The finite rank perturbations of the product of Hankel and Toeplitz operators

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Abstract

In this paper we completely characterize when the product of a Hankel operator and a Toeplitz operator on the Hardy space is a finite rank perturbation of a Hankel operator, and when the commutator of a Hankel operator and a Toeplitz operators has finite rank.

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1. Introduction

Let $D$ be the open unit disk in the complex plane and $\partial D$ the unit circle. Let $d\sigma(z)$ be the normalized Lebesgue measure on the unit circle $\partial D$. Let $L^2$ denote the Lebesgue square integrable functions on the unit circle. The Hardy space $H^2$ is the Hilbert space consisting of the analytic functions on the unit disk $D$ that are also in $L^2$. Let $H^\infty$ denote the set of bounded analytic functions on the unit disk. Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For $f \in L^\infty$, the space of essentially bounded measurable functions on the unit circle, $\partial D$, the Toeplitz operator $T_f$ and the Hankel operator $H_f$ with symbol $f$ are defined by $T_f h = P(fh)$ and $H_f h = P(Uf h)$ for $h$ in $H^2$. Here $U$ is the unitary operator on $L^2$ defined by $Uh(w) = \overline{w}h(w)$.
where \( \hat{h}(w) = h(\hat{w}) \). Clearly, \( H_f^* = H_{f^*} \), where \( f^*(w) = \hat{f}(\hat{w}) \). \( U \) is a unitary operator which maps \( H^2 \) onto \([H^2]_B \) and has useful property: \( UP = (1 - P)U \).

Hankel and Toeplitz operators have both been studied for a long time. A Hankel operator on Hilbert space is one whose matrix representation with respect to an orthonormal basis is constant along the diagonals perpendicular to the main diagonal. \( \{1, z, z^2, \ldots, z^n, \ldots \} \) is an orthonormal basis of \( L^2 \). \( \{1, z, z^2, \ldots, z^n, \ldots \} \) is an orthonormal basis of \( H^2 \).

For \( f \in L^\infty \), we can write \( f = \sum_{k=0}^{\infty} \hat{f}(k)z^k \). Let \( H = (a_{i,j}) \) be the matrix representation of \( H_f \) with respect to basis \( \{1, z, z^2, \ldots, z^n, \ldots \} \). It is easy to calculate that

\[
    a_{i,j} = \langle H_f z^j, z^i \rangle = \hat{f}(-i + j + 1).
\]

Hence \( a_{i,j} = a_{i+1,j-1} \) and \( H_f = H \) is a Hankel matrix with respect to the orthonormal basis \( \{1, z, z^2, \ldots, z^n, \ldots \} \). The operator \( A_f \) is defined by \( A_f h = (1 - P)f h \), for \( h \in H^2 \) that also be said as Hankel operator. It is easy to see that \( H_f h = P(Uf h) = U(1 - P)f h = U A_f h \). Therefore, \( H_f = UA_f \).

An operator on the Hilbert space \( H \) is said to have finite rank if the closure of the range of the operator has finite dimension. Recently, several works [4,6–8] have given the connection between Hankel operators or Toeplitz operators and finite rank operators. The goal of this paper is to give further connection between theory of operators of Toeplitz and theory of operators of finite type. For convenience, we use \( A = B \mod(F) \) to denote that the operator \( A - B \) has finite rank in this paper.

As is well known, Hankel and Toeplitz operators are closely related by the following important facts:

\[
    T_f g - T_f T_g = H_f H_g
\]

and

\[
    H_f g = H_g T_f + T_g H_f.
\]

The second equality implies that if \( g \in H^\infty \), then

\[
    T_g H_f = H_f g = H_f T_g.
\]

Theorem 3.2 gives that the converse of the above statement. A. Brown and P.R. Halmos [2] have shown that the product of two Toeplitz operators \( T_f \) and \( T_g \) is also a Toeplitz operator if and only if \( f \in H^\infty \) or \( g \in H^\infty \). It is easy to see that the product of two Toeplitz operators \( T_f \) and \( T_g \) is a finite rank perturbation of a Toeplitz operator if and only if the semi-commutator \( T_f T_g - T_g T_f \) is of finite rank if and only if the product of two Hankel operators \( H_f H_g \) is of finite rank. S. Axler, A. Chang and D. Sarason [1] have shown that the product \( H_f H_g \) has finite rank if and only if one of the operators \( H_f \) or \( H_g \) has finite rank. Furthermore, S. Axler, A. Chang, D. Sarason [1] and A. Volberg [14] gave the necessary and sufficient condition that the product of two Hankel operators is compact. Recently, D. Zheng [16] gave the another necessary and sufficient condition that the product \( H_f H_g \) is compact. When is the product \( H_f H_g \) of two Hankel operators equal to another Hankel operator? It was shown in [3,13,15], that the product of two Hankel operators is rarely a Hankel operator. When is the product \( H_f H_g \) of two Hankel operators equal to a finite rank perturbation of a Hankel operator? Recently, Ding and Zheng [4] showed that for any \( f, g, h \) in \( L^\infty \), the product \( H_f H_g \) is a finite rank perturbation of a Hankel operator \( H_h \) if and only if both \( H_f H_g \) and \( H_h \) are finite rank operators. A natural question is about the product of a Hankel operator and a Toeplitz operator.
Problem 1.1. When is the product $H_f T_g$ of a Hankel operator and a Toeplitz operator equal to a finite rank perturbation of a Hankel operator?

