# Galois structure and de Rham invariants of elliptic curves

Darren Glass [a],[*], Sonin Kwon [b]

[a] *Department of Mathematics, Gettysburg College, 200 N. Washington Street, Gettysburg, PA 17325, United States*
[b] *Babson Capital Management LLC, Springfield, MA 01115-15189, United States*

## ARTICLE INFO

## ABSTRACT

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Suppose a finite group $G$ acts numerically tamely on a regular scheme $\mathcal{X}$ over $\mathcal{O}_K$. One can then define a de Rham invariant class in the class group $\mathrm{Cl}(\mathcal{O}_K[G])$, which is a refined Euler characteristic of the de Rham complex of $\mathcal{X}$. Our results concern the classification of numerically tame actions and the de Rham invariant classes. We first describe how all Galois étale $G$-covers of a $K$-variety may be built up from finite Galois extensions of $K$ and from geometric covers. When $X$ is a curve of positive genus, we show that a given étale action of $G$ on $X$ extends to a numerically tame action on a regular model if and only if this is possible on the minimal model. Finally, we characterize the classes in $\mathrm{Cl}(\mathcal{O}_K[G])$ which are realizable as the de Rham invariants for minimal models of elliptic curves when $G$ has prime order.

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## 1. Introduction

Suppose $K$ is a number field with ring of integers $\mathcal{O}_K$, and $G$ is a finite group. Much work has been done by number theorists to try to understand the possible extensions $L$ of $K$ which are tamely ramified with Galois group $G$, both in terms of their structure and in terms of the possible Galois module structures of the ring of integers $\mathcal{O}_L$ when viewed as an element of the class group $\mathrm{Cl}(\mathcal{O}_K G)$. The situation becomes even more complicated when one tries to formulate geometric analogues of these results, where it is not even clear what one should mean by a tame action.

The concept of numerical tameness is one attempt to generalize the theory of tame ramification of rings of integers to the setting of coverings of schemes, and was first introduced by Chinburg and Erez in [CE92]. In particular, let $\mathcal{X}$ be a connected regular scheme which is proper, flat, and of finite

type over $\mathcal{O}_K$. An action of $G$ on $\mathcal{X}$ over $\mathcal{O}_K$ is defined to be numerically tame if the inertia subgroup in $G$ of each point $x \in \mathcal{X}$ is of order relatively prime to the residue characteristic of $x$. It was in this setting that Chinburg, Erez, Pappas, and Taylor originally defined de Rham invariants $\chi(\mathcal{X}, G)$ (see Section 4.1 for details) which they decompose into the sum of a root number class and a ramification class [CEPT97]. In recent years numerical tameness has proven to be a useful setting in which to work for a variety of applications such as computing $\epsilon$-constants [Gla04] and constructing a singular homology theory for arithmetic schemes [Sch07].

Given a numerically tame action of a group $G$ on $\mathcal{X}$, we may define $G$-equivariant Euler characteristics for coherent sheaves of $\mathcal{X}$ in the locally free class group $\mathrm{Cl}(\mathcal{O}_K[G])$ (see [Chi94,CE92] for details). The main goals of this paper are to classify numerically tame actions of $G$ and to compute the $G$-equivariant Euler characteristic of the de Rham complex on $\mathcal{X}$ under various hypotheses on $G$ and on $\mathcal{X}$. As one application, we prove a result concerning which classes in the locally free class group $\mathrm{Cl}(\mathcal{O}_K[G])$ are equal to the de Rham invariant studied in [CEPT97] for a numerically tame $G$ action on the minimal model of an elliptic curve. We also consider geometric counterparts of realizability results concerning the relative structure of the stable isomorphism class of the ring of integers in a tamely ramified Galois extension of a number field.

We will call $\mathcal{X}$ a regular model for its generic fiber $X$. It is well known that if the action of $G$ on $\mathcal{X}$ is numerically tame, then the quotient morphism on the generic fiber $X \rightarrow Y = X/G$ is étale. Thus a first step towards classifying $\mathcal{X}$ with a numerically tame action of $G$ is to classify the way finite groups occur as Galois groups of étale coverings of $K$-varieties $Y$. We do this when $Y$ is a smooth projective variety over an arbitrary field $K$ for which $Y$ has a point defined over $K$. Our main result in Section 2, Theorem 2.7, describes how all étale covers of such $Y$ may be built up from Galois extensions of $K$ and geometric étale coverings of $Y$. To illustrate this result, Corollary 2.10 gives a classification of Galois étale coverings of Abelian varieties with Galois group isomorphic to the quaternion group $\mathbb{H}_8$ of order 8. This classification groups such covers into four families according to the interaction between the "arithmetic" and "geometric" Galois groups. This was in part motivated by Fröhlich's classification of Galois extensions of the rational numbers $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{H}_8$ [Frö72].

In Section 3, we assume that $K$ is a number field and that $X$ is a smooth projective $K$-variety with an étale action of a finite group $G$. The next step in our classification problem is to categorize the regular models $\mathcal{X}$ of $X$ having a numerically tame action of $G$ which induces the given action of $G$ on $X$. The existence of a regular model is not known in general. We now specialize to the case in which $X$ is a curve of positive genus. Among all extensions of $X$ to a regular scheme over $\mathcal{O}_K$ there exists a "best" one, the minimal model $\mathcal{X}^{\min}$ of $X$. Proposition 3.3 shows that an action of $G$ on $X$ extends to $\mathcal{X}^{\min}$, and Theorem 3.5 shows that the existence of a regular model of $X$ with a numerically tame action depends only on the action on this minimal model. When $\mathcal{X}^{\min}$ is the minimal model of an elliptic curve $X$, Theorem 3.7 describes which groups of torsion points of $X$ give rise to a numerically tame action. In particular, we show that it is necessary to have good or multiplicative reduction at the places of $\mathcal{O}_K$ whose residue characteristic divides the order of $G$.

The last section of this paper concerns the Galois module structure of the de Rham invariant $\chi(\mathcal{X}, G)$ for a numerically tame action of $G$ on $\mathcal{X}$, which is the projective $G$-equivariant Euler characteristic of differential sheaves on $\mathcal{X}$ in the locally free class group $\mathrm{Cl}(\mathcal{O}_K[G])$. It has been studied in [CEPT97] as a generalization of the stable isomorphism class of the ring of integers $\mathcal{O}_L$ in a tamely ramified $G$-extension $L$ of $K$. In Section 4, we specialize to the case when the minimal model $\mathcal{X}^{\min}$ of an elliptic curve $X$ has at most multiplicative reduction. In this case, we show that $\chi(\mathcal{X}^{\min}, G)$ is the sum of the Euler characteristics of the structure sheaves on the normalizations of the special fibers at the places of bad reduction. To prove this, we use the Lefschetz–Riemann–Roch theorem proven by Baum et al. [BFQ79] to compute the values of the Brauer traces at each $p$-regular element, where $p$ is the residue characteristic of a place of bad reduction.

Suppose finally that $G$ has prime order $\ell$. We give in this case an explicit formula for $\chi(\mathcal{X}^{\min}, G)$ which can be viewed as a geometric counterpart of a realizability result of McCulloh [McC87] concerning the stable isomorphism class of the ring of integers in a tamely ramified $G$-extension of $K$.

We end the introduction by discussing some further problems. The general goal is to describe the subset $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$ of the de Rham invariants in $\mathrm{Cl}(\mathcal{O}_K[G])$. In order to do this one would like

to use Riemann-Roch type theorems to show $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$ is contained in some explicit subset $R_0 \subseteq \mathrm{Cl}(\mathcal{O}_K[G])$ and then show that each class in $R_0$ actually lies in $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$. To do this, one would construct a desired cover locally over each place of $\mathcal{O}_K$, and try to see if it can be a special fiber of some numerically tame cover of a regular model. Results of Raynaud, Harbater, Saidi and others use rigid analytic geometry to construct Galois covers of curves over fields and Dedekind rings (see, for example, [Har94,Ray94,Saï97]). It would be interesting to study how these techniques may be applied to identify classes in $\mathrm{Cl}(\mathcal{O}_K[G])$ which are the de Rham invariants of some cover. Remark 3.8 suggests studying regular models of higher dimensional varieties which have numerically tame actions of finite groups. Abelian varieties and modular varieties are two natural examples to consider, for which regular models are often known. In view of Nakajima's work on Galois module structure of cohomology groups of algebraic varieties over a field (see [Nak84,Nak86]), it would be interesting to formulate an obstruction for a finite group to occur as a Galois group of numerically tame coverings of schemes over a ring of integers.

