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Effects of D-instantons in string amplitudes

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Abstract

We investigate the different energy regimes in the conjectured $SL(2,\mathbf{Z})$ invariant four graviton scattering amplitude that incorporates D-instanton contributions in 10d type IIB superstring theory. We show that the infinite product over $SL(2,\mathbf{Z})$ rotations is convergent in the whole complex plane s,t. For high energies $\alpha' s \gg 1$, fixed scattering angle, and very weak coupling $g_s \ll 1/(\alpha' s)$, the four-graviton amplitude exhibits the usual exponential suppression. As the energy approaches $1/g_s$, the suppression gradually diminishes until there appears a strong amplification near a new pole coming from the exchange of a (p,q) string. At energies $\alpha' s \ll 1/\sqrt{g_s}$, the pure D-instanton contribution to the scattering amplitude is found to produce a factor $A_4^{\text{D-inst}} \cong \exp(c g_s^{3/2} e^{-\frac{2\pi}{g_s}} s^3)$. At energies $1/\sqrt{g_s} \ll \alpha' s \ll 1/g_s$, the D-instanton factor becomes $A_4^{\text{D-inst}} \cong \exp(2 e^{-\frac{2\pi}{g_s} + \pi g_s s^2})$, $\alpha' = 4$. At higher energies $\alpha' s \gg 1/g_s$ the D-instanton contribution becomes very important, and one finds an oscillatory behavior which alternates suppression and amplification. This suggests that non-perturbative effects can lead to a high-energy behavior which is significantly different from the perturbative string behavior.

1. Introduction

A problem of interest is understanding what are the concrete effects that non-perturbative corrections can have in superstring theory, in particular, how they affect the high-energy behavior of string amplitudes. In ten-dimensional type IIB superstring theory, the source of non-perturbative corrections are the D-instantons.

Computing the contribution of multiply-charged D-instantons directly is complicated. However, combining different pieces of information, Green and Gutperle [1] conjectured the exact modular function that multiplies the R^4 term in the type IIB effective action, which exactly incorporates the infinite set of multiply-charged D-instanton corrections.

One of the constraints on the effective action used by Green and Gutperle is precisely $SL(2,\mathbf{Z})$ invari-

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ance. The $SL(2,\mathbf{Z})$ symmetry of type IIB superstring theory requires that the effective action must be invariant under $SL(2,\mathbf{Z})$ transformations to all orders in the α' expansion. In particular, this implies that graviton scattering amplitudes must be $SL(2,\mathbf{Z})$ invariant, since there is a direct correspondence between the terms in the effective action and the momentum expansion of the scattering amplitude.

In [2,3] an $SL(2,\mathbf{Z})$ invariant four graviton amplitude was constructed by applying a simple $SL(2,\mathbf{Z})$ symmetrization of the tree-level string theory four graviton amplitude. The construction follows essentially the same rule used by Green and Gutperle to symmetrize the R^4 term. It was conjectured that this scattering amplitude incorporates the full series of D-instanton corrections with the different D-instanton numbers. This symmetric amplitude satisfies a number of consistency conditions. In particular, corrections of perturbative origin appear with an integer power of g_s^2. This is non-trivial and does not hold for any symmetrization. It is also consistent with the conjecture that high derivative terms in the type II effective action of the form $H^{4k-4}R^4$ should not receive perturbative contributions beyond genus k [4]. By construction, it reproduces the exact R^4 term proposed in [1], and it can be viewed as a tree-level amplitude that accounts for the exchange of (p,q) string states [5].

These (p,q) string states have a simple eleven-dimensional origin [6]. Type IIB superstring theory is obtained from M-theory by compactification on a 2-torus and taking the zero area limit at fixed torus moduli. In this limit, most membrane states get an infinite mass, except a certain set of states that represent the (p,q) strings of uncompactified 10d type IIB string theory. These states are precisely the states that contribute as simple poles in the $SL(2,\mathbf{Z})$ invariant amplitude of [2,3].

In this work we investigate the properties of the $SL(2,\mathbf{Z})$ invariant amplitude. In particular, we factorize the pure D-instanton contribution and study the high energy limit.

