x) \to xg(x)$ of multiplication by $x$ on the Banach space $C([0, 1])$ of continuous functions on $[0, 1]$.

A closed subspace $\mathcal{M}$ of $\mathcal{X}$ is invariant for $T : \mathcal{X} \to \mathcal{X}$ if $Tx \in \mathcal{M}$ for all $x \in \mathcal{M}$. The closed linear span of the orbit of any vector $x$ under $T$ is the smallest closed invariant subspace for $T$ containing $x$. Hence, a vector $x$ is cyclic for $T$ if and only if the smallest closed invariant subspace for $T$ containing $x$ is all of $\mathcal{X}$. The importance of cyclic vectors derives from the long standing study of invariant subspaces of operators and the approximation results they yield.
Cyclic vectors and invariant subspaces of operators on a Hilbert space which are diagonalizable with respect to an orthonormal basis have been well-studied (see, for instance, Wermer [15], Brown, Shields and Zeller [1], Sarason [10,11], Scroggs [12], Sibilev [14], and Nikol’skiĭ [7], amongst others). The purpose of this paper is to study cyclic vectors and invariant subspaces of the corresponding class of operators on the space of functions analytic on an open disk in the complex plane. The preliminaries are as follows.

For each \( R \in (0, \infty) \), we denote by \( \Ca H R \) the vector space of functions analytic on the open disk \( B(0, R) \equiv \{ z \in \C : |z| < R \} \) in the complex plane. It follows from the Radius of Convergence Formula that a function \( f(z) \equiv \sum_{n=0}^{\infty} a_n z^n \) is in \( \Ca H R \) if and only if \( \limsup |a_n|^{1/n} \leq 1/R \). When endowed with the topology of uniform convergence on compact, the space \( \Ca H R \) is an example of a complete locally convex topological vector space (see [9]). Moreover, the topology of \( \Ca H R \) is induced by the invariant metric \( \rho(f,g) \) defined by \( \rho(f,g) \equiv \sum_{n=1}^{\infty} \| f - g \|_n / \{ 2^n (1 + \| f - g \|_n) \} \) where here \( \| h \|_n \equiv \sup \{|h(z)| : |z| \leq R(1 - 1/n) \} \) for all functions \( h \) in \( \Ca H R \) and all \( n \geq 1 \) (see Rudin [9]).

Any linear map \( D \) on \( \Ca H R \) having as eigenvectors the monomials \( z^n \) with associated eigenvalues \( \lambda_n \) is given formally by \( D : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \lambda_n a_n z^n \). The linear map \( D \) defines a continuous linear operator on all of \( \Ca H R \) if and only if \( \limsup |\lambda_n|^{1/n} \leq 1 \) (see Proposition 1). In this paper, any operator \( D \) on \( \Ca H R \) for which there exists a sequence of complex numbers \( \{ \lambda_n \} \) with \( \limsup |\lambda_n|^{1/n} \leq 1 \) and \( D(z^n) = \lambda_n z^n \) for all \( n \geq 0 \) will be called a diagonal operator on \( \Ca H R \) having eigenvalues \( \{ \lambda_n \} \). The purpose of this paper is to study cyclic vectors and invariant subspaces of diagonal operators on \( \Ca H R \). Since every monomial is an eigenvector for every diagonal operator, the closed linear span of any collection of monomials is an invariant subspace for every diagonal operator on \( \Ca H R \). Of particular interest will be conditions on the eigenvalues \( \{ \lambda_n \} \) of a diagonal operator \( D \) on \( \Ca H R \) for the converse to hold. Diagonal operators having this property are said to admit spectral synthesis.

In Section 2, we show that a diagonal operator \( D \) on \( \Ca H R \) is cyclic if and only if the eigenvalues of \( D \) are distinct. In this case, we show \( D \) has a dense set of cyclic vectors.

In Section 3, we show that the uncountable collection of diagonal operators on \( \Ca H R \) each of whose set of eigenvalues are separated has a dense set of common cyclic vectors.

In Section 4, we give equivalent conditions for a cyclic diagonal operator on \( \Ca H R \) to admit spectral synthesis and show that every cyclic diagonal operator on \( \Ca H R \) whose eigenvalues are bounded admits spectral synthesis.

Throughout this paper, we will apply the following results concerning the space \( \Ca H R \) and its dual without further reference (see Rudin [9]): A linear map \( L : \Ca H R \to \C \) is continuous if and only if there exists a sequence \( \{ l_n \} \) of complex numbers for which \( \limsup |l_n|^{1/n} < R \) and \( L(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} l_n c_n \) for every function \( \sum_{n=0}^{\infty} c_n z^n \) in \( \Ca H R \). Moreover, a closed subspace \( \Ca M \) of \( \Ca H R \) is not all of \( \Ca H R \) if and only if there exists a nonzero continuous linear functional \( L : \Ca H R \to \C \) for which \( L(x) \equiv 0 \) for all \( x \in \Ca M \). Finally, the closure of any convex subset of \( \Ca H R \) and the weak closure of the convex subset coincide.

2. Cyclicity results

In this section, we show that a diagonal operator \( D \) on \( \Ca H R \) is cyclic if and only if the eigenvalues of \( D \) are distinct. In this case, we show \( D \) has a dense set of cyclic vectors.

We begin by showing the straightforward result that a linear map \( D \) having eigenvectors \( z^n \) with associated eigenvalues \( \lambda_n \) is continuous on all of \( \Ca H R \) if and only if \( \limsup |\lambda_n|^{1/n} \leq 1 \).
Proposition 1. Let \( \{\lambda_n\} \) be any sequence of complex numbers. Then the linear map
\[
D: \sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} \lambda_n a_n z^n
\]
defines a continuous linear map from \( \mathcal{H}_R \) to \( \mathcal{H}_R \) if and only if \( \limsup |\lambda_n|^{1/n} \leq 1 \).