As is well known, the commutator of two Toeplitz operators $[T_f, T_g] = T_f T_g - T_g T_f$ is the sum of two semi-commutators $T_f T_g - T_{fg}$ and $T_{fg} - T_g T_f$. P.R. Halmos [10] has shown that the commutator $[T_f, T_g]$ of two Toeplitz operators is zero if and only if one of the following holds:

1. $f \in H^\infty$ and $g \in H^\infty$;
2. $\bar{f} \in H^\infty$ and $\bar{g} \in H^\infty$;
3. There are constants $a, b$ and $c$ with $|a| + |b| > 0$ such that $af + bg = c$.

Gorkin and Zheng [5] gave the necessary and sufficient condition that the commutator $[T_f, T_g]$ of two Toeplitz operators is compact. Recently, the author and Zheng [4] showed that the commutator $[T_f, T_g]$ has finite rank if and only if one of the following conditions holds:

1. There is a nonzero analytic polynomial $p$ such that $pf \in H^\infty$ and $pg \in H^\infty$;
2. There is a nonzero analytic polynomial $q$ such that $q \bar{f} \in H^\infty$ and $q \bar{g} \in H^\infty$;
3. There are nonzero analytic polynomials $A_1, A_2, B_1, B_2$ with $|A_1| + |A_2| \neq 0$ and $|B_1| + |B_2| \neq 0$, such that $A_1(z) \bar{B}_1(z) = A_2(z) \bar{B}_2(z)$, $A_1 \bar{g} + A_2 \bar{f} \in H^\infty$ and $B_1 f + B_2 g \in H^\infty$.

The analysis of the commutator $[H_f, T_g]$ turned out to be more difficult than that of the commutator $[T_f, T_g]$. Martinez-Avendaño [11] shows that the commutator $[H_f, T_g]$ is zero if and only if either $f \in H^\infty$ or there exists a constant $\lambda$ such that $g + \lambda f$ is in $H^\infty$, and both $g + \tilde{g}$ and $g \tilde{g}$ are constants. Recently, Guo and Zheng [9] gave the necessary and sufficient condition that the commutator $[H_f, T_g]$ is compact operator.

Naturally, our another question is about the commutator of a Hankel operator and a Toeplitz operator.

Problem 1.2. When does the commutator $[H_f, T_g] = H_f T_g - T_g H_f$ of a Hankel operator $H_f$ and a Toeplitz operator $T_f$ have finite rank?

The Kronecker’s theorem [12] states that for $f \in L^\infty$, $H_f$ is of finite rank if and only if $f$ is the sum of an analytic function $h$ and a rational function $r(z)$ whose poles are not on the unit circle. Thus for a rational $r(z) \in L^\infty$, $H_{r(z)}$ and $H_{\bar{r}(z)}$ both are finite rank operators. This gives that for $f \in L^\infty$ and an analytic polynomial $p(z)$,

$$T_p T_f = T_{pf} \mod(F).$$

Using the same method in [4] we will completely solve Problems 1.1 and 1.2. In Section 2, the theorem will be established for the finite sum of the product of Hankel and Toeplitz operators. In Section 3, we will give the necessary and sufficient condition that the product of a Hankel and a Toeplitz operator is a finite rank perturbation of a Hankel operator. In Section 4, we will completely characterize when the commutator $[H_f, T_g] = H_f T_g - T_g H_f$ has finite rank.
2. The finite sum of the product

We need to introduce some notation. For \( x, y \in H^2, \) \( x \otimes y \) is the operator of rank one defined by

\[
x \otimes y(f) = \langle f, y \rangle x
\]

for every \( f \in H^2. \) It is easy to see that \((x \otimes y)^* = y \otimes x.\)

**Theorem 2.1.** For \( f_i, g_i, h \) in \( L^\infty, \) \( i = 1, 2, \ldots, n, \) if \( \sum_{i=1}^n H_{f_i} T_{g_i} = H_h, \) then there are constants \( A_i, B_i, \) with \( \sum_{i=1}^n |A_i| > 0 \) and \( \sum_{i=1}^n |B_i| > 0, \) such that

\[
\sum_{i=1}^n A_i f_i \in H^\infty \quad \text{or} \quad \sum_{i=1}^n B_i g_i \in H^\infty.
\]

**Proof.** \( \sum_{i=1}^n H_{f_i} T_{g_i} = H_h \) implies that

\[
(H_{f_1} 1 \otimes T_{g_1} + \cdots + H_{f_n} 1 \otimes T_{g_n}) T_z = (H_{f_1} ((1 - T_z T_z^*) T_{g_1} + \cdots + H_{f_n} (1 - T_z T_z^*) T_{g_n}) T_z = (H_{f_1} T_{g_1} + \cdots + H_{f_n} T_{g_n}) T_z - T_z (H_{f_1} T_{g_1} + \cdots + H_{f_n} T_{g_n}) = H_h T_z - T_z H_h = H_{hz} - H_{hz} = 0.
\]

That is

\[
\sum_{i=1}^n H_{f_i} 1 \otimes T_{g_i} 1 = 0.
\]

