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# Deformed Carroll particle from 2 + 1 gravity

Jerzy Kowalski-Glikman *, Tomasz Trześniewski

*Institute for Theoretical Physics, University of Wrocław, Pl. Maksa Borna 9, Pl-50-204 Wrocław, Poland*

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ABSTRACT

We consider a point particle coupled to 2 + 1 gravity, with de Sitter gauge group SO(3, 1). We observe that there are two contraction limits of the gauge group: one resulting in the Poincaré group, and the second with the gauge group having the form $\mathsf{AN}(2) \ltimes \mathsf{an}(2)^*$. The former case was thoroughly discussed in the literature, while the latter leads to the deformed particle action with de Sitter momentum space, like in the case of $\kappa$-Poincaré particle. However, the construction forces the mass shell constraint to have the form $p_0^2 = m^2$, so that the effective particle action describes the deformed Carroll particle.



Gravity in 2 + 1 dimensions seems to be, at first sight, an incredibly dull theory. It does not possess any local degrees of freedom and local Newtonian interactions between masses as well as gravitational waves are absent [1–3]. Therefore it can be described by a topological field theory as firmly established in the seminal paper of Witten [4] (and slightly earlier in [5]). Remarkably, the picture changes dramatically when one adds point particles to pure gravity. Then the gauge degrees of freedom of gravity at the particle's worldlines become dynamical. By solving for [6,7] (or integrating out in the path integral formalism [8,9]) the remaining (gauge) degrees of freedom of gravity we are left with a nontrivial particles dynamics, which includes not only the "pure" particles' degrees of freedom, but also the back reaction resulting from the presence of the gravitational field created by the particles themselves. Since in 2+1 dimensions gravitational "action at a distance" is absent, the only thing that this back reaction can do is to *deform* the original free particles' actions.

In this letter we show that one of the possible deformed actions, which can be obtained from 2 + 1 gravity is an action of the deformed Carroll particle with AN(2) momentum space [10,11] and $\kappa$-Minkowski (noncommutative) spacetime [12,13]. The Carroll particle [14] is a relativistic particle model in the limit in which the velocity of light becomes zero. Such a particle cannot move (… *it takes all the running you can do, to keep in the same place* … [15]) and its relativistic symmetry group is a particular contraction of the Poincaré group [16,17]. The Carroll group has attracted some attention recently, because it seems to become potentially relevant in several distinct fields of theoretical physics, see [18] for discussion and references. It also arises in loop quantum cosmology [19] in the context of so-called asymptotic silence, of which the Carrollian (or ultralocal) limit is a particular realization. More specifically, the Carrollian limit appears at the transition between the low-curvature (Lorentzian) regime and the high-curvature regime in which the metric is Euclidean. It is supposed that at this transition the symmetry group should change from Poincaré to Carroll and eventually to Euclidean.

In all these cases a simple free dynamical relativistic particle model possessing Carroll symmetry is of great interest, because it exhibits physics of the Carrollian world, providing an insight into it. It is reasonable to expect that in this world some remnants of particles' interactions should still be present, and that the situation is similar to that of gravity coupling in 2+1 dimensions. Indeed, in the limit $c \to 0$ the local interactions are frozen, and only the topological sector of gravity can be present. This is exactly the situation that we encounter in 2+1 dimensions, where local degrees of freedom of gravity are not possible. Therefore it could be claimed that the $D + 1$, $D > 2$ analogue of the 2 + 1 deformed particle action, derived below, might be the correct effective description of the Carrollian particle interacting with its own gravitational field.

Let us start with a short discussion of 2 + 1 gravity coupled to particles. According to [4] 2 + 1 gravity can be described by a Chern–Simons theory with an appropriate gauge group; here we will use the 2 + 1-dimensional de Sitter group, with three Lorentz generators $J_a$ and three translational generators $P_a$, satisfying

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \qquad [J_a, P_b] = \epsilon_{ab}{}^c P_c,$$
$$[P_a, P_b] = -\epsilon_{ab}{}^c J_c, \tag{1}$$

where, since the cosmological constant is absorbed into the translational generators, all generators are dimensionless. It is

* Corresponding author.
*E-mail addresses:* jerzy.kowalski-glikman@ift.uni.wroc.pl (J. Kowalski-Glikman), tomasz.trzesniewski@ift.uni.wroc.pl (T. Trześniewski).

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convenient to make use of the time plus space decomposition of spacetime and to accordingly decompose the Chern–Simons connection one-form into

$$\mathsf{A} = A_0 dt + \mathsf{A}_S. \tag{2}$$

The Lagrangian of the Chern–Simons theory of connection $\mathsf{A}$ coupled to a point particle takes the form

$$L = \frac{k}{4\pi}\int \langle \dot{\mathsf{A}}_S \wedge \mathsf{A}_S\rangle - \langle \mathcal{C}h^{-1}\dot{h}\rangle + \int \left\langle A_0, \frac{k}{2\pi}\mathsf{F}_S - h\mathcal{C}h^{-1}\delta^2(\vec{x})\,dx^1 \wedge dx^2\right\rangle, \tag{3}$$

where the spatial curvature $\mathsf{F}_S = d\mathsf{A}_S + [\mathsf{A}_S, \mathsf{A}_S]$. Let us pause for a moment to explain the meaning of the terms in this Lagrangian. The bracket $\langle * \rangle$ denotes an Ad-invariant inner product on the Lie algebra of the gauge group, which in our case is defined to be

$$\langle J_a P_b\rangle = \eta_{ab}, \qquad \langle J_a J_b\rangle = \langle P_a P_b\rangle = 0. \tag{4}$$