Proof. Suppose \( D \) defines a continuous linear map on all of \( \mathcal{H}_R \). Then \( \sum_{n=0}^{\infty} \lambda_n a_n z^n \) is in \( \mathcal{H}_R \) whenever \( \sum_{n=0}^{\infty} \lambda_n a_n z^n \) is in \( \mathcal{H}_R \). Hence \( \limsup |\lambda_n a_n|^{1/n} \leq R \) whenever \( \limsup |\lambda_n a_n|^{1/n} \leq R \) and so \( \limsup |\lambda_n|^{1/n} \leq 1 \). Conversely, if \( \limsup |\lambda_n|^{1/n} \leq 1 \), then \( D: \sum_{n=0}^{\infty} \lambda_n z^n \mapsto \sum_{n=0}^{\infty} \lambda_n a_n z^n \) defines a linear map from \( \mathcal{H}_R \) to \( \mathcal{H}_R \). It follows from the Closed Graph Theorem (see [9, Theorem 2.15, p. 51]) that \( D \) is continuous. \( \square \)

We now derive a simple test for a function in \( \mathcal{H}_R \) to be cyclic for a diagonal operator on \( \mathcal{H}_R \).

Lemma 1. Let \( D \) be a diagonal operator on \( \mathcal{H}_R \) having eigenvalues \( \{\lambda_n\} \) and let \( f(z) \equiv \sum_{n=0}^{\infty} a_n z^n \) be any function in \( \mathcal{H}_R \). The following are equivalent:

(i) \( f \) fails to be cyclic for \( D \).
(ii) the closed linear span of the orbit \( \{\sum_{n=0}^{\infty} a_n \lambda_n^k z^n : k \geq 0\} \) of \( f \) under \( D \) is not all of \( \mathcal{H}_R \), and
(iii) there exists a sequence \( \{l_n\} \) of complex numbers, not all zero, for which \( \limsup |l_n|^{1/n} < R \) and \( 0 \equiv \sum_{n=0}^{\infty} l_n a_n \lambda_n^k \) for all \( k \geq 0 \).

The proof of Lemma 1 follows immediately from the topological vector space version of the Hahn–Banach Theorem.

As an immediate consequence of the preceding lemma, we have that a simple necessary condition for a function \( \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{H}_R \) to be cyclic for a diagonal operator \( D \) is that \( a_n \neq 0 \) for all \( n \geq 0 \). It also follows that a simple necessary condition for a diagonal operator \( D \) to be cyclic is that the eigenvalues \( \{\lambda_n\} \) of \( D \) be distinct (that is, that \( \lambda_m \neq \lambda_n \) whenever \( m \neq n \)). We prove the converse, namely, that a diagonal operator on \( \mathcal{H}_R \) is cyclic whenever its eigenvalues are distinct.

Theorem 1. Let \( D \) be a diagonal operator on \( \mathcal{H}_R \) having eigenvalues \( \{\lambda_n\} \). Then \( D \) is cyclic if and only if \( \lambda_m \neq \lambda_n \) whenever \( m \neq n \). In this case, \( D \) has a dense set of cyclic vectors.

Proof. We have already observed that in order for \( D \) to be cyclic, the eigenvalues of \( D \) must be distinct. Conversely, suppose that \( \lambda_m \neq \lambda_n \) whenever \( m \neq n \). Since \( D \) is continuous, \( \limsup |\lambda_r|^{1/r} \leq 1 \), and so \( |\lambda_r| \leq 2^r \) for all \( r \) sufficiently large. Hence there exists a constant \( c \geq 1 \) such that \( |\lambda_r| \leq c 2^r \) for all \( r \geq 0 \). For each \( n \geq 0 \), define \( \alpha_n \equiv \min\{|\lambda_i - \lambda_j| : 0 \leq i, j \leq n, i \neq j \} \) and \( \beta_n \equiv \min\{1, \alpha_n\} \).

We show that the set \( C \) of functions \( f(z) \equiv \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{H}_R \) for which there exists a constant \( \alpha \) (depending on \( f \)) for which \( 0 < |a_r| \leq \alpha \beta_r^\gamma / (4c)^r r^r \) for all \( r \geq 0 \) is a dense set of cyclic vectors for \( D \).

We begin by showing that every function \( f_0(z) \equiv \sum_{n=0}^{\infty} a_n z^n \) in \( C \) is cyclic for \( D \). To this end, let \( L \) be an arbitrary functional on \( \mathcal{H}_R \). So there exists a sequence of complex numbers \( \{l_n\} \) with \( \limsup |l_n|^{1/n} < R \) and \( L(\sum_{n=0}^{\infty} b_n z^n) = \sum_{n=0}^{\infty} b_n l_n \) for all functions \( \sum_{n=0}^{\infty} b_n z^n \) in \( \mathcal{H}_R \). Since \( \limsup |l_n|^{1/n} < R \), there exists a constant \( \gamma \) for which \( |l_n| \leq \gamma R^n \) for all \( n \geq 0 \).
For each positive integer \( n \) and each \( k \) in \{0, 1, 2, \ldots, n\}, the polynomial \( p_{n,k}(z) \equiv \prod_{i=0, i \neq k}^{n} (z - \lambda_i)/(\lambda_k - \lambda_i) \) is well-defined since the eigenvalues of \( D \) are distinct. Since the sequence \( \{\beta_n\} \) is decreasing and \( \beta_n \leq 1 \) and \( \beta_n \leq \alpha_n \) for all \( n \), we have for all \( r > n \) that

\[
|p_{n,k}(\lambda_r)| \leq \prod_{i=0, i \neq k}^{n} 2(2^r) / \alpha_n \leq \prod_{i=0, i \neq k}^{n} 2(2^r) / \beta_n \leq (2c2^r / \beta_n)^n \leq (2c2^r / \beta_r)^r.
\]