## 2. Galois $G$-covers of $K$-varieties

Let $Y$ be a normal variety over a field $K$. The main goal of this section is to understand the finite quotients of the algebraic fundamental group of $Y$. We will begin by recalling the definition of a Galois $G$-cover from [SGA03, Chapters I and V]. In what follows we will assume that a finite group $G$ acts admissibly on $X$. This means that there is a quotient morphism $f : X \to Y = X/G$. If $f$ is also finite, we will say that $f$ is a $G$-covering. For any scheme $Z$, we denote by $G_Z$ the *constant $Z$-group scheme* $G_Z = \bigsqcup g \in G Z_g$ where for all $g \in G$, $Z_g$ is isomorphic to $Z$. The group scheme structure of $G_Z$ is induced by the identity maps $Z_g \times_Z Z_h \to Z_{gh}$ for $g, h \in G$.

**Definition 2.1.** A $G$-covering $f : X \to Y$ is Galois if $X$ is a $G$-torsor over $Y$, in the sense that $X$ is faithfully flat over $Y$ and the map $(x, g) \mapsto (x, xg)$ defines an isomorphism $X \times_Y G_Y \cong X \times_Y X$.

**Remark 2.2.** A $G$-covering is Galois if and only if its inertia groups are trivial [SGA03, V.2.6]. In particular, Galois $G$-coverings are étale.

We will make the following hypothesis for the rest of this section.

**Hypothesis 2.3.** *Let $K$ be a field of characteristic $p \geqslant 0$ and let $Y$ be a normal $K$-variety with a $K$-point $\infty$.*

**Lemma 2.4.** *Denote a separable closure of $K$ by $K^s$, and denote $Y \times_K K^s$ by $\bar{Y}$. Then under Hypothesis 2.3, $\bar{Y}$ is a normal variety over $K^s$.*

**Proof.** Since there is a unique $K^s$-point of $\bar{Y}$ over $\infty$, $\bar{Y}$ must be connected. Because $Y$ is normal, $\bar{Y}$ is normal by [Mil80, Chapter I, Proposition 3.17]. For every open subset $\mathrm{Spec}\, R \subset Y$, we have an injection $0 \to R \to K(Y)$. Since $K^s$ is flat over $K$, we get $0 \to R \otimes_K K^s \to K(Y) \otimes_K K^s$. By [ZS58, Corollary 2, p. 198], $K(Y) \otimes_K K^s$ is an integral domain. Thus $\bar{Y}$ is irreducible and reduced, so $\bar{Y}$ is a variety over $K^s$. $\quad\square$

Let $y$ be the generic point of $\bar{Y}$. Define a geometric point $\bar{y} = \mathrm{Spec}\, K^s(\bar{Y})^s$, where $K^s(\bar{Y})^s$ is a separable closure of the function field $K^s(\bar{Y})$ of $\bar{Y}$. Let $\Omega$ be the union of all finite extensions $K'$ of $K(Y)$ which are contained in $K^s(\bar{Y})^s$ such that the normalization of $Y$ in $K'$ is étale over $Y$. Then $\Omega$ contains $K^s(\bar{Y})$. Since $K$ is algebraically closed in $K(Y)$, we have a natural exact sequence

$$1 \to \mathrm{Gal}\big(\Omega/K^s(\bar{Y})\big) \to \mathrm{Gal}\big(\Omega/K(Y)\big) \xrightarrow{\pi} \mathrm{Gal}\big(K^s/K\big) \to 1 \tag{1}$$

in which $\pi$ is the restriction of automorphisms from $\Omega$ to $K^s$. By [Mil80, Remark 5.1(b), p. 41] and [SGA03, IX, 6.1, 6.4], the sequence (1) is naturally identified with the sequence of algebraic fundamental groups

$$1 \rightarrow \pi_1(\bar{Y}, \bar{y}) \rightarrow \pi_1(Y, \bar{y}) \rightarrow \text{Gal}(K^s/K) \rightarrow 1. \tag{2}$$

The following lemma is clear from the fact that we may view $\infty$ as a discrete valuation of $K(Y)$ which has residue field $K$.

**Lemma 2.5.** *Let $\bar{\infty}$ be a discrete valuation of $\Omega$ which extends $\infty$, and let $G_{\bar{\infty}} \subseteq \text{Gal}(\Omega/K(Y))$ be the decomposition group of $\bar{\infty}$. Restricting automorphisms from $\Omega$ to $K^s$ induces an isomorphism $t : G_{\bar{\infty}} \rightarrow \text{Gal}(K^s/K)$ whose inverse $s : \text{Gal}(K^s/K) \rightarrow G_{\bar{\infty}} \subseteq \text{Gal}(\Omega/K(Y))$ is a section of the homomorphism $\pi$ in the sequence* (1). *Thus* (1) *is split exact.*

Given a finite group $G$, we will now show how to construct all irreducible Galois $G$-covers of $Y$ from Galois extensions of $K$ and Galois coverings of $\bar{Y}$.

**Construction 2.6.** *Choose a normal subgroup $H_1$ and a subgroup $H_2$ of $G$ which together generate $G$ with the following properties*:

(a) *There is a Galois extension $L/K$ of fields with Galois group $H_2$.*
(b) *There is a $K$-variety $Y'$ such that $\bar{Y}'$ is an irreducible Galois $H_1$-cover of $\bar{Y}$. Thus $H_1 = \text{Gal}(K^s(Y')/K^s(Y))$.*
(c) *The function field $L(Y')$ of the irreducible variety $Y' \times_K L$ is Galois over $K(Y)$ with Galois group $H_1 \rtimes H_2$, where $H_1$ acts on $L(Y')$ via its action on $K^s(Y')$.*

*Define an irreducible Galois $G$-cover $X$ to be the normalization of $Y$ in $L(Y')^H$, where $H$ is the normal subgroup $\{(g, g^{-1}) \in H_1 \rtimes H_2 \mid g \in H_1 \cap H_2\}$ of $H_1 \rtimes H_2$.*
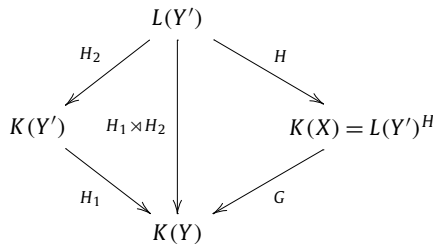


The following proposition shows that we obtain all irreducible Galois G-covers in this way.

**Theorem 2.7.** *An irreducible Galois $G$-cover $X$ of $Y$ is a quotient of an irreducible variety $Y' \times_K L$ for some Galois extension $L/K$ of fields and a $K$-variety $Y'$ as in Construction 2.6. The function field $K(X)$ of $X$ is an intermediate field $L(Y')^H$ of $L(Y')$ over $K(Y)$ and the corresponding homomorphism $\rho : H_1 \rtimes H_2 \twoheadrightarrow G$ is an injection on both $H_1$ and $H_2$.*

**Proof.** By Lemma 2.5, the Galois group $\text{Gal}(\Omega/K(Y))$ is the semi-direct product $\Gamma = \text{Gal}(\Omega/K^s(\bar{Y})) \rtimes G_{\bar{\infty}}$, where $G_{\bar{\infty}}$ is isomorphic to $\text{Gal}(K^s/K)$. The irreducible Galois $G$-cover $X$ of $Y$ is the normalization of $Y$ in $\Omega^{\ker(\rho)}$, for some surjective homomorphism $\rho : \Gamma \rightarrow G$.