The conjecture of [1] has withstood different tests and has been generalized in different directions [7–19]. The idea of organizing type IIB perturbation theory in $SL(2,\mathbf{Z})$ invariant way was also suggested by [20,21]. Scattering amplitudes at high energies incorporating higher genus effects were investigated by [22] and [23].

2. $SL(2,\mathbf{Z})$ invariant amplitude

The four-graviton scattering amplitude for 10d type IIB superstring introduced in [2,3] is given by the following formula:

$$A_4 = \kappa^2 K A_4^{sl(2)}(s,t), \tag{2.1}$$

$$A_4^{sl(2)}(s,t) = \frac{1}{stu}\prod_{(p,q)'}\frac{\Gamma(1-s_{pq})\Gamma(1-t_{pq})\Gamma(1-u_{pq})}{\Gamma(1+s_{pq})\Gamma(1+t_{pq})\Gamma(1+u_{pq})}, \tag{2.2}$$

$$s_{pq} = \frac{\alpha' s}{4|p+q\tau|}, \qquad t_{pq} = \frac{\alpha' t}{4|p+q\tau|},$$
$$u_{pq} = \frac{\alpha' u}{4|p+q\tau|}, \qquad s_{pq}+t_{pq}+u_{pq}=0, \tag{2.3}$$

where p and q are relatively prime, $\tau = C^{(0)} + i g_s^{-1}$ is the usual coupling of type IIB superstring theory, and K is the same kinematical factor depending on the momenta and polarization of the external states appearing in the tree-level Virasoro amplitude (see, e.g., [24])

$$K = \zeta_1^{AA'}\zeta_2^{BB'}\zeta_3^{CC'}\zeta_4^{DD'} K_{ABCD}(k_i) K_{A'B'C'D'}(k_i),$$
$$K_{ABCD} = -\frac{1}{4} st\eta_{AC}\eta_{BD} + \cdots.$$

The scattering amplitude (2.2) can also be written as

$$A_4^{sl(2)}(s,t) = \frac{1}{stu} e^{\delta(s,t)}, \tag{2.4}$$

with

$$\delta(s,t) = 2\sum_{k=1}^{\infty}\frac{\zeta(2k+1) g_s^{k+1/2} E_{k+1/2}(\tau)}{2k+1} \times \left(\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}\right),$$
$$\bar{s} = \frac{1}{4}\alpha' s, \qquad \bar{t} = \frac{1}{4}\alpha' t,$$
$$\bar{u} = \frac{1}{4}\alpha' u, \qquad \bar{s}+\bar{t}+\bar{u} = 0, \tag{2.5}$$

and $E_r(\tau)$ is the non-holomorphic Eisenstein series, given by $(\mathrm{Re}\, r > 1)$

$$E_r(\tau) = \sum_{(p,q)'}\frac{\tau_2^r}{|p+q\tau|^{2r}}. \tag{2.6}$$

For very small coupling $g_s \ll 1$, the terms with $q \neq 0$ are negligible in the sum (2.5), so that $g_s^{k+1/2} \times E_{k+1/2}(\tau) \to 1$. One recovers the tree-level four-graviton Virasoro amplitude,

$$A_4(s,t) = \kappa^2 K A_4^0(s,t),$$
$$A_4^0(s,t) = \frac{1}{stu} e^{\delta_0(s,t)}, \tag{2.7}$$
$$\delta_0(s,t) = 2\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}\left(\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}\right). \tag{2.8}$$

This gives

$$A_4^0(s,t) = \frac{1}{stu} \frac{\Gamma(1-\bar{s})\Gamma(1-\bar{t})\Gamma(1-\bar{u})}{\Gamma(1+\bar{s})\Gamma(1+\bar{t})\Gamma(1+\bar{u})}. \tag{2.9}$$