Let \( k \) be any nonnegative integer. We have that \( p_{n,k}(\lambda_r) = 0 \) for all \( r \leq n \) with \( r \neq k \) and \( p_{n,k}(\lambda_k) = 1 \) for all \( n \) with \( k \leq n \). Whenever \( n \) is sufficiently large, we have for \( r > n \) that \( \{2^rR/r^r\} \leq 1 \) and so

\[
|L(p_{n,k}(D)f_0 - a_kz^k)| = \left| \sum_{r>n} p_{n,k}(\lambda_r)a_lr \right| \leq \sum_{r>n} \left[ \frac{2c2^r}{\beta_r} \right] \cdot \left[ \frac{\alpha \beta_r^r}{(4c)^r r^r} \right] \cdot \gamma R^r 
\]

which tends to zero as \( n \) tends to infinity. Since \( L \) is an arbitrary functional on \( \mathcal{H}_R \), \( a_kz^k \) is in the weak closure of the orbit of \( f_0 \) under \( D \) and hence in the closure of the orbit of \( f_0 \) under \( D \). Since \( f_0 \) is in \( C \), \( a_k \neq 0 \) and so \( z^k \) is in the closure of the orbit of \( f_0 \) under \( D \) for all \( k \geq 0 \). Since the monomials have dense linear span in \( \mathcal{H}_R \), it follows that \( f_0 \) is cyclic for \( D \).

We now show that \( C \) is dense in \( \mathcal{H}_R \). To this end, let \( g(z) \equiv \sum_{r=0}^\infty a_r z^r \) be an arbitrary function in \( \mathcal{H}_R \). We define a sequence of functions \( \{g_n\} \) in \( \mathcal{H}_R \) converging uniformly to \( g \) in the topology of \( \mathcal{H}_R \), that is, converging uniformly on compact subsets of \( B(0, R) \). Let \( K \) be any compact subset of \( B(0, R) \). So \( \tilde{R} \equiv \sup\{|z| : z \in K\} < R \).

Since \( g \) is in \( \mathcal{H}_R \), we have that \( \lim \sup |a_r|^{1/r} \leq 1 / \tilde{R} \). Let \( K \) be any constant in \( (1/R, 1/\tilde{R}) \). So \( |a_r| \leq K^r \) for all \( r \) sufficiently large and so there exists a constant \( \gamma \) for which \( |a_r| \leq \gamma K^r \) for all \( r > 0 \). Define \( a_{n,k} \equiv a_n \) whenever \( k < n \) and \( a_{n,k} \equiv \beta_k^r / [(4c)^r k^r] \) whenever \( k \geq n \). The functions \( g_n(z) \equiv \sum_{k=0}^{\infty} a_{n,k}z^k \) are in \( C \). Recall that \( 1 < c \). Since \( K^\tilde{R} < 1 \) and \( \beta_r < 1 \) for all \( r > 0 \), we have that

\[
\sup\{|g_n(z) - g(z)| : z \in K\} = \sup\left\{ \left| \sum_{r>n} (a_{n,r} - a_r)z^r \right| : z \in K \right\}
\]

\[
\leq \sum_{r>n} \left\{ \frac{\beta_r^r}{(4c)^r r^r} + \gamma K \right\} \tilde{R}^r
\]

\[
\leq \sum_{r>n} \left\{ \left( \frac{\tilde{R}}{r^r} \right)^r + \gamma (K^\tilde{R})^r \right\}
\]

which tends to zero as \( n \) tends to infinity. It follows that \( \{g_n\} \) converges uniformly to \( g \) on \( K \) and so \( C \) is dense in \( \mathcal{H}_R \). \( \Box \)

The rate of decay of the coefficients defining the collection \( C \) of functions in the preceding proof may be improved significantly.

3. Common cyclic vectors

We say that a vector \( x \) in a complete metrizable topological vector space \( \mathcal{X} \) is a common cyclic vector for a set \( D \) of cyclic operators on \( \mathcal{X} \) if \( x \) is a cyclic vector for each operator \( D \) in \( D \). Herrero has shown that a cyclic operator on a Banach space has a dense set of cyclic vectors if and only if the point spectrum of its adjoint has empty interior (see [2, Theorem 1, p. 918]).
Moreover, Shields has shown that the set of cyclic vectors of an operator on a Banach space is a $G_δ$ set (see [13, Proposition 40, p. 411]). Hence by the Baire Category Theorem any countable collection of cyclic operators on a Banach space the point spectra of all of whose adjoints have empty interior has a dense set of common cyclic vectors.

In this section, we show that the uncountable collection of cyclic diagonal operators on $\mathcal{H}_R$ each of whose eigenvalues are separated (in a sense made precise below) has a dense set of common cyclic vectors.

**Theorem 2.** Let $D_0$ denote the collection of cyclic diagonal operators on $\mathcal{H}_R$ each of whose set of eigenvalues $\{\lambda_n\}$ is such that $\inf(|\lambda_i - \lambda_j|: i \neq j) > 0$. Then $D_0$ has a dense set of common cyclic vectors.