Note that \( T_{zg} 1 = P_{zg} 1 = P_UG^* 1 = H_{g^*} 1, \) where \( g^*(z) = g(\overline{z}). \) If there is an \( i_0 \) such that \( T_{zg_{i_0}} 1 = 0, \) then \( g_{i_0} \in H^\infty. \) Thus we have

\[
0 \cdot g_1 + \cdots + 1 \cdot g_{i_0} + \cdots + 0 \cdot g_n \in H^\infty.
\]

If none of \( T_{zg_i} 1 \) is zero, then there exists a \( \lambda_0 \in D \) such that \( T_{zg_i} 1(\lambda_0) \neq 0 \) for all \( 1 \leq i \leq n. \) Let \( K_\lambda(z) \) be the reproducing kernel at \( \lambda \in D \) and \( A_i = T_{zg_i} 1(\lambda_0), \) then

\[
\sum_{i=1}^n H_{f_i} 1 \otimes T_{zg_i} 1(K_{\lambda_0}) = \sum_{i=1}^n (K_{\lambda_0} T_{zg_i} 1) H_{f_i} 1 = \sum_{i=1}^n T_{zg_i} 1(\lambda_0) H_{f_i} 1 = \sum_{i=1}^n A_i H_{f_i} 1
\]

\[
= H_{\sum_{i=1}^n A_i f_i} 1 = 0.
\]

Hence \( \sum_{i=1}^n A_i f_i \in H^\infty. \) This completes the proof of the theorem. \( \square \)

**Theorem 2.2.** For \( f_i, g_i, h \) in \( L^\infty, \) \( i = 1, 2, \ldots, n, \) if \( \sum_{i=1}^n H_{f_i} T_{g_i} - H_h \) has rank \( k, \) then there are analytic polynomials \( A_i(z), B_i(z) \) with \( \max \{\deg A_i(z): 1 \leq i \leq n\} = k, \) and \( \max \{\deg B_i(z): 1 \leq i \leq n\} = k, \) such that \( \sum_{i=1}^n A_i f_i \in H^\infty \) or \( \sum_{i=1}^n B_i g_i \in H^\infty. \)
**Proof.** For \( k = 0 \), the result is true by Theorem 2.1. Next we assume \( k \geq 1 \).

We write

\[
\sum_{i=1}^{n} H_{f_i} T_{g_i} - H_h = \sum_{j=1}^{k} x_j \otimes y_j,
\]

where \( x_j, y_j \) are in \( H^2 \) and \( \dim \text{span}\{x_1 \cdots x_k\} = \dim \text{span}\{y_1 \cdots y_k\} = k \).

We have

\[
\left\{ \sum_{i=1}^{n} H_{f_i} (1 \otimes 1) T_{g_i} \right\} T_{\bar{z}} = \left\{ \sum_{i=1}^{n} H_{f_i} (1 - T_{\bar{z}} T_{\bar{z}}) T_{g_i} \right\} T_{\bar{z}}
\]

\[
= \sum_{i=1}^{n} H_{f_i} T_{g_i} T_{\bar{z}} - T_{\bar{z}} \sum_{i=1}^{n} H_{f_i} T_{g_i}
\]

\[
= \left( H_h + \sum_{j=1}^{k} x_j \otimes y_j \right) T_{\bar{z}} - T_{\bar{z}} \left( H_h + \sum_{j=1}^{k} x_j \otimes y_j \right)
\]

\[
= H_h T_{\bar{z}} - T_{\bar{z}} H_h + \sum_{j=1}^{k} x_j \otimes T_{\bar{z}} y_j - \sum_{j=1}^{k} T_{\bar{z}} x_j \otimes y_j
\]

\[
= \sum_{j=1}^{k} x_j \otimes T_{\bar{z}} y_j - \sum_{j=1}^{k} T_{\bar{z}} x_j \otimes y_j.
\]

That is

\[
\sum_{i=1}^{n} H_{f_i} \otimes T_{\bar{z} g_i} 1 = \sum_{j=1}^{n} x_j \otimes T_{\bar{z}} y_j - \sum_{j=1}^{n} T_{\bar{z}} x_j \otimes y_j.
\] (2.1)

Applying the \( y_1 \) to both sides of the above equation gives

\[
\sum_{i=1}^{n} (y_1, T_{\bar{z} g_i} 1) H_{f_i} 1 = \sum_{j=1}^{k} (y_1, T_{\bar{z}} y_j) x_j - \sum_{j=1}^{k} (y_1, y_j) T_{\bar{z}} x_j
\]

\[
= \sum_{j=1}^{k} (y_1, T_{\bar{z}} y_j) x_j - \sum_{j=1}^{k} (y_1, y_j) (\bar{z} x_j - \bar{z} x_j (0))
\]

\[
= \sum_{j=1}^{k} [(y_1, T_{\bar{z}} y_j) - (y_1, y_j) \bar{z}] x_j + \sum_{j=1}^{k} (y_1, y_j) x_j (0) \bar{z}.
\]