Let $I$ (resp. $J$) be the kernel of the restriction of $\rho$ to $\text{Gal}(\Omega/K^s(\bar{Y}))$ (resp. $G_{\bar{\infty}}$). The extension $L = (K^s)^J$ is a Galois extension of $K$ with group $H_2 = G_{\bar{\infty}}/J$. The group $G_{\bar{\infty}}$ acts by conjugation on $I$. Let $Y'$ be the normalization of $Y$ in $\Omega^{\Gamma_I}$, where $\Gamma_I = I \rtimes G_{\bar{\infty}}$. The point of $Y'$ determined by $\bar{\infty}$ has residue field $K$. Therefore $\bar{Y}'$ is an irreducible Galois cover of $\bar{Y}$ with Galois group $H_1 = \text{Gal}(\Omega/K^s(\bar{Y}))/I$.

The group $I \rtimes J$ is the kernel of the restriction of $\rho$ to the normal subgroup $\mathrm{Gal}(\Omega/K^s(\bar{Y})) \rtimes J$ of $\Gamma$. Hence $I \rtimes J$ is normal in $\Gamma$, and $\Omega^{I \rtimes J}$ is the function field $L(Y')$ of $Y' \times_K L$. Thus $L(Y')$ is Galois over $K(Y)$ with group $\Gamma/(I \rtimes J) = H_1 \rtimes H_2$. Since $I \rtimes J$ is in the kernel of $\rho$, it acts trivially on the function field $K(X)$ of $X$. Thus $K(X)$ is a subfield of $L(Y')$, and we may view $\rho$ as a surjection $\rho : H_1 \rtimes H_2 \twoheadrightarrow G$. Both $H_1$ and $H_2$ inject into $G$ under $\rho$, so their images correspond to subgroups which together generate $G$. Since $H$ is the kernel of $\rho$, we see that the function field $K(X)$ of $X$ is $L(Y')^H$. $\quad\square$

**Remark 2.8.** We note the following facts about the relevant function fields:

(i) The function field of $X$ and the function field of $Y'$ are isomorphic over $L$ (i.e. $L(Y') = L \cdot K(Y) \cdot K(X) = L(X)$).
(ii) Let $F$ be the subfield of $L$ fixed by the subgroup $H_1 \cap H_2$ of $H_2 = \mathrm{Gal}(L/K)$. The extension $K(X)/K(Y)$ has an intermediate field $F(Y)$, where $F$ is the field of constants of $X$ (i.e. $F = K^s \cap K(X)$).

**Corollary 2.9.** *Suppose that $\pi_1(\bar{Y}, \bar{y})$ has no nontrivial quotient group isomorphic to a normal subgroup of $G$. Then every Galois $G$-cover of $Y$ has the form $Y \times_K L$ for some Galois $G$-extension $L/K$ of fields. In particular, the condition holds if $G$ is a non-Abelian simple group and if $Y$ is an Abelian variety, or more generally if $\pi_1(\bar{Y}, \bar{y})$ is solvable.*

**Proof.** The group $H_1$ appearing in Theorem 2.7 must be trivial, so Corollary 2.9 follows from Theorem 2.7. $\quad\square$

Fröhlich [Frö72] and others have considered the problem of classifying and constructing the Galois field extensions $L$ of a given field $K$ for which $\mathrm{Gal}(L/K)$ is isomorphic to the quaternion group $\mathbb{H}_8$ of order 8. Such $L/K$ will be called quaternion extensions. As an example of Theorem 2.7, we classify the irreducible Galois $\mathbb{H}_8$-covers of a given Abelian variety $Y$ over $K$. Write $\mathbb{H}_8 = \langle \sigma, \tau \rangle$ with relations $\tau^4 = 1$, $\sigma^2 = \tau^2$, $\sigma\tau\sigma^{-1} = \tau^{-1}$. Denote by $C_2$ the center of $\mathbb{H}_8$ so that $C_2 \cong \mathbb{Z}/2\mathbb{Z}$. The possible combinations of subgroups $H_1$ and $H_2$ as in Theorem 2.7 are pairs $(H_1, H_2) = (1, \mathbb{H}_8)$, $(C_2, \mathbb{H}_8)$, $(C_4, \mathbb{H}_8)$, or $(C_4, C_4')$, where $C_4$ and $C_4'$ are distinct subgroups of $\mathbb{H}_8$ of order four.

**Example 2.10.** Any irreducible Galois $\mathbb{H}_8$-cover of a given Abelian variety $Y$ over $K$ comes from one of the following four cases.

(i) Constant field extension: $(H_1, H_2) = (1, \mathbb{H}_8)$.
   Given the function field $K(Y)$ of an Abelian variety $Y$ over $K$, we extend its constant field $K$ to a quaternion extension $L/K$. This gives $L \cdot K(Y) = L(Y)$ as the function field of an irreducible Galois $\mathbb{H}_8$-cover $X$ of $Y$.
(ii) Quadratic twist of a quaternion field: $(H_1, H_2) = (C_2, \mathbb{H}_8)$.
   Since $K(Y')/K(Y)$ is an extension of fields of degree 2 in this case, it is Galois. Let $L/K$ be a quaternion extension. Then the group $\mathrm{Gal}(L(Y')/K(Y))$ is isomorphic to $C_2 \times \mathbb{H}_8$ and there is a unique Galois $\mathbb{H}_8$-extension of $K(Y)$ in $L(Y')$ other than $L(Y)$. It is the subfield fixed by the subgroup of $C_2 \times \mathbb{H}_8$ generated by $(-1, \tau^2)$. In view of Remark 2.8, the field of constants of $X$ is a biquadratic subextension of a quaternion extension $L/K$. The tower of field extensions $K(X)/F(Y)/K(Y)$ corresponds to a central extension $1 \to \mathbb{Z}/2 \to \mathbb{H}_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$.
(iii) Quartic twist of a quaternion field: $(H_1, H_2) = (C_4, \mathbb{H}_8)$.
   Let $L/K$ be a quaternion extension. In this case, $K(Y')$ is a degree four field extension of $K(Y)$. The group $\mathrm{Gal}(L(Y')/K(Y))$ is isomorphic to $C_4 \rtimes \mathbb{H}_8$, where $\mathbb{H}_8$ acts on the group $C_4$ by conjugation. Let $\tau$ be a generator of $C_4$. Let $H$ be the cyclic subgroup of $C_4 \rtimes \mathbb{H}_8$ generated by $(\tau, \tau^{-1})$. Then $K(X) = L(Y')^H$ is a Galois $\mathbb{H}_8$ extension of $K(Y)$. Note that the fixed field of $C_4 \subseteq \mathbb{H}_8 = \mathrm{Gal}(L/K)$ is a quadratic extension $F/K$ such that $F(Y')/F(Y)$ is a cyclic Galois extension of degree four whose Galois group is identified with $(C_4 \rtimes C_4)/1 \rtimes C_4$.

(iv) Quartic twist of a cyclic quartic field: $(H_1, H_2) = (C_4, C_4')$.

Let $L/K$ be a cyclic Galois extension of degree four. As in case (iii), $K(Y')$ is a degree four field extension of $K(Y)$. The group $\mathrm{Gal}(L(Y')/K(Y))$ is now isomorphic to $C_4 \rtimes C_4'$, where $C_4'$ acts on the group $C_4$ by conjugation. There is a unique Galois $\mathbb{H}_8$-extension of $K(Y)$ in $L(Y')$. It is the subfield fixed by the subgroup of $C_4 \rtimes C_4'$ generated by $(\tau^2, \tau^2)$. Note that the fixed field of $\tau^2 \subseteq C_4' = \mathrm{Gal}(L/K)$ is a quadratic extension $F/K$ such that $F(Y')/F(Y)$ is a cyclic Galois extension of degree four whose Galois group is identified with $(C_4 \rtimes C_2)/1 \rtimes C_2$.

In response to a question of H. Darmon, we will discuss one further example.