**Proof.** We show that the set $C$ of functions $\sum_{k=0}^{\infty} a_k z^k$ in $\mathcal{H}_R$ for which there exists a constant $\alpha$ with $0 < |a_r| \leq \alpha / r^r$ for all $r \geq 0$ is a dense set of common cyclic vectors for $D_0$. To this end, let $f_0(z) \equiv \sum_{k=0}^{\infty} a_k z^k$ be an arbitrary function in $C$ and let $D$ be an arbitrary diagonal operator in $D_0$. We show that $f_0$ is cyclic for $D$. As in the proof of Theorem 1, it suffices to show that $a_k z^k$ is in the closure of the orbit of $f_0$ under $D$ for all $k \geq 0$. To this end, let $L$ be an arbitrary functional on $\mathcal{H}_R$. So there exists a sequence $\{l_i\}$ of complex numbers for which $\sup |l_i|^{1/n} < R$ and $L(\sum_{k=0}^{\infty} c_k z^k) = \sum_{k=0}^{\infty} l_k c_k$ for all functions $\sum_{k=0}^{\infty} c_k z^k$ in $\mathcal{H}_R$. Let $B$ be any constant in $(\limsup |l_i|^{1/n}, R)$. So $|l_i| \leq B^n$ for all $n$ sufficiently large. It follows that there exists a constant $\gamma$ for which $|l_i| \leq \gamma B^n$ for all $n \geq 0$.

For each positive integer $n$ and each $k$ in $\{0, 1, \ldots, n\}$, the polynomial

$$p_{n,k}(z) \equiv \prod_{i=0, i \neq k}^{n} \frac{z - \lambda_i}{\lambda_k - \lambda_i}$$

is well-defined since $D$, being cyclic, has distinct eigenvalues. We have that $p_{n,k}(\lambda_r) = 0$ for all $r \leq n$ with $r \neq k$ and $p_{n,k}(\lambda_k) = 1$ for all $n$ with $k \leq n$. Since $D$ is continuous, $\limsup |\lambda_i|^{1/r} \leq 1$ and so $|\lambda_r| \leq 2^r$ for all $r$ sufficiently large. Hence there exists a constant $c \geq 1$ for which $|\lambda_r| \leq c2^r$ for all $r > 0$. Since $D$ is in $D_0$, $\delta \equiv \min\{1, \inf(|\lambda_k - \lambda_i|: i \neq k)\} > 0$. It follows that $|p_{n,k}(\lambda_r)| \leq (2c2^r/\delta)^n$ for all $r > n$.

Whenever $n$ is sufficiently large, we have for $r > n$ that $\{2c2^r/\delta r^r\} < 1/2$ and so

$$|L(p_{n,k}(D) f_0 - a_k z^k)| = \left| \sum_{r>n} p_{n,k}(\lambda_r) a_r l_r \right| \leq \sum_{r>n} \left\{ \frac{2c2^r}{\delta} \right\}^{n} \left\{ \frac{\alpha}{r^r} \right\} \gamma B^r \leq \sum_{r>n} \left\{ \frac{2c2^r}{\delta} \right\}^{r} \gamma B^r \leq \alpha \gamma \sum_{r>n} \frac{1}{2^r}$$

which tends to zero as $n$ tends to infinity. Since $L$ is an arbitrary functional, we have that $a_k z^k$ is in the closure of the orbit of $f_0$ under $D$. Since $f_0$ is in $C$, $a_k \neq 0$ and so $z^k$ is in the closure of the orbit of $f_0$ under $D$ for all $k \geq 0$. Since the monomials have dense linear span in $\mathcal{H}_R$, it follows that $f_0$ is cyclic for $D$. An argument similar to one presented in the proof of Theorem 1 shows that $C$ is dense in $\mathcal{H}_R$. The result follows. \Box

The rate of decay of the coefficients defining the collection $C$ of functions in $\mathcal{H}_R$ in the preceding proof may be improved significantly.

It is not known if the set of all cyclic diagonal operators on $\mathcal{H}_R$ has a common cyclic vector.
4. Spectral synthesis

A continuous linear operator \( T : \mathcal{X} \to \mathcal{X} \) on a complete metrizable topological vector space \( \mathcal{X} \) is said to **admit spectral synthesis** if every closed invariant subspace \( \mathcal{M} \) for \( T \) equals the closed linear span of the eigenvectors for \( T \) contained in \( \mathcal{M} \). By definition, a diagonal operator on \( \mathcal{H}_R \) having eigenvalues \( \{ \lambda_n \} \) has as eigenvectors the monomials \( z^n \). If \( D \) is cyclic, then the eigenvalues are distinct and the monomials are the only eigenvectors for \( D \). Hence a cyclic diagonal operator on \( \mathcal{H}_R \) admits spectral synthesis if and only if the lattice of closed invariant subspaces of \( D \) consists precisely of the closed linear span of sets \( \{ z^n : n \in \mathbb{N} \} \) of monomials where \( N \) is an arbitrary subset of nonnegative integers.

Theorem 3 of this section gives various equivalent conditions for a cyclic diagonal operator on \( \mathcal{H}_R \) to admit spectral synthesis. Recall that a simple necessary condition for a function \( \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{H}_R \) to be cyclic for a cyclic diagonal operator \( D \) on \( \mathcal{H}(\mathbb{C}) \) is that \( a_n \neq 0 \) for all \( n \geq 0 \). The result shows, for instance, that the converse holds only for those cyclic diagonal operators \( D \) on \( \mathcal{H}_R \) admitting spectral synthesis.

We begin with the following technical lemma.

**Lemma 2.** Let \( \mathcal{M} \) be any closed subspace of \( \mathcal{H}_R \) other than the whole space \( \mathcal{H}_R \) or \( \{ 0 \} \) and define \( K \) to be the set of nonnegative integers \( k \) for which there exists a function \( \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{M} \) with \( a_k \neq 0 \). Then there exists a function \( \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{M} \) with \( a_k \neq 0 \) for all \( k \in K \).