The scattering amplitude $A_4^{sl(2)}$ adds to the Virasoro amplitude perturbative and non-perturbative contributions. They are seen explicitly by expanding the Eisenstein functions at large $\tau_2 = g_s^{-1}$ [25],

$$E_r(\tau) = \tau_2^r + \gamma_r \tau_2^{1-r} + \frac{4\tau_2^{1/2}\pi^r}{\zeta(2r)\Gamma(r)} \sum_{n,w=1}^{\infty} \left(\frac{w}{n}\right)^{r-1/2} \times \cos(2\pi w n \tau_1) K_{r-1/2}(2\pi w n \tau_2),$$
$$\gamma_r = \frac{\sqrt{\pi}\,\Gamma(r-1/2)\zeta(2r-1)}{\Gamma(r)\zeta(2r)}. \tag{2.10}$$

Using the asymptotic expansion for the Bessel function $K_{r-1/2}$,

$$K_{r-1/2}(2\pi w n \tau_2) = \frac{1}{\sqrt{4wn\tau_2}} e^{-2\pi w n \tau_2} \times \sum_{m=0}^{\infty} \frac{1}{(4\pi w n \tau_2)^m} \frac{\Gamma(r+m)}{\Gamma(r-m)m!}, \tag{2.11}$$

we see that the $E_{k+1/2}(\tau)$ terms in the amplitude are of the form

$$g_s^{k+1/2} E_{k+1/2}(\tau) = 1 + \gamma_{k+1/2} g_s^{2k} + O\left(e^{-2\pi/g_s}\right). \tag{2.12}$$

Note that the non-perturbative contributions are $O(e^{-\frac{2\pi m}{g_s}})$, where $m = wn$ is an integer number. The coefficient $2\pi m$ is crucial in order to have a one-to-one correspondence between these terms and instanton contributions. It is a remarkable fact that the product over $SL(2,\mathbf{Z})$ rotations automatically generates the full series of D-instanton contributions.

We summarize the main properties of $A_4^{sl(2)}$:

(1) It is $SL(2,\mathbf{Z})$ invariant. This is explicit in the Einstein frame, $g^E_{\mu\nu} = g_B^{-1/2} g_{\mu\nu}$, so that $s_E = g_B^{1/2} s$, $t_E = g_B^{1/2} t$, $u_E = g_B^{1/2} u$, and s_E, t_E, u_E remain fixed under $SL(2,\mathbf{Z})$ transformations.

(2) It adds perturbative g_s^{2k} and non-perturbative $O(e^{-\frac{2\pi m}{g_s}})$ corrections to the Virasoro amplitude.

(3) It has simple poles in the s–t–u channels at $s_{pq} = n$, $t_{pq} = n$, $u_{pq} = n$, $n = 0, 1, 2, \ldots$ corresponding to a tree-level exchange of particles with masses

$$\frac{1}{4}\alpha' M^2 = n|p + q\tau|. \tag{2.13}$$

(4) It reproduces the exact (proportional to $E_{3/2}(\tau)$) R^4 term conjectured in [1], containing a one-loop correction and the full D-instanton contributions. It also reproduces the exact $\zeta(5)E_{5/2}(\tau)\nabla^4 R^4$ term conjectured in [18] (moreover, in [17] there was a calculation of a genus one term in $\nabla^6 R^4$ which was found proportional to $2\zeta(3)\zeta(2)$, which differs from the prediction of the $A_4^{sl(2)}$ amplitude only by a factor of 2).

The spectrum (2.13) is the spectrum of (p,q) string states [5]:

$$M^2 = 4\pi T_{pq}(N_R + N_L) = \frac{2}{\alpha'}|p + q\tau|(N_R + N_L), \quad N_R = N_L. \tag{2.14}$$

This spectrum corresponds to the zero winding sector of the spectrum studied in [6,26] for the nine-dimensional type IIB string theory. Setting $\tau_1 = C^{(0)} = 0$, the full spectrum in $D = 9$ is given by

$$M_9^2 = \frac{n^2}{R_{10}^2}\left(p^2 + \frac{q^2}{g_s^2}\right) + \frac{w_{10}R_{10}}{\alpha'^2} + \frac{2}{\alpha'}\sqrt{p^2 + \frac{q^2}{g_s^2}}(N_R + N_L),$$
$$N_R - N_L = n w_{10}, \tag{2.15}$$

where R_{10} is the radius of the compact tenth dimension. In the limit $R_{10} \to \infty$, one must set, as usual, the winding number w_{10} to zero to have finite mass. The term proportional to $1/R_{10}^2$ becomes the continuous 10d component of the momentum p_{10}, so that $M^2 = M_9^2 - p_{10}^2$ and one gets Eq. (2.14). It is important to note that a charged D string has mass $M = O(1/g_s)$, as seen from (2.15). The neutral (p,q) strings of ten dimensions have masses given by (2.14) of order $M = O(1/\sqrt{g_s})$ for $q \neq 0$. This is why the product over $SL(2,\mathbf{Z})$ rotations produces poles at $\alpha' s = O(1/g_s)$.