**Example 2.11.** Suppose that $p$ is an odd prime and that $G$ is the Heisenberg group of all upper triangular unipotent matrices in $GL_3(\mathbb{Z}/p)$. Denote by $Z(G)$ ($\cong \mathbb{Z}/p$) the center of $G$. Let $C_p$ and $C_p'$ be two distinct subgroups of $G$ of order $p$. Let $X \to Y$ be an irreducible Galois $G$-covering. Suppose that the function field $K(X)$ of $X$ is not a constant field $G$-extension of $K(Y)$, and that $K$ is not algebraically closed in $K(X)$. Then in view of Theorem 2.7, the possible combinations for $H_1 \rtimes H_2$ are:

  (i) $Z(G) \times G$,
 (ii) $(Z(G) \times C_p) \rtimes G$,
(iii) $(Z(G) \times C_p) \rtimes (Z(G) \times C_p')$, or
(iv) $(Z(G) \times C_p) \rtimes C_p'$ ($\cong G$).

Suppose now that $Y$ is an Abelian variety over $K$. The irreducible $H_1$-covering $\bar{Y}' \to \bar{Y}$ is an isogeny of Abelian varieties, and its kernel is a subgroup $\mathbb{Z}/p \times \mathbb{Z}/n$, $n = 1$ or $p$ of the group of $p$-torsion points which is identified with $H_1$. Note that if n=p and $Y$ is an elliptic curve, then $\bar{Y}'$ is isomorphic to $\bar{Y}$ and the above isogeny is just multiplication by $p$ map on $\bar{Y}$.

The subgroup $\mathbb{Z}/p \times \mathbb{Z}/n$ of $p$-torsion points on $\bar{Y}'$ is defined over $H_2$-extension $L$ of $K$. The action of $H_2 = \mathrm{Gal}(L/K)$ on $\mathbb{Z}/p \times \mathbb{Z}/n$ corresponds to the conjugation action of $H_2$ on $H_1$. Since the subgroup $\mathbb{Z}/p \subseteq \mathbb{Z}/p \times \mathbb{Z}/n$ is identified with the center $Z(G)$ of $G$, it is defined over $K$. When n=p, the subgroup $\mathbb{Z}/n \subseteq \mathbb{Z}/p \times \mathbb{Z}/n$ is defined over the subfield $F$ of $L$, where $\mathrm{Gal}(F/K) \cong \mathbb{Z}/p$ (cf. Remark 2.8). We can choose generators of $\mathbb{Z}/p = \langle a \rangle$ and $\mathbb{Z}/n = \langle b \rangle$ so that $\mathrm{Gal}(F/K)$ acts on $\langle b \rangle$ as the translation by the point $a$ [Sil94, Chapter V, Proposition 6.1].

In this example, the irreducible Galois $G$-cover in case (i) corresponds to a central extension $1 \to \mathbb{Z}/p \to G \to \mathbb{Z}/p \times \mathbb{Z}/p \to 1$.

## 3. Numerically tame covers of regular models

Suppose $f_K : X \to Y$ is an irreducible Galois $G$-covering of smooth projective varieties over a number field $K$. Let $\mathcal{X}$ be a connected regular scheme which is proper, flat, and of finite type over the ring $\mathcal{O}_K$ of integers of $K$. We will call $\mathcal{X}$ a regular model of $X$ if the generic fiber $\mathcal{X} \times_{\mathcal{O}_K} K$ of $\mathcal{X}$ is isomorphic to $X$. Recall from the introduction that an action of a finite group $G$ is said to be numerically tame on $\mathcal{X}$ if the order of the inertia group in $G$ of each point $x \in \mathcal{X}$ is relatively prime to the residue characteristic of $x$.

In this section, we address the following question. What are necessary and/or sufficient conditions for the existence of a numerically tame $G$-covering $f : \mathcal{X} \to \mathcal{Y}$ over the ring of integers $\mathcal{O}_K$ which restricts to $f_K$ on the generic fiber?

**Remark 3.1.** We note that it follows from [CE92, Theorem A.2] that a $G$-covering $\mathcal{X} \to \mathcal{Y}$ of normal schemes which is tamely ramified in codimension 1 with respect to some divisor with normal crossings is numerically tame. Moreover, it follows from the definition that numerical tameness is a local condition and can be detected by an étale base change.

**Proposition 3.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a numerically tame $G$-covering over* $\operatorname{Spec} \mathcal{O}_K$. *Then $f$ induces an (étale) Galois $G$-covering $\mathcal{X} \times_{\mathcal{O}_K} K \to \mathcal{Y} \times_{\mathcal{O}_K} K$ on the generic fiber.*

**Proof.** Suppose that there is a point $x \in \mathcal{X} \times_{\mathcal{O}_K} K$ with the nontrivial inertia group $I_x$. Let $p$ be a prime number which divides the order of $I_x$. Let $v$ be a place of $\mathcal{O}_K$ over $p$ and denote by $\mathcal{X}_v$ the special fiber of $\mathcal{X}$ over $v$. Since $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ is proper, there is a point $x'$ of $\mathcal{X}_v$ lying on the Zariski closure of $x$ in $\mathcal{X}$. We have an inclusion of local rings $\mathcal{O}_{\mathcal{X},x} \subset \mathcal{O}_{\mathcal{X},x'}$, so an inclusion of inertia groups $I_x \subset I_{x'}$. Since $p$ divides the order of the group $I_x$, it divides the order of $I_{x'}$. The fact that the residue characteristic of $x'$ is $p$ contradicts our hypothesis that $f : \mathcal{X} \to \mathcal{Y}$ is numerically tame.　□

The next two propositions show that if a group acts on a smooth projective curve of genus $g > 0$ then the action extends to the minimal model of the curve, and further that if the action of the group on the minimal model is numerically tame then the original action was as well.

**Proposition 3.3.** *Suppose $Y$ is a smooth projective curve over a number field $K$ of positive genus on which a finite group $G$ acts. This action extends to an action of $G$ on the minimal model $\mathcal{Y}$ of $Y$ over $R$.*

We begin by noting that the quotient curve $V = Y/G$ is normal, and hence smooth over $K$. Let $\mathcal{V}$ be a regular model of $V$. Then $G$ acts on the normalization $\mathcal{Y}_0$ of $\mathcal{V}$ in the function field $K(Y)$, since $K(Y)/K(V)$ is a Galois $G$-extension. We now use Lipman's desingularization process. Beginning with $n = 0$, suppose that $\mathcal{Y}_n$ is a normal curve over $R$ with generic fiber $Y$; we will suppose in addition that there is an action of $G$ on $\mathcal{Y}_n$ which extends the action of $G$ on $Y$. We now define $\mathcal{Y}_{n+1}$ to be the normalization of the blow-up of $\mathcal{Y}_n$ at the reduced singular locus of $\mathcal{Y}_n$. Since the reduced singular locus is $G$-equivariant, we see that $G$ acts on $\mathcal{Y}_{n+1}$. Lipman proves that this process stops, in the sense that $\mathcal{Y}_m$ is regular for large enough $m$ (see [Lip69] for details). We have thus obtained a regular curve $\mathcal{Y}_m$ over $R$ with these properties:

(a) The generic fiber of $\mathcal{Y}_m$ is $Y$.
(b) The action of $G$ on $Y$ extends to an action of $G$ on $\mathcal{Y}_m$, and
(c) There is a proper birational $G$-equivariant morphism $\mathcal{Y}_m \to \mathcal{Y}_0$ which is an isomorphism on the generic fiber $Y$ of $\mathcal{Y}_m$.

We now need the following lemma. Recall that an exceptional divisor $E$ on a regular projective curve $\mathcal{Z}$ over $R$ is an irreducible fibral divisor isomorphic to $P^1_{k'}$ over $k' = H^0(E, \mathcal{O}_E)$ and such that the self intersection $E^{(2)}$ with respect to $k'$ is $-1$. Here $k' = k$ since $k'$ is a finite extension of $k$ and $k$ is algebraically closed.