**Proof.** By means of contradiction, suppose that no such function in \( \mathcal{M} \) exists. Then \( \mathcal{M} = \bigcup_{k \in K} \mathcal{M}_k \) where \( \mathcal{M}_k \equiv \{ h(z) \equiv \sum_{r=0}^{\infty} a_r z^r : a_k = 0 \} \). Since \( \mathcal{M} \) is closed in \( \mathcal{H}_R \), it is complete, and hence of second category in \( \mathcal{M} \). In order to obtain a contradiction to the Baire Category Theorem, we need only show that \( \mathcal{M}_k \) is of first category in \( \mathcal{M} \) for each \( k \in K \). To this end, let \( k \) be any nonnegative integer in \( K \). We show that \( \mathcal{M}_k \) is, in fact, nowhere dense in \( \mathcal{M} \). By means of contradiction, suppose that the interior \( (\mathcal{M}_k)^{\circ} \) of \( \mathcal{M}_k \) in \( \mathcal{M} \) is nonempty. Hence there exists a nonempty open set \( \Theta \) in \( \mathcal{H}_R \) for which \( (\mathcal{M}_k)^{\circ} = \Theta \cap \mathcal{M} \).

Let \( h \) be any function in \( (\mathcal{M}_k)^{\circ} = \Theta \cap \mathcal{M} \). Since \( k \in K \), there exists a function \( f_k(z) \equiv \sum_{r=0}^{\infty} a_r z^r \) in \( \mathcal{M} \) for which \( a_k \neq 0 \). Since \( \Theta \) is open, there exists a positive number \( \epsilon \) for which the open ball \( B(h, \epsilon) \) in \( \mathcal{H}_R \) with center \( h \) and radius \( \epsilon \) is a subset of \( \Theta \). We show that \( h + cf_k \) is in \( \Theta \) whenever \( c \) is sufficiently small. For any function \( g \) in \( \mathcal{H}_R \) and any positive integer \( i \), we denote \( \| g \|_i \equiv \sup \{ |g(z)| : |z| \leq R(1-1/i) \} \). By the Maximum Modulus Principle, we have that \( \| g \|_i \leq \| g \|_j \) whenever \( i < j \). Let \( N \) be any positive integer for which \( \sum_{i=N+1}^{\infty} 1/2^i < \epsilon/2 \). For any \( c \in (0, \epsilon/(2\| f_k \|_N)) \), we have that the distance between \( h \) and \( h + cf_k \) in \( \mathcal{H}_R \) is

\[
\rho(h, h + cf_k) = \sum_{i=0}^{\infty} \frac{\| cf_k \|_i}{2^i (1 + \| cf_k \|_i)} \leq \sum_{i=0}^{N} \frac{\| cf_k \|_i}{2^i} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \leq c \| f_k \|_N + \frac{\epsilon}{2} < \epsilon.
\]

Hence \( h + cf_k \) is in \( B(h, \epsilon) \subseteq \Theta \) whenever \( c \) is in \( (0, \epsilon/(2\| f_k \|_N)) \). Since \( f_k(z) = \sum_{r=0}^{\infty} a_r z^r \) where \( a_k \neq 0 \), there exists a constant \( c \) in \( (0, \epsilon/(2\| f_k \|_N)) \) with \( h(z) + cf_k \equiv \sum_{r=0}^{\infty} b_r z^r \) where \( b_k \neq 0 \). Since \( h \) is in \( (\mathcal{M}_k)^{\circ} \subseteq \Theta \cap \mathcal{M} \) and \( f_k \) is in \( \mathcal{M} \), we have that \( h + cf_k \) is in \( \mathcal{M} \). Moreover, \( h + cf_k \) is in \( \Theta \). Hence \( h + cf_k \) is in \( \Theta \cap \mathcal{M} = \mathcal{M}_k \cap \mathcal{M} \). Hence by definition of \( \mathcal{M}_k \), we have that \( b_k = 0 \), a contradiction. That is \( (\mathcal{M}_k)^{\circ} \) is empty and so \( \mathcal{M}_k \) is nowhere dense. The result follows. \( \square \)

**Theorem 3.** Let \( D \) be the cyclic diagonal operator on \( \mathcal{H}_R \) having distinct eigenvalues \( \{ \lambda_n \} \). Then the following are equivalent:

- \( D \) admits spectral synthesis.
- The set of nonnegative integers \( k \) for which \( \sum_{n=0}^{\infty} a_n z^n \) with \( a_k \neq 0 \) is nonempty.
- For any \( c \in (0, \epsilon/(2\| f_k \|_N)) \), the distance between \( h \) and \( h + cf_k \) in \( \mathcal{H}_R \) is less than \( \epsilon/2 \).
- \( \mathcal{M}_k \) is nowhere dense in \( \mathcal{M} \).
- \( (\mathcal{M}_k)^{\circ} \) is empty for all \( k \in K \).
- \( \Theta \cap \mathcal{M} \leq \mathcal{M}_k \subseteq \mathcal{M}_k \).
(i) $D$ admits spectral synthesis,
(ii) every closed invariant subspace of $D$ is the closed linear span of $\{z^n : n \in N\}$ where $N$ is an arbitrary set of nonnegative integers,
(iii) every closed invariant subspace for $D$ (other than the empty set and $\{0\}$) contains at least one monomial $z^n$ for some $n \geq 0$,
(iv) every function $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}_R$ with $a_n \neq 0$ for all $n \geq 0$ is cyclic for $D$,
(v) there does not exist a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$.

If, in addition, $\{\lambda_n/n : n \geq 1\}$ is bounded, then $\sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ is analytic on the open ball $B(0, \epsilon)$ containing the origin whenever $\{w_n\}$ is a sequence of complex numbers for which $\limsup |w_n|^{1/n} < 1$ where $\epsilon \equiv |\ln(1/\limsup |w_n|^{1/n})|/|\sup\{|\lambda_n/n\}|$.

In this case, conditions (i)–(vi) are equivalent to
(vi) there does not exist a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ for all $z$ in the open ball $B(0, \epsilon)$.