Let \( a_{ij} = (y_1, y_j), b_{ij} = (y_1, T_{\bar{z}} y_j) - (y_1, y_j) \bar{z}, c_{ij} = (y_1, T_{\bar{z} g_i} 1) \) we have

\[
\begin{pmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{21} & c_{22} & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{k1} & c_{k2} & \cdots & c_{kn}
\end{pmatrix}
\begin{pmatrix}
  H_{f_1} 1 \\
  H_{f_2} 1 \\
  \vdots \\
  H_{f_n} 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  H_{f_1} 1 \\
  H_{f_2} 1 \\
  \vdots \\
  H_{f_n} 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
    b_{11} & b_{12} & \cdots & b_{1k} \\
    b_{21} & b_{22} & \cdots & b_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{k1} & b_{k2} & \cdots & b_{kk}
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_k
\end{pmatrix}
+ \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1k} \\
    a_{21} & a_{22} & \cdots & a_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k1} & a_{k2} & \cdots & a_{kk}
\end{pmatrix}
\begin{pmatrix}
    x_1(0) \\
    x_2(0) \\
    \vdots \\
    x_k(0)
\end{pmatrix} = \bar{z}.
\]

That is
\[
CH_1 = BX + \bar{z}AX(0)
\]
where \( C = (c_{ij}) \), \( B = (b_{ij}) \), \( A = (a_{ij}) \), \( H_1 = (H_{f_1} \cdots H_{f_k})^T \), \( X = (x_1 \cdots x_k)^T \), \( X = (x_1(0) \cdots x_k(0))^T \).

The determinant of matrix \( B = (b_{ij})_{k \times k} \) is
\[
B(z) = \det(b_{ij}) = (-1)^k a\bar{z}^k + a_1\bar{z}^{k-1} + \cdots + a_k
\]
where \( a = \det(a_{ij}) \neq 0 \) since \( y_1, y_2, \ldots, y_k \) are linearly independent, and \( a_i \) are constants. Hence \( \text{rank } B(z) = k \), \( B(z) \) is a co-analytic polynomial in \( z \).

The adjoint of the matrix \( B \) is
\[
\text{adj } B = \begin{pmatrix}
    B_{11} & B_{21} & \cdots & B_{k1} \\
    B_{12} & B_{22} & \cdots & B_{k2} \\
    \vdots & \vdots & \ddots & \vdots \\
    B_{1k} & B_{2k} & \cdots & B_{kk}
\end{pmatrix}
\]
where \( B_{ij} \) denotes the cofactor of \( b_{ij} \) and is a co-analytic polynomial in \( z \) with degree at most \( k - 1 \). So
\[
(\text{adj } B)CH_1 = B(z)X + (\text{adj } B)AX(0)\bar{z}.
\]

Let
\[
(C_{li}(z)) = (\text{adj } B)C
\]
where \( C_{li}(z) \) are co-analytic polynomials in \( z \) with degree at most \( k - 1 \).

Applying the projection \( P \) to both sides of the above equation gives
\[
P[(C_{li}(z)H_1)] = PB(z)X.
\]

It follows that
\[
\begin{pmatrix}
    H_{\sum_{i=1}^{n} c_{li}(z)f_i}^1 \\
    \vdots \\
    H_{\sum_{i=1}^{n} c_{li}(z)f_i}^1
\end{pmatrix} = \begin{pmatrix}
    T_{B(z)x_1} \\
    \vdots \\
    T_{B(z)x_k}
\end{pmatrix}.
\]

We have also
\[
\sum_{i=1}^{n} T_{z_i^1} \otimes H_{f_i} 1 = \left( \sum_{i=1}^{n} H_{f_i} 1 \otimes T_{z_i^1} \right)^*
\]
and \((x \otimes y)^* = y \otimes x\). So
\[
\sum_{i=1}^{n} T_{z_i^1} \otimes H_{f_i} 1 = \sum_{j=1}^{k} T_{z_j} y_j \otimes x_j - \sum_{j=1}^{k} y_j \otimes T_{z_j} x_j.
\]
(2.2)
Hence
\[ n \sum_{i=1}^{n} \langle x_i, H_{f_1} \rangle T_{\mathcal{E}}^{i} 1 = \sum_{j=1}^{k} \langle x_i, x_j \rangle T_{\mathcal{E}} y_j - \sum_{j=1}^{k} \langle x_i, T_{\mathcal{E}} x_j \rangle y_j. \]

By the same argument, we also have
\[ P \left[ (u_{li} (z)) T_{\mathcal{E}}^{i} 1 \right] = P \left( E(z) Y \right). \]

It follows that
\[
\begin{pmatrix}
T_{\sum_{i=1}^{n} u_{li}(z) T_{\mathcal{E}}^{i} 1} \\
\vdots \\
T_{\sum_{i=1}^{n} u_{kl}(z) T_{\mathcal{E}}^{i} 1}
\end{pmatrix}
= \begin{pmatrix}
TE(z) y_1 \\
\vdots \\
TE(z) y_k
\end{pmatrix}
\]
where $u_{li}(z)$ are co-analytic polynomials in $z$ with degree at most $k - 1$ and $E(z)$ is a co-analytic polynomial in $z$ with degree $k$.