**Lemma 3.4.** *Suppose $\mathcal{Z}$ is a regular flat projective curve over $R$ such that the generic fiber $Z$ of $\mathcal{Z}$ is an (irreducible) smooth curve over $K$ of positive genus. Then no two exceptional curves on $\mathcal{Z}$ intersect.*

Before proving this lemma, we note that we can use it along with the Castelnuovo criterion to complete the proof of Proposition 3.3. In particular, elements of $G$ permute the exceptional curves on $\mathcal{Y}_m$, and by Lemma 3.4, none of these exceptional curves intersect. Therefore, blowing down an exceptional curve sends a different exceptional curve to an exceptional curve on the image of the blow-down. Thus blowing down all the exceptional curves on $\mathcal{Y}_m$ at once gives the same result as blowing them down one at a time. However, because the exceptional curves are permuted by $G$, blowing them all down at once gives a $G$-equivariant morphism. If we replace $\mathcal{Y}_m$ by the image of the blow-down and repeat this process, then by the minimal models theorem, after finitely many steps, there will be no more exceptional curves and we will have the minimal model $\mathcal{Y}$ of $Y$. We have thus produced an action of $G$ on $\mathcal{Y}$ compatible with the action of $G$ on $Y$.

**Proof of Lemma 3.4.** Suppose $E$ and $E'$ are exceptional curves which intersect at the point $y$. Define $m_E$ and $m_{E'}$ to be the multiplicities of $E$ and $E'$ in the special fiber $F$ of $\mathcal{Z}$. Let $\{F_i\}_i$ be the set of other fiber components of $F$, and let $m_i$ be the multiplicity of $F_i$. As divisors, we then have

$$F = m_E \cdot E + m_{E'} \cdot E' + \sum_i m_i \cdot F_i.$$

Without loss of generality, we can suppose $m_E \leqslant m_{E'}$. Now

$$
\begin{aligned}
0 &= E \cdot F \\
&= m_E(E \cdot E) + m_{E'}(E \cdot E') + \sum_i m_i(E \cdot F_i) \\
&= -m_E + m_{E'}(E \cdot E') + \sum_i m_i(E \cdot F_i).
\end{aligned}
\tag{3}
$$

Here $E \cdot E' \geqslant 1$ and $E \cdot F_i \geqslant 0$, so we conclude that $E \cdot E' = 1$, $m_{E'} = m_E$ and $E \cdot F_i = 0$ for all $i$. A similar argument shows that $E' \cdot F_i = 0$ for all $i$. However, the special fiber of $\mathcal{Z}$ is connected by the Zariski connectedness theorem, so we see that $E$ and $E'$ are in fact the only components of the special fiber. Since $E'$ is exceptional, we can blow down $E'$ on $\mathcal{Z}$ by the Castelnuovo criterion to arrive at a regular flat projective curve $\mathcal{Z}'$ over $R$ with the following properties. The unique irreducible divisor on the special fiber of $\mathcal{Z}'$ is the image $E_0$ of $E$, which is isomorphic to $E$. The multiplicity of $E_0$ is $m_E$. Let $\pi$ be a uniformizer in $R$. Since $\mathcal{Z}'$ is flat over $R$, the arithmetic genus of the general fiber $Z$ of $\mathcal{Z}'$ is equal to

$$
\begin{aligned}
g(Z) &= 1 - \mathrm{length}\big(\chi(\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'})\big) \\
&= 1 - \mathrm{length}\big(H^0(\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'})\big) + \mathrm{length}\big(H^1(\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'})\big)
\end{aligned}
\tag{4}
$$

where $\mathrm{length}(M)$ is the length of an Artinian $R$-module $M$. Let $I = I_{E_0}$ be the ideal sheaf of $E_0$. The coherent sheaf $\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'}$ has a filtration in which the successive quotients are isomorphic to $I^j/I^{j+1}$ for $j = 0, \ldots, m_E - 1$. Therefore

$$\mathrm{length}\big(\chi(\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'})\big) = \sum_{i=0}^{m_E-1} \mathrm{length}\big(\chi(I^j/I^{j+1})\big). \tag{5}$$

Since the special fiber of $\mathcal{Z}'$ is equal to $m_E E_0$ as a divisor, we find

$$0 = E_0 \cdot E_0 = \deg_k\big(I^{-1}|_{E_0}\big). \tag{6}$$

Thus the restriction $I|_{E_0} = I/I^2$ of $I_0$ to $E_0$ also has degree 0 as a line bundle on $E_0 = P_k^1$, so $I/I^2$ is isomorphic to the trivial line bundle $\mathcal{O}_{P_k^1}$. Since $I^j/I^{j+1}$ is isomorphic to $(I/I^2)^{\otimes j}$, we conclude that each of the line bundles $I^j/I^{j+1}$ are isomorphic to $\mathcal{O}_{P_k^1}$. (We are ignoring all group actions at this point.) Now (5) and (4) give

$$
\begin{aligned}
g(Z) &= 1 - \mathrm{length}\big(\chi(\mathcal{O}_{\mathcal{Z}'}/\pi\mathcal{O}_{\mathcal{Z}'})\big) \\
&= 1 - \sum_{j=0}^{m_E-1} \chi(\mathcal{O}_{P_k^1}) \\
&= 1 - m_E.
\end{aligned}
\tag{7}
$$

However, this is impossible since $m_E \geqslant 1$ and we assumed $g(Z) > 0$. The contradiction completes the proof of Lemma 3.4, and thus of Proposition 3.3. $\quad\square$

**Proposition 3.5.** *Let $\mathcal{X}$ be a regular model of $X$ over $\mathcal{O}_K$ with an action of a finite group $G$. Then the action of $G$ on $\mathcal{X}$ is numerically tame if and only if its action on $\mathcal{X}^{\min}$ is.*

**Proof.** Since $\mathcal{X}$ is a regular model of $\mathcal{X}^{\min} \times_{\mathcal{O}_K} K$, there is a proper morphism $\phi : \mathcal{X} \to \mathcal{X}^{\min}$ which factors as a finite number of $G$-equivariant blow downs of exceptional curves [Chi86, Theorem 1.2]. Let $v$ be a place of $\mathcal{O}_K$. Denote by $R$ the strict henselization of the local ring $\mathcal{O}_{K,v}$, and by $k$ the (algebraically closed) residue field of $R$ with characteristic $p$. Then $\phi$ induces a proper morphism $\phi_R : \mathcal{C}' = \mathcal{X} \times_{\mathcal{O}_K} R \to \mathcal{C} = \mathcal{X}^{\min} \times_{\mathcal{O}_K} R$. A point $c'$ on the closed fiber of $\mathcal{C}'$ has the same inertia group as its image in $\mathcal{X}$, and similarly for points on the closed fiber of $\mathcal{C}$. Hence we may work with $\phi_R$ to detect numerical tameness.

For any $c' \in \mathcal{C}'$, $I_{c'} \subseteq I_{\phi_R(c')}$. Therefore, if $p$ divides the order $|I_{c'}|$ of the inertia group of a point $c' \in \mathcal{C}'$ then $p$ divides $|I_{\phi_R(c')}|$. Conversely, suppose that there is a point $c \in \mathcal{C}$ with an element $g \in I_c$ of order divisible by $p$. We will show that $g \in I_{c'}$ for some $c' \in \phi_R^{-1}(c)$, which will complete the proof of Theorem 3.5.