The collection of states (2.14) are the only quantum states of M-theory compactified on a 2-torus that remain of finite mass after taking the zero-area limit of the torus that leads to ten-dimensional type IIB string theory [3]. The scattering amplitude $A_4^{sl(2)}$ can thus be viewed as a tree-level scattering amplitude where all these states are exchanged.

The scattering amplitude $A_4^{sl(2)}$ does not describe loop effects such as discontinuity cuts (see [3,17] for discussions). In particular, it should not be a good approximation of the full scattering amplitude at $g_s = O(1)$ and $\alpha' s$ large. It represents an improvement of the Virasoro amplitude at $g_s \ll 1$ (or at the S-dual situation, $g_s \gg 1$), where D-instanton (and some perturbative) contributions have been incorporated.

The exchange of (p,q) string states is clear from the pole structure of (2.2). It becomes manifest by writing

$$\delta = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \log \frac{M_{mn}^2 + s}{M_{mn}^2 - s} + (s \to t) + (s \to u), \tag{2.16}$$

where

$$\alpha' M_{mn}^2 = 4|m + n\tau|.$$

This generalizes the analogous formula for the Virasoro amplitude, with δ_0 written in the form

$$\delta_0 = \sum_{m=1}^{\infty} \delta_{(m)},$$

$$\delta_{(m)} = \log \frac{M_m^2 + s}{M_m^2 - s} + (s \to t) + (s \to u),$$

$$\alpha' M_m^2 = 4m. \tag{2.17}$$

Now the sum in (2.16) contains not only the terms $\alpha' M_m^2 = 4m$, but all the terms M_{mn}^2, representing all (p,q) string states.

3. Convergence properties

The scattering amplitude $A_4^{sl(2)}$ is defined through an infinite product (2.2) over a pair (p,q) of relatively prime integers, i.e., integers (p,q) having greatest common divisor equal to one. An important issue is what are the convergence properties of this product.

To study the convergence, we write

$$\delta(s,t) = \sum_{(p,q)'} \log \frac{\Gamma(1 - s_{pq})}{\Gamma(1 + s_{pq})} + (s \to t) + (s \to u). \tag{3.1}$$

We have to look at the behavior of terms with large p, q. For any given s, t, u, there are positive integers (p_0, q_0) such that all terms with $p > p_0, q > q_0$ have $|p + q\tau| \gg s, t, u$ and s_{pq}, t_{pq}, u_{pq} small. These terms have the behavior

$$\log \frac{\Gamma(1 - s_{pq})\Gamma(1 - t_{pq})\Gamma(1 - u_{pq})}{\Gamma(1 + s_{pq})\Gamma(1 + t_{pq})\Gamma(1 + u_{pq})} \cong \frac{2\zeta(3)}{3|p + q\tau|^3}\left(\bar{s}^3 + \bar{t}^3 + \bar{u}^3\right), \tag{3.2}$$

where we have used $s + t + u = 0$. The sum over p, q of $|p + q\tau|^{-3}$ is known to be convergent [25] (in particular, the full sum over p, q of $|p + q\tau|^{-3}$ defines $E_{3/2}(\tau)$, see (2.6)). Therefore the sum in (3.1) is convergent. Since we have made no assumption about s, t, the series (3.1) has infinite radius of convergence.

We have also investigated the convergence numerically, by explicit calculation of the infinite product in different sectors of the complex planes s and t, for generic values of the coupling τ. As an additional check, we have also computed the amplitude in the representation (2.16), obtaining the same (finite) numerical results.