If $T_{\mathcal{E}}^{i} 1, T_{\mathcal{E}}^{2} 1, \ldots, T_{\mathcal{E}}^{n} 1, y_1, y_2, \ldots, y_k$ are linearly dependent, then there exist constants $a_i, b_j, \text{not all are zero, such that}$
\[ \sum_{i=1}^{n} a_i T_{\mathcal{E}}^{i} 1 + \sum_{j=1}^{k} b_j y_j = 0. \]

One of the $a_1, a_2, \ldots, a_n$ must be nonzero since $y_1, \ldots, y_k$ are linearly independent. Without loss of generality, assume
\[ T_{\mathcal{E}}^{i} 1 = a_1 T_{\mathcal{E}}^{i} 1 + \cdots + a_{n-1} T_{\mathcal{E}}^{i} 1 + b_1 y_1 + \cdots + b_k y_k. \]

Then we have
\[
TE(z) T_{\mathcal{E}}^{i} 1 = TE(z) T_{\mathcal{E}}^{i} 1 = \sum_{i=1}^{n-1} a_i \bar{T_{\mathcal{E}}}^{i} E(z) g_i 1 + \sum_{j=1}^{k} b_j T_{\mathcal{E}} y_j
\]
\[ = \sum_{i=1}^{n-1} a_i \bar{T_{\mathcal{E}}}^{i} E(z) g_i 1 + \sum_{j=1}^{k} b_j T_{\mathcal{E}}^{i} \sum_{j=1}^{n} a_{ji}(z) T_{\mathcal{E}}^{j} 1
\]
\[ = \sum_{i=1}^{n-1} a_i \bar{T_{\mathcal{E}}}^{i} E(z) g_i 1 + \sum_{i=1}^{n} T_{\mathcal{E}}^{i} \sum_{j=1}^{n} b_{ji}(z) T_{\mathcal{E}}^{j} 1. \]

Therefore,
\[
T_{\mathcal{E}} \left[ \left( E(z) - \sum_{j=1}^{k} b_j u_{jz}(z) \right) g_n - \sum_{i=1}^{n-1} \left[ a_i E(z) + \sum_{j=1}^{k} b_{ji}(z) \right] g_i(z) \right] 1 = 0.
\]

It follows from the above equation, we have
\[
\left( E(z) - \sum_{j=1}^{k} b_j u_{jz}(z) \right) g_n - \sum_{i=1}^{n-1} \left[ a_i E(z) + \sum_{j=1}^{k} b_{ji}(z) \right] g_i(z) \in H^\infty.
\]

Let
\[ B_n(z) = \left[ E(z) - \sum_{j=1}^{k} b_j u_j(z) \right], \]

\[ B_i(z) = \left[ a_i E(z) + \sum_{j=1}^{k} b_j u_j(z) \right], \quad 1 \leq i \leq n - 1. \]

Then \( B_i(z) \) are analytic polynomials in \( z \) with degree \( B_n(z) = k \), \( \deg B_i(z) \leq k \), \( 1 \leq i \leq n - 1 \), and \( \sum_{i=1}^{n} B_i g_i \in H^\infty \).

If \( H_{f_1}, H_{f_2}, \ldots, H_{f_n}, x_1, \ldots, x_n \) are linearly dependent, without loss of generality, we may assume that

\[ H_{f_n} = e_1 H_{f_1} + \cdots + e_{n-1} H_{f_{n-1}} + d_1 x_1 + \cdots + d_k x_k. \]

Then

\[ T_B(z) H_{f_n} = H_{\tilde{B}(z) f_n} = \sum_{i=1}^{n-1} e_i H_{\tilde{B}(z) f_i} + \sum_{j=1}^{k} d_j T_{B(z)} x_j \]

\[ = \sum_{i=1}^{n-1} e_i H_{\tilde{B}(z) f_i} + \sum_{j=1}^{k} \sum_{i=1}^{n} H_{\tilde{C}_{ji}(z) f_i} \]

\[ = \sum_{i=1}^{n-1} e_i H_{\tilde{B}(z) f_i} + \sum_{i=1}^{n} H_{\sum_{j=1}^{k} d_j \tilde{C}_{ji}(z) f_i}. \]

This gives

\[ H_{[\tilde{B}(z) - \sum_{j=1}^{k} d_j \tilde{C}_{jn}(z)] f_n - \sum_{m=1}^{n-1} [e_i \tilde{B}(z) + \sum_{j=1}^{k} d_j \tilde{C}_{ji}(z)] f_i} = 0. \]

Let

\[ A_n(z) = \tilde{B}(z) - \sum_{j=1}^{k} d_j \tilde{C}_{jn}(z), \]

\[ A_i(z) = e_i \tilde{B}(z) + \sum_{j=1}^{k} d_j \tilde{C}_{ji}(z), \quad 1 \leq i \leq n - 1. \]

Then \( A_i(z) \) are analytic polynomials in \( z \) and \( \deg A_n(z) = k \), \( \deg A_i(z) \leq k \) for \( 1 \leq i \leq n - 1 \), and \( \sum_{i=1}^{n} A_i f_i \in H^\infty \).

Now we assume that \( H_{f_1}, \ldots, H_{f_n}, x_1, \ldots, x_k \) are linearly independent and \( T_{\xi_j 1}, \ldots, T_{\xi_k 1}, y_1, \ldots, y_k \) are also linearly independent. We will derive a contradiction.