**Proof.** The equivalence of conditions (i) and (ii) was demonstrated in the remarks preceding the lemma.

(ii) $\Rightarrow$ (iii). Let $\mathcal{M}$ be any closed invariant subspace for $D$ other than the empty set or $\{0\}$. By (ii), there exists a set $N$ of nonnegative integers for which $\mathcal{M}$ is the closed linear span of $\{z^n : n \in N\}$. Since $\mathcal{M}$ is not the empty set or $\{0\}$, $N$ is nonempty. Hence $\mathcal{M}$ contains every monomial $z^n$ with $n \in N$.

(iii) $\Rightarrow$ (ii). Let $\mathcal{M}$ be any closed invariant subspace for $D$. Define $N$ to be the set of all nonnegative integers $n$ for which $z^n$ is in $\mathcal{M}$ and define $\mathcal{M}_0$ to be the set of all functions $\sum_{n \notin N} a_n z^n$ in $\mathcal{H}_R$ for which $\sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{M}$. Since $\mathcal{M}$ is invariant for $D$ and $z^n$ is in $\mathcal{M}$ for all $n \in N$, we have that $\mathcal{M}_0$ is invariant for $D$. Hence by (iii) and the definition of $N$, we have that $\mathcal{M}_0$ is the empty set or $\{0\}$. That is, $\mathcal{M}$ equals the closed linear span of $\{z^n : n \in N\}$.

(ii) $\Rightarrow$ (iv). Let $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ be any function in $\mathcal{H}_R$ for which $a_n \neq 0$ for all $n \geq 0$. Since $a_n \neq 0$ for all $n \geq 0$, we have by (ii) that the only closed invariant subspace for $D$ containing $f$ is $\mathcal{H}_R$. That is, $f$ is cyclic for $D$.

(iv) $\Rightarrow$ (ii). Let $\mathcal{M}$ be an arbitrary closed invariant subspace for $D$ other than the empty set or $\{0\}$. Define $K$ to be the set of all nonnegative integers $k$ for which there exists a function $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{M}$ with $a_k \neq 0$. Clearly $\mathcal{M}$ is a subset of the closed linear span of $\{z^k : k \in K\}$. We show equality. By Lemma 2, there exists a function $f_1(z) \equiv \sum_{k \in K} a_k z^k$ in $\mathcal{M}$ with $a_k \neq 0$ for all $k \in K$. Define $a_n \equiv 1/n^n$ for each $n$ in $K^c$ and define $f_2(z) \equiv \sum_{n \in K^c} a_n z^n$. By (iv), $f_1 + f_2$ is cyclic for $D$. That is, the closed linear span of $\{D^k(f_1 + f_2) : k \in K\}$ is all of $\mathcal{H}_R$. Since $D^k(f_1 + f_2) = D^k(f_1) + D^k(f_2)$, $D^k(f_1)$ is in $\mathcal{M}$ and $D^k(f_2)$ is in $\mathcal{M}$, the closed linear span of $\{z^k : k \in K\}$, we have that the closed linear span of $D^k(f_1)$ must be the closed linear span of $\{z^k : k \in K\}$. Since $f_1$ is in $\mathcal{M}$, we have that $\mathcal{M}$ is the closed linear span of $\{z^k : k \in K\}$.

(iv) $\Rightarrow$ (v). By means of contradiction, assume that condition (v) fails. So there exists a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$. Since $\limsup |w_n|^{1/n} < 1$, it follows that there exists a constant $\gamma < 1$ and a constant $c$ for which $|w_n| \leq c \gamma^n$ for all $n \geq 0$. Define $a_n \equiv 1/R^n$ and $l_n \equiv w_n/a_n$ for all $n \geq 0$. Define $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$. Since $\limsup |a_n|^{1/n} = 1/R$, we have that $f(z)$ is in $\mathcal{H}_R$. Since $a_n \neq 0$ for all $n$, it follows from (iv) that $f(z)$ is a cyclic vector for $D$. That is, $\sqrt{\{D^k(f) = \sum_{n=0}^{\infty} a_n \lambda_n^k z^n : k \geq 0\}} = \mathcal{H}_R$. Since $\limsup l_n^{1/n} = R \cdot \limsup_{n=0}^{\infty} |w_n|^{1/n} <
\(\gamma R < R\), we have that \(L(\sum_{n=0}^{\infty} b_n z^n) = \sum_{n=0}^{\infty} b_n l_n\) defines a continuous linear functional on all of \(H_R\). Since the sequence of complex numbers \(\{w_n\}\) is not identically zero, \(L\) is not the zero functional on \(H_R\). However, \(L(D^k(f)) = \sum_{n=0}^{\infty} a_n l_n \lambda_n^k = \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\) and since \(\sqrt{\{D^k(f) = \sum_{n=0}^{\infty} a_n \lambda_n^k z^n: k \geq 0\}} = H_R\), we have that \(L \equiv 0\), a contradiction.

\((\nu) \Rightarrow (iv)\). Let \(f(z) = \sum_{n=0}^{\infty} a_n e^{\lambda_n z}\) be any function in \(H_R\) for which \(a_n \neq 0\) for all \(n \geq 0\). Hence \(\lim sup |a_n|^{1/n} < 1/R\). If \(f\) is not cyclic for \(D\), then there exists a nonzero functional \(L\) annihilating \(D(f)\) for all \(k \geq 0\). Since \(L\) is a functional, there exists a sequence \(\{l_n\}\) of complex numbers, not all zero, for which \(L(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} l_n c_n\) for all \(\sum_{n=0}^{\infty} c_n z^n\) in \(H_R\) where \(\lim sup |l_n|^{1/n} < R\). Hence \(0 \equiv L(D^k(f)) = L(\sum_{n=0}^{\infty} a_n \lambda_n^k z^n) = \sum_{n=0}^{\infty} l_n a_n \lambda_n^k = \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\), where \(w_n = a_n l_n\) for all \(n \geq 0\). Since \(\lim sup |l_n|^{1/n} < R\), we have that there exists a constant \(B\) in \((1/R, 1/B)\). Since \(\lim sup |a_n|^{1/n} < 1/R < K\), we have that \(|a_n| \leq K^n\) for all \(n \geq 0\) sufficiently large. Hence there exists a constant \(\beta\) for which \(|a_n| \leq \beta K^n\) for all \(n \geq 0\) and so \(\lim sup |w_n|^{1/n} \leq \lim sup |a|^{1/n} |B^n| = \gamma\) for all \(n \geq 0\) where \(\gamma = BK < 1\), contradicting (\(\nu\)).