Note that the series (2.5) defining the amplitude has a finite radius of convergence if we write $\zeta(2k+1) = \sum_m m^{-2k-1}$ and perform first the sum over k. The sum over k is the series of a logarithm and it diverges when s, t or u meet the first pole. The same of course applies for the Virasoro amplitude written in the form (2.8).

4. Approach to the different scales

Let us assume $g_s \ll 1$, and examine the different scales that appear as the center-of-mass energy is increased from zero. We set $C^{(0)} = \tau_1 = 0$, so that $\tau = i\tau_2 = i g_s^{-1}$.

4.1. Region $\alpha' s \ll \frac{1}{g_s}$

For $\alpha' s \ll 1$, one has $A_4^{sl(2)}(s,t) \to \frac{1}{stu}$, and one recovers the supergravity tree-level four graviton amplitude.

In general, for any $\alpha' s \ll \frac{1}{g_s}$, one has $A_4^{sl(2)}(s,t) \to A_4^0(s,t)$, one recovers the Virasoro amplitude (2.9) with its usual properties: simple poles at $\bar s = n$, $\bar t = n$, $\bar u = n$, with n positive (recall that in the physical region of elastic scattering s is positive, while t, u are negative). The high energy behavior is as follows:

(i) High $\alpha' s$, fixed scattering angle φ:

$$A_4^{sl(2)}(s,t) \to A_4^0(s,t) \cong \frac{1}{stu} e^{-\alpha' a_0 s},$$
$$a_0 = \frac{1}{2}\left| \sin^2\frac{\varphi}{2} \log\sin^2\frac{\varphi}{2} + \cos^2\frac{\varphi}{2} \log\cos^2\frac{\varphi}{2} \right|,$$
$$t = -s\sin^2\frac{\varphi}{2}, \qquad u = -s\cos^2\frac{\varphi}{2}. \tag{4.1}$$

(ii) High $\alpha' s$, fixed t:

$$A_4^{sl(2)}(s,t) \to A_4^0(s,t) \cong \frac{1}{stu}(-1)^t s^{-\frac{\alpha'}{2}|t|} \times \frac{\Gamma(1-\frac{\alpha' t}{4})}{\Gamma(1+\frac{\alpha' t}{4})}. \tag{4.2}$$

It is useful to split the scattering amplitude $A_4^{sl(2)}$ in three factors:

$$A_4^{sl(2)}(s,t) = A_4^0 \times A_4^{\text{pert}} \times A_4^{\text{D-inst}} = \frac{1}{stu} e^{\delta_0 + \delta_{\text{pert}} + \delta_{\text{D-inst}}}. \tag{4.3}$$

Here A_4^{pert} represents perturbative corrections to the Virasoro amplitude of the form g_s^{2k} coming from the term $\gamma_r \tau_2^{1-r}$ in the expansion (2.10). The remaining factor $A_4^{\text{D-inst}}$ represents the pure D-instanton contribution, terms proportional to $e^{-2\pi wn/g_s}$ coming $K_{r-1/2}$ in (2.10). They are given by

$$\delta_{\text{pert}} = \sqrt{\pi} \sum_{k=1}^{\infty} \frac{(k-1)!\zeta(2k)}{\Gamma(k+\frac{3}{2})} g_s^{2k} \times \left(\bar s^{2k+1} + \bar t^{2k+1} + \bar u^{2k+1}\right), \tag{4.4}$$

$$\delta_{\text{D-inst}} = 4\sqrt{\pi} \sum_{n,w,k=1}^{\infty} \left(\frac{w}{n}\right)^k \frac{\pi^k g_s^k}{\Gamma(k+\frac{3}{2})} \times K_k\left(\frac{2\pi wn}{g_s}\right)\left(\bar s^{2k+1} + \bar t^{2k+1} + \bar u^{2k+1}\right). \tag{4.5}$$

These series converge for $\bar s < \frac{1}{g_s}$. In the region $\bar s \ll \frac{1}{g_s}$, the leading behavior of δ_{pert} is just given by the first term in the series (4.4), $\delta_{\text{pert}} \cong \frac{2\pi^2}{9}(\bar s^3 + \bar t^3 + \bar u^3)$.