First we claim that

\[ \dim \text{span}\{x_1, \ldots, x_k, T_{\xi_1 x_1}, \ldots, T_{\xi_k x_k}\} \geq k + n. \]

In fact, since \( T_{\xi_j 1}, \ldots, T_{\xi_k 1} \) are linearly independent, there is a vector \( \xi \in H^2 \), such that \( \langle \xi, T_{\xi_j 1} \rangle = 1 \) and \( \langle \xi, T_{\xi_i 1} \rangle = 0 \) for all \( j \neq i \).

Hence

\[ H_{f_i} = \sum_{j=1}^{k} \langle \xi, T_{\xi_j} \rangle x_j - \sum_{j=1}^{k} \langle \xi, y_j \rangle T_{\xi x_j}. \]
by Eq. (2.1). This implies that $Hf_1 \in \text{span}\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\}$. This gives
$$\text{span}\{Hf_1, \ldots, Hf_n, x_1, \ldots, x_k\} \subseteq \text{span}\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\}.$$ Thus
$$\dim \text{span}\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\} \geq \dim \text{span}\{Hf_1, \ldots, Hf_n, x_1, \ldots, x_k\} = k + n.$$ Since
$$\dim \text{span}\{x_1, \ldots, x_k\} = k < k + n,$$ there is a nonzero vector $\xi$ in $\text{span}\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\}$ such that
$$\xi \perp \{x_1, \ldots, x_k\}.$$ It follows that
$$\sum_{i=1}^{n} \langle \xi, Hf_i \rangle T_\xi g_i 1 = -\sum_{j=1}^{k} \langle \xi, T_\xi x_j \rangle y_j$$ by Eq. (2.2). Not all of $\{(\xi, T_\xi x_j)\}_{j=1}^{k}$ are zero since $\xi \in \text{span}\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\}$. Otherwise $\xi$ is orthogonal to $\{x_1, \ldots, x_k, T_\xi x_1, \ldots, T_\xi x_k\}$ it would imply that $\xi = 0$. This gives that $T_\xi g_1, \ldots, T_\xi g_n, y_1, \ldots, y_k$ are linearly dependent. We have obtained a contradiction to complete the proof.

3. The product of Hankel and Toeplitz operators

**Theorem 3.1.** For $f, g, h \in L^\infty$, $Hf Tg = Hh \mod(F)$ if and only if one of the following conditions holds:

1. $Hf$ and $Hh$ have finite rank;
2. $Hg$ and $Hfg - h$ have finite rank.

**Proof.** First we prove the “only if” part. Suppose $Hf Tg = Hh \mod(F)$. By Lemma 2.2, there are nonzero analytic polynomials $A(z)$ and $B(z)$ such that $A(z)f \in H^\infty$ or $B(z)g \in H^\infty$. If $A(z)f \in H^\infty$, then $Hf$ has finite rank, and $Hf Tg = Hh \mod(F)$ implies that $Hh$ also has finite rank. If $B(z)g \in H^\infty$, then $Hg$ has finite rank. Because $Hf Tg = Hfg - Tg Hh = Hfg \mod(F)$, so $Hfg = Hh \mod(F)$. Thus $Hfg - h$ is a finite rank operator.

Next we prove the “if” part.

1. If $Hf$ and $Hh$ have finite rank, then $Hf Tg = Hh \mod(F)$ is obvious.
2. If $Hg$ and $Hfg - h$ have finite rank, then
$$Hf Tg = Hfg - Tg Hh = Hfg = Hh \mod(F).$$

This completes the proof of the theorem.

**Theorem 3.2.** For $f, g, h \in L^\infty$, $Hf Tg = Hh$ if and only if one of the following conditions holds:

1. $f$ and $h$ in $H^\infty$;
2. $g$ and $fg - h$ in $H^\infty$. 
Proof. Obviously, “if” part is true.
We only prove the “only if” part. Assume $H_f T_g = H_h$. If $H_f = 0$, then $H_h = 0$. It follows that $f$ and $h$ in $H^\infty$. If $H_f \neq 0$, then
\[
H_f 1 \otimes T_g z = H_f 1 \otimes T_{\overline{fz}} 1 = H_f (1 - T_z T_{\overline{z}}) T_g z = H_f T_g z - T_z H_f T_g
\]

This implies that $H_f 1 = 0$ or $T_{\overline{fz}} 1 = 0$. Because $H_f \neq 0$, so $T_z T_{\overline{z}} = 0$. Thus $g \in H^\infty$ and $H_f T_g = H_f h$. It follows that $fg - h \in H^\infty$. This completes the proof of the theorem.

4. The commutator of Hankel and Toeplitz operators

We begin the following lemma which be known in [4].

Lemma 4.1. Let $A$ be a bounded linear operator on $H^2$. Suppose that $p(z)$ and $q(z)$ are nonzero analytic polynomials. Then $T_p^* A T_q$ has finite rank if and only if $A$ has finite rank.

We prove the following theorem which encompasses the difficulty in the proofs of our main results.

Theorem 4.2. For $f_1, f_2, g_1, g_2, h$ in $L^\infty$, none of the $H_{f_1}, H_{f_2}, H_{g_1}, H_{g_2}$ has finite rank, then
\[
H_{f_1} T_{g_1} + H_{f_2} T_{g_2} = H_h \mod(F)
\]
if and only if following conditions hold:

(1) there are nonzero analytic polynomials $A_i(z), B_i(z)$ such that $A_1 f_1 + A_2 f_2 \in H^\infty$ and $B_1 g_1 + B_2 g_2 = h_1 \in H^\infty$;
(2) $A_1 B_1 + A_2 B_2 = 0$ and $HA_2 f_2 h_1 + A_1 B_1 h = 0 \mod(F)$.