Since $g$ fixes the point $c$, $g$ acts on $\phi_R^{-1}(c)$. If $c$ is a point of codimension 1, then $\phi_R$ is an isomorphism over an open neighborhood of $c$ in $\mathcal{C}$. Hence there is a point $c'$ of codimension 1 in $\phi_R^{-1}(c)$ such that $g \in I_{c'}$. Therefore, we may assume that $c$ is a closed point. Then $\phi_R^{-1}(c)$ is isomorphic either to $c$ or to a connected curve over $k$ whose irreducible components are each isomorphic to $\mathbb{P}_k^1$. The dual graph of $\phi_R^{-1}(c)$ is a finite tree (a graph without cycles). By the following lemma, there is a closed point $c' \in \phi_R^{-1}(c)$ for which $g \in I_{c'}$. $\quad\square$

**Lemma 3.6.** *Let $C$ be a connected curve over an algebraically closed field $k$ whose irreducible components are each rational. Assume that the dual graph $\Gamma$ of $C$ is a finite tree. Then for any nontrivial finite group $G$ acting on $C$, there is a closed point of $C$ with nontrivial inertia group.*

**Proof.** Consider the action of $G$ on $\Gamma$. Let $V$ be the set of vertices of $\Gamma$ and let $E$ be the set of edges. Then $\chi(\Gamma) = |V| - |E| = 1$, since $\Gamma$ is a finite tree. If $G$ neither fixes an element of $V$ nor an element of $E$, then $|G|$ divides $\chi(\Gamma)$, contradicting $|G| > 1$. Hence, there is a vertex or an edge of $\Gamma$ fixed by $G$. A vertex of $\Gamma$ corresponds to an irreducible component of $C$ which has no fixed point free action, since it is rational. An edge corresponds to an intersection point of two irreducible components. Therefore, there is a closed point fixed by a nontrivial element of $G$, and the inertia group of this point is nontrivial, since $k$ is algebraically closed. $\quad\square$

The following theorem concerns the case when a subgroup $G$ of torsion points acts on an elliptic curve $X$ by translations. Suppose that the action of $G$ on $X$ extends to an action on a regular model $\mathcal{X}$. By Theorem 3.5, to check whether the extended action is numerically tame, we may assume that $\mathcal{X}$ is the minimal model $\mathcal{X}^{\min}$.

**Theorem 3.7.** *Let $X$ be an elliptic curve over $K$ and let $\mathcal{X}$ be the minimal model of $X$ over $\mathcal{O}_K$. Consider the action of a group $G \cong \mathbb{Z}/n \times \mathbb{Z}/m \subset X(K)$ of torsion points on $\mathcal{X}$. Then the action of $G$ on $\mathcal{X}$ is numerically tame if and only if for each place $v$ of $\mathcal{O}_K$ whose residue characteristic $p$ divides the order of $G$, the following conditions are satisfied*:

(i) *The minimal model $\mathcal{X}$ has good or multiplicative reduction at $v$.*
(ii) *The Zariski closure in $\mathcal{X}$ of the $p$-Sylow subgroup $G_p$ of $G$ is smooth over $\operatorname{Spec} \mathcal{O}_K$.*

*Under these conditions, $\gcd(n, m) = 1$.*

**Proof.** Let $v$ be a place of $\mathcal{O}_K$ over a prime $p$ dividing the order of $G$. Since the reduction type and the smoothness of a subscheme does not change under an étale base change to the strict henselization of the local ring $\mathcal{O}_{K,v}$, we may assume that the residue field $k$ of $v$ is algebraically closed.

We first show that condition (i) is necessary. Suppose $\mathcal{X}$ has additive reduction at $v$. By [Tat75], each irreducible component of the special fiber $\mathcal{X}_v$ of $\mathcal{X}$ over $v$ is isomorphic to $\mathbb{P}^1_k$ and the dual graph $\Gamma$ of the special fiber is a finite tree. By Lemma 3.6, for a nontrivial element $g \in G_p$ there is a point of $\mathcal{X}_v$ such that $g$ lies in its inertia group, hence also in the inertia group of a point of $\mathcal{X}$. Thus the action of $G$ on $\mathcal{X}$ is not numerically tame.

Suppose that $\mathcal{X}$ has good reduction at $v$. Condition (ii) implies that the induced action of $G$ on the special fiber $\mathcal{X}_v$ is étale. Otherwise, an element of $G$ of order $p$ specializes to the identity element on on $\mathcal{X}_v$. Let $x$ be the generic point of $\mathcal{X}_v$. Then $p$ divides the order of the inertia group of $x$.

Now suppose that $\mathcal{X}$ has multiplicative reduction at $v$. Condition (ii) now implies then that the Zariski closure of $G_p$ specializes to points on distinct irreducible components, and it acts freely on $\mathcal{X}_v$. Hence, the induced action of $G_p$ on $\mathcal{X}_v$ is étale, and the action of $G$ is numerically tame. Otherwise, an element of $G$ of order $p$ specializes to the identity element on the identity component of $\mathcal{X}_v$. Let $x$ be the generic point of the identity component. Then $p$ divides the order of the inertia group of $x$.

In each case, the number of distinct $p$-torsion points on the special fiber $\mathcal{X}_v$ is at most $p$. Hence to satisfy condition (ii), it is necessary that $p \nmid \gcd(n, m)$, for all $p$.   $\square$

**Remark 3.8.** For higher dimensional Abelian varieties, it is not known whether regular models exist in general. However, a certain regular model $\mathcal{X}$ for Abelian surfaces with potential good reduction is constructed in [JM94]. The dual complex $\Gamma$ of the special fiber of $\mathcal{X}$ is a finite tree. Each vertex of $\Gamma$ corresponds to $\mathbb{P}^2$ and each edge corresponds to $\mathbb{P}^1$. Hence, there is no fixed point free action on the special fiber of $\mathcal{X}$. Therefore, if the residue characteristic of the special fiber divides the order of the group $G$ of torsion points acting on $\mathcal{X}$, then the action is not numerically tame.

## 4. $G$-equivariant Euler characteristics

In the classical case of number field extensions, we denote by $R(\mathcal{O}_K[G])$ those stable isomorphism classes of $\mathcal{O}_K[G]$-modules which contain the ring of integers of some tamely ramified extension $L/K$ with $\mathrm{Gal}(L/K) \cong G$. In [McC87], McCulloh gives an idelic description of describes of $\mathrm{Cl}(\mathcal{O}_K[G])$ which then allows him to describe $R(\mathcal{O}_K[G])$ in terms of the action on $\mathrm{Cl}(\mathcal{O}_K[G])$ of the Stickelberger ideal $\mathcal{J}$ whenever $G$ is a finite Abelian group. The main goal of this section is to formulate and study an a geometric counterpart of McCulloh's results.

In particular, given a regular model $\mathcal{X}$ of an elliptic curve defined over $\mathcal{O}_K$ with a numerically tame action of a finite group $G$, Chinburg et al. give in [CEPT97] a definition for the de Rham invariant of an elliptic curve, which will be an element $\chi(\mathcal{X}, G) \in \mathrm{CL}(\mathcal{O}_K[G])$. We recap this definition in Section 4.1 and under certain hypotheses we give an explicit decomposition of this element. In Section 4.2 we define $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$ to be the subset of $\mathrm{Cl}(\mathcal{O}_K[G])$ consisting of those classes which are de Rham invariants of elliptic curves with a numerically tame $G$ action. The remainder of the section shows that when $G \cong \mathbb{Z}/\ell\mathbb{Z}$ for prime $\ell$ we are able to use the decompositions of $\chi(\mathcal{X}, G)$ to describe $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$.

### 4.1. De Rham invariant of an elliptic curve

Let $\mathcal{X}$ be a regular model over $\mathcal{O}_K$ with a numerically tame action of a finite group $G$, and let $\mathcal{F}$ be a $G$-equivariant coherent sheaf on $\mathcal{X}$. We follow Chinburg et al. and define the projective Euler characteristic $\chi^p(\mathcal{F})$ of $\mathcal{F}$ in the locally free class group $\mathrm{Cl}(\mathcal{O}_K[G])$ [Chi94,CEPT97].

A coherent (resp. coherent locally free) $\mathcal{O}_{\mathcal{X}}$-$G$-module is a coherent (resp. locally free coherent) sheaf of $\mathcal{O}_{\mathcal{X}}$-modules with compatible $G$ action. Denote by $G_0(G, \mathcal{X})$ (resp. $K_0(G, \mathcal{X})$) the Grothendieck group of coherent (resp. coherent locally free) $\mathcal{O}_{\mathcal{X}}$-$G$-modules. The group $K_0(G, \mathcal{X})$ has a $\lambda$-ring structure via $[\mathcal{F}][\mathcal{G}] = [\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}]$, $\lambda^i([\mathcal{F}]) = [\Lambda^i \mathcal{F}]$, when $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_{\mathcal{X}}$-$G$-modules which are locally free as $\mathcal{O}_{\mathcal{X}}$-modules. Because $\mathcal{X}$ is regular, the $\lambda^i$ operations can be extended to $G_0(G, \mathcal{X})$ (see [CPT00] for details).