Assuming $g_s \ll 1$, one can use the asymptotic form of the Bessel function (2.11). Then one obtains that in this region $\alpha' s \ll \frac{1}{g_s}$, the D-instanton contribution is given by

$$\begin{aligned}\delta_{\text{D-inst}} &= 2\bar s\sqrt{\pi g_s} e^{-\frac{2\pi}{g_s}} \sum_{k=1}^{\infty} \frac{(\pi g_s \bar s^2)^k}{\Gamma(k+\frac{3}{2})} + (s\to t) + (s\to u) \\ &= 2\,\text{sign}(s) e^{-\frac{2\pi}{g_s}} e^{\pi g_s \bar s^2} \text{Erf}\left(\sqrt{\pi g_s \bar s^2}\right) + (s\to t) + (s\to u),\end{aligned} \tag{4.6}$$

where Erf is the error function. There are two regimes, $\bar s \ll 1/\sqrt{g_s}$, so that $\pi g_s \bar s^2 \ll 1$, and $1/\sqrt{g_s} \ll \bar s \ll 1/g_s$. In the first case, we get

$$\delta_{\text{D-inst}} \cong \frac{8\pi}{3} g^{3/2} e^{-\frac{2\pi}{g_s}}\left(\bar s^3 + \bar t^3 + \bar u^3\right), \quad \bar s \ll \frac{1}{\sqrt{g_s}}. \tag{4.7}$$

This is a positive contribution in the physical region of the Mandelstam parameters, but it is negligible compared to δ_0 and δ_{pert}. In the second case, we get

$$\delta_{\text{D-inst}} \cong 2e^{-\frac{2\pi}{g_s}}\left(e^{\pi g_s \bar s^2} - e^{\pi g_s \bar t^2} - e^{\pi g_s \bar u^2}\right), \quad \frac{1}{\sqrt{g_s}} \ll \bar s \ll \frac{1}{g_s}. \tag{4.8}$$

This is still tiny, since in this region $\pi g_s \bar s^2 \ll 2\pi/g_s$.

4.2. Region $\alpha' s = O(\frac{1}{g_s})$

In this case one begins to see simple poles at $\alpha' s = \sqrt{m^2 + n^2/g_s^2}$ with $n \neq 0$. Since g_s is small, there is an accumulation of poles with $m = 0, 1, 2, \ldots$ near the pole at $n = 1$, at $n = 2$, etc. These poles are not seen in a coarse grain plot of the amplitude, since they appear at special points.

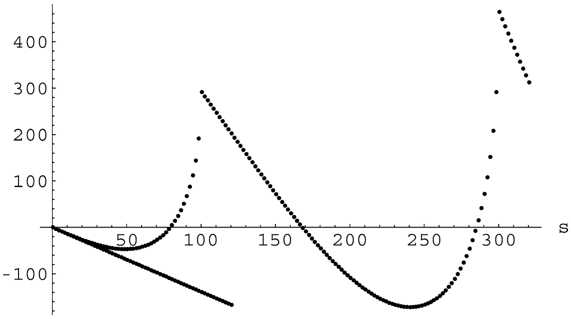

Fig. 1. $\delta(s,t)$ as a function of $\bar{s} = \frac{\alpha' s}{4}$ at fixed scattering angle $\varphi = \frac{\pi}{2}$ for $g_s = 0.01$. The straight line is $\delta_0(s,t)$.

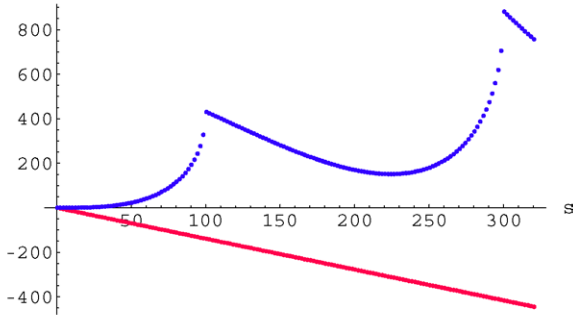

Fig. 2. The separate contributions to $A_4^{sl(2)}$ for $g_s = 0.01$: (a) The tree-level Virasoro part δ_0 is a straight line. (b) The perturbative part δ_{pert} has cusps and is positive, giving an amplification effect. (c) The D-instanton part $\delta_{\text{D-inst}}$ is still negligible at $\bar{s} < 320$.