Proof. First we prove the “only if” part. Suppose
\[
H_{f_1} T_{g_1} + H_{f_2} T_{g_2} = H_h \mod(F).
\]
By Theorem 2.2, there are analytic polynomials $A_i, B_i$, such that
\[
A_1 f_1 + A_2 f_2 \in H^\infty \quad \text{or} \quad B_1 g_1 + B_2 g_2 \in H^\infty.
\]
Here none of $A_1, A_2, B_1$ and $B_2$ is zero since none of the $H_{f_1}, H_{f_2}, H_{g_1}, H_{g_2}$ has finite rank.

(1) Suppose that $A_1 f_1 + A_2 f_2 \in H^\infty$ and $A_1 \cdot A_2 \neq 0$. Then
\[
T_{A_1} (H_{f_1} T_{g_1} + H_{f_2} T_{g_2} - H_h) = H_{A_1 f_1} T_{g_1} + H_{A_1 f_2} T_{g_2} - H_{A_1 h}
\]
\[
= -H_{A_2 f_2} T_{g_1} + H_{A_1 f_2} T_{g_2} - H_{A_1 h}
\]
\[
= -H_{f_2} T_{A_1 g_1} + H_{f_2} T_{A_1 g_2} - H_{A_1 h}
\]
\[
= H_{f_2} (T_{A_1 g_2 - A_2 g_1}) - H_{A_1 h} \mod(F).
\]
The last equality comes from that $T_{A_2} T_{g_1} = T_{A_2 g_1} \mod(F)$ and $T_{A_1} T_{g_2} = T_{A_1 g_2} \mod(F)$.

Thus
\[
H_{f_2} T_{A_1 g_2 - A_2 g_1} = H_{A_1 h} \mod(F).
\]
Hence there is a nonzero analytic polynomial $B(z)$ such that $B(A_1g_2 - A_2g_1) \in H^\infty$ by Theorem 2.2 and $H_{f_2}$ has not finite rank of hypothesis of the theorem. Let $B_1 = -BA_2$, $B_2 = BA_1$, then

$$B_1g_1 + B_2g_2 \in H^\infty.$$  

(2) Assume that $h_1 = B_1g_1 + B_2g_2 \in H^\infty$ and $B_1B_2 \neq 0$, then

$$Hf_1 T_{g_1} + Hf_2 T_{g_2} - Hh)T_{B_1} = Hf_1 T_{g_1} + Hf_2 T_{g_2} - HhB_1$$

$$= Hf_1(T_{g_1} - B_2 g_2) + Hf_2 T_{g_2} - HhB_1$$

$$= Hf_1 T_{g_1} - Hf_1 T_{B_2 g_2} + Hf_2 T_{g_2} - HhB_1$$

$$= -Hf_1 T_{B_2 g_2} + Hf_2 T_{g_2} - Hf_1 h_1 - hB_1$$

$$= -Hf_1 T_{B_2 g_2} + Hf_2 T_{B_1 g_2} + Hf_1 h_1 - hB_1 \mod(F)$$

$$= H - B_2 f_1 + B_1 f_2 T_{g_2} + Hf_1 h_1 - hB_1.$$  

That is $Hf_2 B_1 - f_1 B_2 T_{g_2} = HhB_1 - f_1 h_1 \mod(F)$. Hence there is a nonzero analytic polynomial $A$ such that $A(f_2 B_1 - f_1 B_2) \in H^\infty$ since Theorem 2.2 and $H_{g_2}$ has not finite rank.

Let $A_1 = -A B_2$, $A_2 = A B_1$, then

$$A_1 f_1 + A_2 f_2 \in H^\infty.$$  

Now we already prove that $A_1 f_1 + A_2 f_2 \in H^\infty$ and $h_1 = B_1g_1 + B_2g_2 \in H^\infty$, where $A_1$, $A_2$, $B_1$, $B_2$ all are nonzero analytic polynomials.

Also we have

$$T_{A_1}(Hf_1 T_{g_1} + Hf_2 T_{g_2} - Hh)T_{B_1} = H_{A_1} f_1 T_{g_1} B_1 + H_{A_1} f_2 T_{g_2} B_1 - H_{A_1} h B_1$$

$$= H_{A_2} f_2 T_{B_2 g_2} - h_1 + H_{A_1} f_2 T_{B_2 g_2} - H_{A_1} h B_1$$

$$= H_{A_2} f_2 T_{B_2 g_2} + H_{A_1} f_2 T_{B_2 g_2} - H_{A_2} f_2 h_1 + h_1 B_1$$

$$= H_{f_2} (T_{A_2} T_{B_2 g_2} + T_{A_1} T_{B_1 g_2}) - H_{A_2} f_2 h_1 + h_1 B_1$$

$$= H_{f_2} (T_{A_2} T_{B_2 g_2} + T_{A_1} T_{B_1 g_2}) - H_{A_2} f_2 h_1 + h_1 B_1 \mod(F)$$

Thus $H_{f_2} T_{(A_1 B_1 + A_2 B_2) g_2} - H_{A_2} f_2 h_1 + A_1 B_1 h$ has finite rank. By Theorem 3.1, one of $H_{f_2}$ and $H_{(A_1 B_1 + A_2 B_2) g_2} = H_{g_2} T_{(A_1 B_1 + A_2 B_2)}$ must be finite rank operator. But none of $H_{f_2}$ and $H_{g_2}$ has finite rank, so $A_1 B_1 + A_2 B_2 = 0$. This implies that $H_{A_2} f_2 h_1 + A_1 B_1 h$ has finite rank. This completes the proof of “only if” part.