Denote by $K_0(\mathcal{O}_K[G])$ the Grothendieck group of all finitely generated projective $\mathcal{O}_K[G]$-modules. The reduced Grothendieck group $K_0(\mathcal{O}_K[G])^{\text{red}}$ is the quotient of $K_0(\mathcal{O}_K[G])$ by the subgroup generated by the class of $(\mathcal{O}_K[G])$. The 'forgetful' homomorphism $K_0(\mathcal{O}_K[G]) \to CT(\mathcal{O}_K[G])$ induces an isomorphism between the locally free class group $\text{Cl}(\mathcal{O}_K[G])$ and $K_0(\mathcal{O}_K[G])^{\text{red}}$ [Ser77, Example 16.4, Theorem 34]. For a numerically tame $G$ action on $\mathcal{X}$, we have a $G$-cohomologically trivial Euler characteristic morphism $\chi : G_0(G, \mathcal{X}) \to K_0(\mathcal{O}_K[G])$ defined in [CE92, Theorem 5.2]. We will write $\chi^p(\mathcal{F})$ for the image of $\chi(\mathcal{F})$ in $Cl(\mathcal{O}_K[G])$.

**Definition 4.1.** (See [CEPT97].) The de Rham invariant of $\mathcal{X}$ with a numerically tame action of $G$ is the class

$$\chi(\mathcal{X}, G) := \sum_{i=0}^{d} (-1)^i \chi^p\big(\lambda^i\big(\Omega^1_{\mathcal{X}/\mathcal{O}_K}\big)\big)$$

in $\text{Cl}(\mathcal{O}_K[G])$, where $\Omega^1_{\mathcal{X}/\mathcal{O}_K}$ is the sheaf of relative differential 1-forms on $\mathcal{X}$ and $d+1$ is the dimension of $\mathcal{X}$.

**Example 4.2.**

(i) Let $L/K$ be a tamely ramified Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$. Let $\mathcal{X}$ be $\text{Spec}\,\mathcal{O}_L$. Then $\chi(\mathcal{X}, G) = (\mathcal{O}_L)$.

(ii) Let $\mathcal{X}$ be a regular model of a curve over some number field $K$. Then $\chi(\mathcal{X}, G) = \chi^p(\mathcal{O}_\mathcal{X}) - \chi^p(\Omega^1_{\mathcal{X}/\mathcal{O}_K})$.

The remainder of this paper is concerned with computing $\chi(\mathcal{X}, G) = \chi^p(\mathcal{O}_\mathcal{X}) - \chi^p(\Omega^1_{\mathcal{X}/0_K})$ for suitable choices integral models of elliptic curves. We remark that work of Chinburg et al. [CEPT99] and Pappas [Pap00] consider the problem of directly computing $\chi(\mathcal{O}_\mathcal{X})$ when the group $G$ acts tamely on a suitable model of a curve of any genus.

We will make the following hypothesis for the rest of this section.

**Hypothesis 4.3.** *Let $\mathcal{X} = \mathcal{X}^{\text{min}}$ be the minimal model over $\mathcal{O}_K$ of an elliptic curve $X$ over $K$. Denote by $T$ the finite set of places in $\mathcal{O}_K$ where $\mathcal{X}$ has bad reduction. Assume that $\mathcal{X}$ has multiplicative reduction at places in $T$ and that the action on $\mathcal{X}$ of a finite group $G$ is numerically tame.*

Let $\tilde{\mathcal{X}}_v \to \mathcal{X}_v$ be the normalization of the special fiber $\mathcal{X}_v$ of $\mathcal{X}$ over $v \in T$. The action of $G$ on $\mathcal{X}_v$, and hence on $\tilde{\mathcal{X}}_v$, is numerically tame. Therefore if $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathcal{X}_v} - G$-module or $\mathcal{O}_{\tilde{\mathcal{X}}_v} - G$-module, one has a $G$-cohomologically trivial Euler characteristic $\chi(\mathcal{F})$ in $K_0(k(v)[G])$.

Let $f : \mathcal{X} \to \mathcal{X}/G = \mathcal{Y}$. Denote by $\mathcal{Y}_T$ the disjoint union of fibers $\mathcal{Y}_v$ for $v \in T$, and by $K_\mathcal{Y}$ a canonical divisor representing the relative dualizing sheaf on $\mathcal{Y}$. In [CEPT97], the class $\chi(\mathcal{X}, G)$ is decomposed into a sum of two classes, $\chi_1 = \chi^p(\mathcal{O}_\mathcal{X}) - \chi^p(\mathcal{L})$ and $\chi_2 = \chi^p(\mathcal{L}) - \chi^p(\Omega^1_{\mathcal{X}/\mathcal{O}_K})$, where $\mathcal{L} = \mathcal{O}_\mathcal{X}(f^*(K_\mathcal{Y} + \mathcal{Y}_T))$. Then the calculation of the second term $\chi_2$ is reduced to the calculation of the projective Euler characteristic of a sheaf supported on special fibers. In the case when $X$ is an elliptic curve, we will show that the first term $\chi_1$ can also be calculated by a sheaf on special fibers.

**Definition 4.4.** Suppose $B \to A$ is a homomorphism of Noetherian rings such that $A$ is a finitely generated module over the image of $B$ and $B$ is regular. Restriction of operators from $A[G]$ to $B[G]$ then induces homomorphisms $\text{Res}_{B \to A} : K_0(A[G]) \to K_0(B[G])$ and $\text{Res}^{\text{stab}}_{B \to A} : K_0(A[G]) \to \text{Cl}(B[G])$.

**Proposition 4.5.** *Assume Hypothesis* 4.3. *Then*

$$\chi(\mathcal{X}, G) = \bigoplus_{v \in T} \text{Res}^{\text{stab}}_{\mathcal{O}_K \to k(v)} \chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$$

*in* $\text{Cl}(\mathcal{O}_K[G])$.

**Proof.** Let $\mathcal{X}_T$ be the disjoint union of the special fibers $\mathcal{X}_v$ for $v \in T$ and let $i : \mathcal{X}_T \to \mathcal{X}$ be the closed imbedding. Let $U$ be the open complement $\mathcal{X} \setminus \mathcal{X}_T$. Since $\mathcal{O}_{\mathcal{X}}|_U \simeq \Omega^1_{\mathcal{X}/\mathcal{O}_K}|_U$, there is an $\mathcal{F} = (\mathcal{F}_v)_{v \in T}$ in $G_0(\mathcal{X}_T, G)$ such that $i_*[\mathcal{F}] = [\mathcal{O}_{\mathcal{X}}] - [\Omega^1_{\mathcal{X}/\mathcal{O}_K}]$ by the localization sequence [Tho87, Theorem 2.7].

By making a flat base change to the strict henselization and tensoring $k(v)$-modules with an algebraic closure $k$ of $k(v)$, we may assume that $k(v)$ and $k$ are equal. For a finitely generated projective $k$-module $M$, denote by $BTr(M)$ the Brauer trace of $M$, and by $BTr(M)(g)$ its value at $g \in G$. (For the Brauer traces, see [Ser77, Part III].) We will compare the Brauer traces of $\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$ and of $\chi(\mathcal{F}_v)$. Let $p$ be the residue characteristic of $k$. Denote by $W$ the ring of Witt vectors of $k$ and by $\mathcal{X}_W$ the base change $\mathcal{X} \times_{\mathcal{O}_K} W$.

One can use the relative dualising sheaf of $\mathcal{X}_W/W$ to construct a translation invariant global section of $\Omega^1_{\mathcal{X}_W/W}$. This allows us to conclude that the natural map $\mathcal{O}_{\mathcal{X}_W} \to \Omega^1_{\mathcal{X}_W/W}$ is injective and the cokernel $\mathcal{F}$ is supported at singular points (cf. [Blo87]). Since all singular points of $\mathcal{X}_v$ are ordinary double points, the stalk of $\mathcal{F}$ at each singular point is $k$. Let $c_v$ be the number of irreducible components of $\mathcal{X}_v$. Then, $\dim_k \chi(\mathcal{F}) = c_v$. On the other hand, under Hypothesis 4.3, the normalization $\tilde{\mathcal{X}}_v$ of $\mathcal{X}_v$ is isomorphic to the disjoint union of $c_v$ copies of $\mathbb{P}^1_k$, and $\dim_k \chi(\mathcal{O}_{\tilde{\mathcal{X}}_v}) = c_v$. Therefore, we have $BTr(\chi(\mathcal{F}_v))(1) = BTr_h(\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v}))(1)$, for the trivial element $1 \in G$.