Fig. 1 shows $\delta(s,t)$ (the logarithm of the amplitude, see (2.4)) as a function of s for large s, t, u and fixed scattering angle $\varphi = \frac{\pi}{2}$ and for $g_s = 0.01$. One can see that at the beginning there is the straight line with negative slope as in (4.1), reproducing the usual suppression of the Virasoro amplitude. Then $\delta(s,t)$ becomes positive, producing an amplification of the amplitude near $\bar{s} = 1/g_s$ (see Fig. 2). As s is further increased, the amplitude diminishes; then it is amplified again near $\bar{s} = 3/g_s$.

The behavior can be understood as a combination of the effects of δ_0 and δ_{pert}, since the D-instanton contribution $\delta_{\text{D-inst}}$ is still negligible in this region for $g_s = 0.01$. Fig. 2 shows the two contributions separately.

4.3. Region $\alpha' s \gg \frac{1}{g_s}$

To examine the behavior in this region, we again consider the three contributions $\delta_0, \delta_{\text{pert}}, \delta_{\text{D-inst}}$ separately.

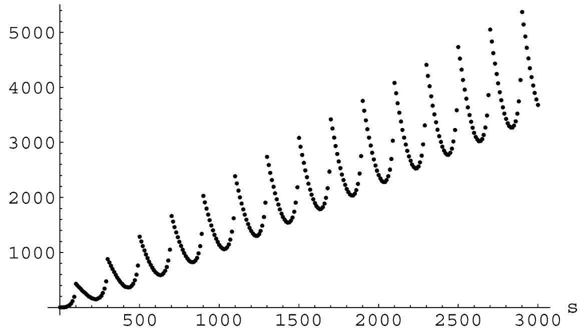

Fig. 3. The perturbative part δ_{pert} computed at ultra high energies. It grows linearly with s.

The perturbative part (4.4) can be resummed explicitly, with the result [3]:

$$\delta_{\text{pert}} = -4 \sum_{m=1}^{\infty} \sqrt{\frac{m^2}{g_s^2} - \bar{s}^2} \arcsin \frac{\bar{s} g_s}{m} + (s \to t) + (s \to u). \tag{4.9}$$

Its high energy behavior is shown in Fig. 3, which indicates a behavior $\delta_{\text{pert}} \cong \text{const}\, s$. More precisely, it is bounded between two straight lines: $1.22\bar{s} < \delta_{\text{pert}} < 1.69\bar{s}$. On the other hand, we have from (4.1):

$$\delta_0 \cong -\alpha' a_0 s. \tag{4.10}$$

The pure D-instanton part $\delta_{\text{D-inst}}$ can be computed from $\delta_{\text{D-inst}} = \delta - \delta_0 - \delta_{\text{pert}}$, where $\delta(s,t)$ and δ_{pert} are computed from the convergent sums (3.1) and (4.9). Numerically, one finds that $\delta_{\text{D-inst}}$ oscillates between negative and positive values, which are of the same order of magnitude as $\delta_0, \delta_{\text{pert}}$. This gives rise to a behavior which alternates strong suppression and amplification of the amplitude as s is increased.

The asymptotic behavior at very large s is unclear since numerical precision is worst at high s. It is plausible that at $\alpha' s \gg 1/g_s$ there are higher genus corrections not contained in $A_4^{sl(2)}$ which become important. Among the different types of corrections, there are gravitational corrections corresponding to multiple exchange of gravitons [23,27]. In the present case of high energy and fixed scattering angle, the dominant genus h contribution is known [22], though it is unclear how to resum the full series [28].

We find remarkable that the product over $SL(2, \mathbf{Z})$ rotations produces a convergent, mathematically well defined amplitude, and that the infinite D-instanton sum produces significant changes in the high energy behavior. It would be interesting to understand how to incorporate higher genus corrections to $A_4^{sl(2)}$ in an $SL(2, \mathbf{Z})$ invariant way.

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