Now we prove the sufficient part. Assume conditions (1) and (2) hold. We have

$$T_{A_1} (Hf_1 T_{g_1} + Hf_2 T_{g_2} - Hh)T_{B_1} = H_{A_1} f_1 T_{g_1} B_1 + H_{A_1 f_2 T_{g_2} B_1} - H_{A_1 B_1 h}$$

$$= H_{A_2} f_2 T_{B_2 g_2} - h_1 + H_{A_1} f_2 T_{B_2 g_2} - H_{A_1 B_1 h}$$

$$= H_{f_2} (T_{A_2} T_{B_2 g_2} - T_{A_1} T_{B_1 g_2}) - H_{A_2} f_2 h_1 + h_1 B_1$$

$$= H_{f_2} (T_{A_2} T_{B_2 g_2} - T_{A_1} T_{B_1 g_2}) - H_{A_2} f_2 h_1 + h_1 B_1 \mod(F)$$

$$= 0 \mod(F).$$

That is $T_{A_1} (Hf_1 T_{g_1} + Hf_2 T_{g_2} - Hh)T_{B_1}$ has finite rank. Hence Lemma 4.1 gives that $Hf_1 T_{g_1} + Hf_2 T_{g_2} = Hh \mod(F)$. This completes the proof of the theorem.  

Next we consider the commutator $[H_f, T_g] = H_f T_g - T_g H_f$. The following theorem is our main result.
Theorem 4.3. For \( f, g \) in \( L^\infty \), the commutator \( H_f T_g - T_g H_f \) has finite rank if and only if one of the following conditions hold:

1. \( H_f \) has finite rank;
2. \( H_g \) and \( T_g H_f - H_f g \) have finite rank;
3. \( H_\tilde{g} \) and \( H_f T_g - H_f \tilde{g} \) have finite rank;
4. None of \( H_f, H_g \) and \( H_\tilde{g} \) has finite rank. There are nonzero analytic polynomials \( A_i, B_i \) such that

\[
A_1 f + A_2 \tilde{g} \in H^\infty \quad \text{and} \quad B_1 g + B_2 f = h_1 \in H^\infty,
\]

moreover,

\[
A_1 B_1 + A_2 B_2 = 0 \quad \text{and} \quad H_{A_1 B_1 \tilde{g} f + A_2 \tilde{g} h_1} = 0 \mod(F).
\]

Proof. Suppose that one of conditions (1)–(4) holds. We are going to show that the commutator \( H_f T_g - T_g H_f \) has finite rank. Obviously condition (1) implies that \( H_f T_g - T_g H_f \) has finite rank.

If the condition (2) holds, then \( H_g \) has finite rank. This implies that there is a nonzero analytic polynomial \( p(z) \) such that \( pg \in H^\infty \) by Kronecker’s theorem. Thus

\[
[H_f, T_g]T_p = H_f T_g p - T_g H_f T_p = H_{fg} p - T_g H_f T_p = (H_{fg} - T_g H_f)T_p.
\]

Therefore \( [H_f, T_g]T_p \) has finite rank. It follows \( H_f T_g - T_g H_f \) has finite rank by Lemma 4.1.

If the condition (3) holds, then \( H_\tilde{g} \) has finite rank. Since \( H_\tilde{g} T_f + T_g H_f = H_f \tilde{g} \), \( T_g H_f = H_f \tilde{g} \mod(F) \). Hence

\[
H_f T_g - T_g H_f = H_f T_g - H_f \tilde{g} \mod(F).
\]

Since \( H_f T_g - H_f \tilde{g} \) has finite rank, \( H_f T_g - T_g H_f \) has finite rank.

If the condition (4) holds, because \( H_\tilde{g} T_f + T_g H_f = H_f \tilde{g} \),

\[
H_f T_g - T_g H_f = H_f T_g + H_\tilde{g} T_f - H_f \tilde{g}.
\]

Let \( f_1 = f, f_2 = \tilde{g}, g_1 = g, g_2 = f, h = f \tilde{g} \), so \( H_f T_g - T_g H_f \) has finite rank by Theorem 4.2.

Conversely, suppose that \( H_f T_g - T_g H_f \) has finite rank. It is easy to see that one of the following conditions holds:

(a) \( H_f \) has finite rank;
(b) \( H_g \) has finite rank;
(c) \( H_\tilde{g} \) has finite rank;
(d) None of \( H_f, H_g \) and \( H_\tilde{g} \) has finite rank.

It follows that one of the conditions (1)–(4) holds. This completes the proof of the theorem. \( \Box \)

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