\((\nu) \Leftrightarrow (vi)\). Let \(\{\lambda_n/n\}\) be bounded and let \(\{w_n\}\) be any sequence for which \(\lim sup |w_n|^{1/n} < 1\). Then the series \(g(k) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}\) converges uniformly absolutely on every compact subset of the open ball \(B(0, \epsilon)\) by the Root Test where \(\epsilon \equiv [\ln(1/\lim sup |w_n|^{1/n})]/[\sup |\lambda_n/n|]\). Hence \(g(k) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}\) is analytic on the open ball \(B(0, \epsilon)\). Moreover, \(g^{(k)}(0) = \sum_{n=0}^{\infty} w_n \lambda_n^k\) and so \(\sum_{n=0}^{\infty} w_n e^{\lambda_n z} = 0\) for all \(z\) in some open ball \(B(0, r)\) if and only if \(0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\). The result follows.

Regarding the extra hypothesis preceding condition (vi) of Theorem 3, it is worth noting that if \(\{\lambda_n/n: n \geq 1\}\) is not bounded, then there exists a sequence \(\{w_n\}\) of complex numbers for which \(\lim sup |w_n|^{1/n} < 1\) but for which \(\sum_{n=0}^{\infty} w_n e^{\lambda_n z}\) is not analytic on any open ball containing the origin.

Regarding condition (v) of Theorem 3, in 1921 Wolff [16] gave the first example of a sequence \(\{w_n\}\) of complex numbers, not all zero, and a sequence \(\{\lambda_n\}\) of distinct complex numbers for which \(0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\). In Wolff’s example, the sequence \(\{\lambda_n\}\) is bounded (and so \(\lim sup |\lambda_n/n|^{1/n} < 1\)) and \(\{w_n\}\) is in \(\ell^1\). In 1952, Wermer showed that the condition \(0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\) is equivalent to the operator \(D\) on a separable complex Hilbert space \(H\) diagonalizable with respect to an orthonormal basis \(\{e_n\}\) for \(H\) and satisfying \(D e_n = \lambda_n e_n\) for all \(n \geq 0\) failing to admit spectral synthesis (see [15, Theorem 1, p. 270]). In fact, much more is now known. The following result is an analogue of Theorem 3 for diagonalizable operators on a Hilbert space which helps illustrate the differences between the study of spectral synthesis for diagonal operators on the space \(H_R\) and for diagonalizable operators on a separable complex Hilbert space (see Wermer [15, Theorem 1, p. 270], Sarason [10,11], Nikol’ski [6, pp. 106–107] and [7, p. 141], Brown, Shields and Zeller [1, Theorem 3, p. 167], Sibielev [14, Propositions 1, 2 and Corollary 1]). In particular, it is the precise rate of decay of the coefficients \(\{w_n\}\) occurring in the condition that \(0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k\) for all \(k \geq 0\) that is critical. For instance, condition (v) of Theorem 3, which pertains to diagonal operators on \(H_R\), requires that \(\lim sup |w_n|^{1/n} < 1\) whereas the analogous condition (v) of Theorem 4 below, which pertains to diagonal operators on Hilbert spaces, only requires the weaker condition that \(\{w_n\}\) be in \(\ell^1\).

**Theorem 4.** Let \(H\) be a separable complex Hilbert space and let \(D\) be any bounded linear operator of \(H\) for which there exist an orthonormal basis \(\{e_n\}\) for \(H\) and a sequence \(\{\lambda_n\}\) of
complex numbers for which \( D e_n = \lambda_n e_n \) for all \( n \geq 0 \). Then \( \{\lambda_n\} \) is bounded. Moreover, \( D \) is cyclic if and only if \( \lambda_m \neq \lambda_n \) for all \( m \neq n \), and in this case, the following are equivalent:

(i) \( D \) admits spectral synthesis,

(ii) there does not exist a sequence \( \{w_n\} \) of complex numbers in \( \ell^1 \), not all zero, for which

\[
0 \equiv \sum_{n=0}^{\infty} w_n \lambda^k_n \text{ for all } k \geq 0, \tag{i}
\]

(iii) there does not exist a sequence \( \{w_n\} \) of complex numbers in \( \ell^1 \), not all zero, for which the Wolff–Denjoy series

\[
\sum_{n=0}^{\infty} \frac{w_n}{z^\lambda_n} \equiv 0 \text{ for all } z \text{ with } |z| > \sup |\lambda_n|, \tag{ii}
\]

(iv) there does not exist a sequence \( \{w_n\} \) of complex numbers in \( \ell^1 \), not all zero, for which the exponential series

\[
\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0 \text{ on the complex plane}, \tag{iii}
\]

(v) there does not exist a sequence \( \{w_n\} \) of complex numbers in \( \ell^1 \), not all zero, for which the exponential series

\[
\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0 \text{ on the complex plane}. \tag{iv}
\]

(vi) every closed invariant subspace of \( D \) is invariant for the adjoint \( D^* \) of \( D \), and

(vii) the adjoint \( D^* \) of \( D \) is in the weakly closed algebra generated by the identity operator and \( D \).