The fact that for any nontrivial $p$-regular element $h \in G$, the Brauer traces of $\chi(\mathcal{F}_v)$ and $\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$ evaluated at $h$ are equal follows from the Lefschetz–Riemann–Roch theorem stated as Theorem 8.3.3 of [CEPT97]. This in turn implies that $\chi(\mathcal{F}_v) = \chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$ in $K_0(k(v)[G])$, so Proposition 4.5 follows [Ser77, Corollary 1, p. 149 and Theorem 36, p. 133]. $\square$

### 4.2. Realizable classes

Suppose that $G$ is of prime order $\ell$. Recall that in the classical setting of number field extensions, McCulloh describes the set of realizable classes $R(\mathcal{O}_K[G])$ in terms of the action on $\mathrm{Cl}(\mathcal{O}_K[G])$ of the Stickelberger ideal $\mathcal{J}$. We now formulate an elliptic curve counterpart of $R(\mathcal{O}_K[G])$.

**Definition 4.6.** Fix a number field $K$ and a prime number $\ell$. Let $\mathcal{X}$ vary over all minimal models of elliptic curves satisfying Hypothesis 4.3 for some isomorphism of $G$ with the group $\mathbb{Z}/\ell\mathbb{Z}$. Then the de Rham invariant $\chi(\mathcal{X}, G)$ ranges over a set $R^{\mathrm{Ell}}(\mathcal{O}_K[G]) \subseteq \mathrm{Cl}(\mathcal{O}_K[G])$. We call an element of $R^{\mathrm{Ell}}(\mathcal{O}_K[G])$ a *realizable class* in $\mathrm{Cl}(\mathcal{O}_K[G])$.

**Theorem 4.7.** *Assume Hypothesis* 4.3. *Suppose that the group* $G \cong \mathbb{Z}/\ell\mathbb{Z}$ *is isomorphic to a subgroup of torsion points acting as translations. Denote by* $\Delta_v$ *the minimal discriminant of $X$ at a finite place $v$ of* $\mathcal{O}_K$ *and denote by* $D_{X/K}$ *the minimal discriminant ideal* $\prod_v \mathcal{P}_v^{\mathrm{ord}_v(\Delta_v)}$ *of $X$. Denote by $S$ the set of places $v \in T$ such that the action of $G$ on $\mathcal{X}_v$ is étale. Then* $\chi(\mathcal{X}, G) = \bigoplus_{v \in S} \mathrm{Res}^{\mathrm{stab}}_{\mathcal{O}_K \to k(v)}(\frac{c_v}{\ell} r_G) \oplus \bigoplus_{v \notin S} \mathrm{Res}^{\mathrm{stab}}_{\mathcal{O}_K \to k(v)}(c_v \epsilon_G)$.

**Proof.** By Proposition 4.5, to compute $\chi(\mathcal{X}, G)$ we need to compute $\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$, for each $v \in T$. As in the proof of Proposition 4.5, we may assume that the residue field $k(v)$ of $v$ is algebraically closed and the reduction splits at $v$. Recall from the proof of Proposition 4.5, that $\dim_{k(v)} \chi(\mathcal{O}_{\tilde{\mathcal{X}}_v}) = c_v$.

If the action of $G$ on the special fiber $\mathcal{X}_v$ is étale, i.e. $v \in S \subseteq T$, then the Euler characteristic $\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})$ is a free $k(v)[G]$-module [Nak84]. In this case, the Zariski closure in $\mathcal{X}$ of $G$ considered now as a subgroup of torsion points of the general fibre of $\mathcal{X}$ must specialize to distinct irreducible components of $\mathcal{X}_v$ by Theorem 3.7. Hence, the order $\ell$ of $G$ divides $c_v$. Therefore,

$$BT_r\big(\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})\big) = \frac{c_v}{\ell} r_G, \quad \text{for } v \in S, \tag{8}$$

where $r_G$ is the character of the regular representation of $G$.

Suppose now that the action of $G$ on the special fiber $\mathcal{X}_v$ is not étale. Since the $G$ action is numerically tame, the order $\ell$ of $G$ is not the characteristic $p$ of $k(v)$, and every $1 \neq h \in G$ is $p$-regular. The Zariski closure in $\mathcal{X}$ of $G$ specializes to the identity component of $\mathcal{X}_v$ as the group of $\ell$-th roots

of unity. The action of $G$ stabilizes each irreducible component of $\mathcal{X}_v$ and it is the multiplication by an $\ell$th root of unity on the identity component of $\mathcal{X}_v$. The set of singular points of $\mathcal{X}_v$ is the set of fixed points. Each singular point of $\mathcal{X}_v$ lies on exactly two irreducible components of $\mathcal{X}_v$ and if $\chi$ is the character attached to one of the two components then $\chi^{-1}$ is the one on the other component.

Let $\chi$ be the nontrivial character of $G$ giving the action of $G$ on the canonical bundle of the identity component of $\mathcal{X}_v$. The calculation in [CEPT97, Remark 8.4.4] now gives that

$$BT_r\big(\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})\big)(h) = c_v\left(\frac{1}{1-\chi(h)} + \frac{1}{1-\chi^{-1}(h)}\right), \quad \forall h \in G, \ h \neq 1. \tag{9}$$

It can be computed directly that the term in the parentheses on the right-hand side of Eq. 9 is equal to 1. Therefore $BT_r(\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v}))(h) = c_v$, and we can conclude that

$$BT_r\big(\chi(\mathcal{O}_{\tilde{\mathcal{X}}_v})\big) = c_v \epsilon_G \tag{10}$$

where $\epsilon_G$ is the identity character of $G$.

Together, (8) and (10) give the formula:

$$\chi(\mathcal{X}, G) = \bigoplus_{v \in S} \mathrm{Res}^{\mathrm{stab}}_{\mathcal{O}_K \to k(v)}\left(\frac{c_v}{\ell} r_G\right) \oplus \bigoplus_{v \notin S} \mathrm{Res}^{\mathrm{stab}}_{\mathcal{O}_K \to k(v)}(c_v \epsilon_G). \tag{11}$$

The first term on the right-hand side of (11) is the class of an $\mathcal{O}_K G$-module given by the fractional ideal $(\prod_{v \in S} \mathcal{P}_v^{\frac{c_v}{\ell}}) \otimes_{\mathcal{O}_K} \mathcal{O}_K G$.  □

**Corollary 4.8.** *If $K$ is the field of rational numbers, then $R^{\mathrm{Ell}}(\mathbb{Z}[G])$ is trivial.*

**Proof.** Because $K = \mathbb{Q}$, the ideals $\mathcal{P}_v$ are principal and thus the first term on the right-hand side of Eq. (11) is trivial. Furthermore, we can interpret the second term as a sum of Swan classes, and in this case we know that they are all trivial because $G$ is a cyclic group [Tay84, Corollary 1.5].  □

**Corollary 4.9.** *If $G$ is trivial, then $\chi(\mathcal{X}, G) = [D_{X/K}]$ in $\mathrm{Cl}(\mathcal{O}_K)$. In particular, $R^{\mathrm{Ell}}(\mathcal{O}_K[G]) = 12\,\mathrm{Cl}(\mathcal{O}_K)$.*

**Proof.** Denote by $W_{X/K}$ the Weierstrass class of $X$. Then for any given ideal class $c \in \mathrm{Cl}(\mathcal{O}_K)$, there is an elliptic curve $X/K$ with $W_{X/K} = c$ [Sil84]. Our corollary now follows from Theorem 4.7 and the fact that $12 W_{X/K} = D_{X/K}$ in $\mathrm{Cl}(\mathcal{O}_K)$ (see [Sil86, p. 224] for details).  □

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