If, in addition, the \( \lambda_n \) lie inside a Jordan region \( G \) and accumulate only on the boundary of \( G \), then conditions (i)–(vii) are equivalent to

\[
\sup\{|f(z)| : z \in G\} = \sup\{|f(\lambda_n)| : n \geq 0\}. \tag{viii}
\]

The study of Wolff–Denjoy series has a long and rich history. Of particular interest has been conditions for an analytic function to be representable as a Wolff–Denjoy series, and conditions for such a representation, if one exists, to be unique. Borel, Beurling, and Carleman all gave sufficient conditions for the representation of an analytic function as a Wolff–Denjoy series to be unique in terms of the rate of decay of the coefficients in the representing series. Sibilev in 1995 gave a definitive uniqueness theorem of this type (see Sibilev [14]). Wolff–Denjoy series have also been studied extensively by Poincaré, Wolff, Borel, Carleman, and Beurling, amongst others, mainly in connection with quasianalyticity and analytic continuation (see the recent monograph of Ross and Shapiro [8]).

Wolff’s example of a nontrivial sequence \( \{w_n\} \) in \( \ell^1 \) and bounded sequence of distinct complex numbers \( \{\lambda_n\} \) for which

\[
0 = \sum_{n=0}^{\infty} w_n \lambda^k_n \text{ for all } k \geq 0 \text{ has been extended to sequences } \{\lambda_n\} \text{ of distinct complex numbers which are unbounded. For instance, in 1936, Natanson showed that there exists a sequence } \{w_n\} \text{ of complex numbers for which } \sum_{n=0}^{\infty} |w_n| |\lambda_n|^k < \infty \text{ and } 0 = \sum_{n=0}^{\infty} w_n \lambda^k_n \text{ for all } k \geq 0 \text{ in the special case } \lambda_n = n \text{ for all } n \geq 0 \text{ (see 5.7.8c(v) on p. 128 of Nikol’skii [6]). In 1959, Makarov generalized Natanson’s example to include any sequence } \{\lambda_n\} \text{ of complex numbers for which } |\lambda_n| \to \infty \text{ (see 5.7.8c(vi) on p. 128 of Nikol’skii [6]).}

However, we will see as consequences of Corollary 1 and Theorem 5 below that the coefficients \( \{w_n\} \) which occur in Wolff’s example and in Natanson’s example fail to satisfy the condition that \( \limsup |w_n|^{1/n} < 1 \). In fact, it remains an open question as to whether or not every cyclic diagonal operator on \( \mathcal{H}_R \) admits spectral synthesis. That is, it is not known if there exists a sequence \( \{\lambda_n\} \) of distinct complex numbers for which \( \limsup |\lambda_n|^{1/n} < 1 \) and a nontrivial sequence \( \{w_n\} \) of complex numbers for which \( \limsup |w_n|^{1/n} < 1 \) with \( 0 \equiv \sum_{n=0}^{\infty} w_n \lambda^k_n \) for all \( k \geq 0 \).

We show, however, that every cyclic diagonal operator \( D \) on \( \mathcal{H}_R \) whose eigenvalues \( \{\lambda_n\} \) are bounded admits spectral synthesis using the uniqueness result of Sibilev mentioned earlier.
Corollary 1. Every cyclic diagonal operator $D$ on $\mathcal{H}_R$ whose eigenvalues $\{\lambda_n\}$ are bounded admits spectral synthesis.

Proof. By means of contradiction, assume that $D$ is a cyclic diagonal operator on $\mathcal{H}_R$ whose eigenvalues $\{\lambda_n\}$ are bounded but which fails spectral synthesis. Without loss of generality, we may assume that $|\lambda_n| < 1$ for all $n \geq 0$. By Theorem 3(v), there exists a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\lim \sup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$. In particular, $\{w_n\}$ is in $\ell^1$ and so $g(z) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ is an entire function. Since $0 = \sum_{n=0}^{\infty} w_n \lambda_n^k = g^{(k)}(0)$ for all $k \geq 0$, we have that $g(z) \equiv 0$ for all complex numbers $z$. Hence by Proposition 2 of Sibilev [14, p. 147], $0 \equiv \sum_{n=0}^{\infty} w_n / (z - \lambda_n)$ whenever $|z| > 1$. Since $\sum (\ln(c^{\gamma_n})) / n^2 = -\infty$, we have that $w_n \equiv 0$ for all $n \geq 0$ by the theorem on p. 146 of Sibilev [14], a contradiction. $\square$

It follows from Corollary 1 that there exist cyclic diagonal operators on $\mathcal{H}_R$ admitting spectral synthesis the closure of whose eigenvalues $\{\lambda_n\}$ have nonempty interior. This is in contrast to the case for diagonalizable operators on a separable complex Hilbert space (see Scroggs [12, Corollary 3.1, p. 104]).

It also follows from Corollary 1 that the coefficients $\{w_n\}$ in Wolff’s example do not satisfy the condition that $\lim \sup |w_n|^{1/n} < 1$.

We conclude this paper with an application of Leontev’s work (see [3] and [4]) on (the uniqueness of) representations of analytic functions as exponential series $\sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ which shows that there exist diagonal operators on $\mathcal{H}_R$ admitting spectral synthesis whose eigenvalues are unbounded.

Theorem 5. Let $D$ be any diagonal operator on $\mathcal{H}_R$ having eigenvalues $\{\lambda_n\}$. If $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{n \to \infty} \lambda_n = \infty$, and $\lim \sup_{n \to \infty} \lambda_n / n < \infty$, then $D$ admits spectral synthesis.

Proof. By Theorem 3(iv), we need only show that every function $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}_R$ for which $a_n \neq 0$ for all $n \geq 0$ is a cyclic vector for $D$. But this follows directly from the work of Leontev (see [3] or [5, Theorem 2, p. 7]). $\square$

It follows from Theorem 5 that the coefficients $\{w_n\}$ in Natanson’s example do not satisfy the condition that $\lim \sup |w_n|^{1/n} < 1$.

References