The coupling constant $k$ can be related to the physical parameters, the Planck mass $\kappa$ (which in $2+1$ dimensions is purely classical) and the cosmological constant as follows

$$\frac{k}{2\pi} = \frac{\kappa}{\sqrt{\Lambda}}. \tag{5}$$

Then, the first term in (3) describes the pure gravity while the second one is the particle term. More precisely, $h$ includes the translation and Lorentz transformation acting on the particle and providing it with an arbitrary position, momentum, and angular momentum, while $\mathcal{C}$ is a gauge algebra element characterizing the particle at rest at the origin,

$$\mathcal{C} = \frac{m}{\sqrt{\Lambda}} J_0 + s P_0, \tag{6}$$

where $m$ is the mass and $s$ the spin of the particle in the rest frame.

Last but not least, the integrand in the third term in (3) can be seen as a constraint relating the curvature with the mass/spin of the particle

$$\frac{k}{2\pi}\mathsf{F}_S = h\mathcal{C}h^{-1}\delta^2(\vec{x})\,dx^1 \wedge dx^2, \tag{7}$$

enforced by the Lagrange multiplier $A_0$. It bounds together the gravitational and particle degrees of freedom, therefore we may use it to solve for the latter in terms of the former.

In order to proceed further we divide the space into two regions: the disc $\mathcal{D}$ with the particle at its center, on which we introduce the coordinates $r \in [0,1]$, $\phi \in [0, 2\pi]$, and the asymptotic empty region $\mathcal{E}$ (with $r \geq 1$). They share the boundary $\Gamma$ at $r = 1$. Then by virtue of (7) in the asymptotic region the connection is flat and has the form

$$\mathsf{A}_S^{(\mathcal{D})} = \gamma d\gamma^{-1}, \tag{8}$$

where $\gamma$ is an element of the gauge group. In the particle region $\mathcal{D}$ the general solution of (7) can also be found and it is given by

$$\mathsf{A}_S^{(\mathcal{E})} = \bar\gamma \frac{1}{k}\mathcal{C}d\phi\bar\gamma^{-1} + \bar\gamma d\bar\gamma^{-1}, \qquad \bar\gamma(0) = h \tag{9}$$

(note that $dd\phi = 2\pi\delta^2(\vec{x})\,dx^1 \wedge dx^2$). Substitution of (8) into the Lagrangian yields the boundary term and the so-called WZW term. The same is the case for the second term in (9). Contribution from the first term in (9) can be rewritten as

$$\begin{aligned} &2\langle \partial_0\big(\bar\gamma\mathcal{C}\bar\gamma^{-1}d\phi\big) \wedge \bar\gamma d\bar\gamma^{-1}\rangle \\ &\quad = -2\partial_0\langle \mathcal{C}d\phi \wedge \bar\gamma^{-1}d\bar\gamma\rangle - 2d\langle \mathcal{C}d\phi\bar\gamma^{-1}\dot{\bar\gamma}\rangle \\ &\qquad + 4\pi\delta(\vec{x})\,dx^1 \wedge dx^2\langle \mathcal{C}\bar\gamma^{-1}\dot{\bar\gamma}\rangle, \end{aligned} \tag{10}$$

where the first term can be neglected being a total time derivative and the last one cancels the particle term in (3). Summing all the contributions and adopting the opposite orientation of the boundary $\Gamma$ for the terms coming from the disc $\mathcal{D}$ we obtain the total Lagrangian

$$\begin{aligned} L = {}& \frac{k}{4\pi}\int_\Gamma \left\langle \dot\gamma\gamma^{-1}d\gamma\gamma^{-1} - \dot{\bar\gamma}\bar\gamma^{-1}d\bar\gamma\bar\gamma^{-1} + \frac{2}{k}\mathcal{C}d\phi\dot{\bar\gamma}\bar\gamma^{-1}\right\rangle \\ &+ \frac{k}{4\pi}\int_\mathcal{E} \langle \dot\gamma\gamma^{-1}d\gamma\gamma^{-1} \wedge d\gamma\gamma^{-1}\rangle \\ &+ \frac{k}{4\pi}\int_\mathcal{D} \langle \dot{\bar\gamma}\bar\gamma^{-1}d\bar\gamma\bar\gamma^{-1} \wedge d\bar\gamma\bar\gamma^{-1}\rangle. \end{aligned} \tag{11}$$

In the next step we impose the continuity condition on the boundary $\Gamma$, $\mathsf{A}_S^{(\mathcal{D})}|_\Gamma = \mathsf{A}_S^{(\mathcal{E})}|_\Gamma$. Solving this equation we find the expression

$$\gamma^{-1}|_\Gamma = Ne^{\frac{1}{k}\mathcal{C}\phi}\bar\gamma^{-1}|_\Gamma, \qquad dN = 0, \tag{12}$$

where $N = N(t)$ is an arbitrary gauge group element.

The idea now is to use the continuity condition (12) to simplify the Lagrangian (11). Unfortunately, this condition is very difficult to disentangle in the case of the gauge group $\mathrm{SO}(3,1)$. Therefore in what follows we will consider only contractions of this gauge group, leading to the effective gauge group having the form of the semidirect product of some new group and the dual of its algebra $G \ltimes \mathfrak{g}^*$, see [20].

Before deriving the main result of this paper, let us pause for a moment to recall the well known construction in the case of the standard contraction of de Sitter group $\mathrm{SO}(3,1)$ to the Poincaré group, which can be presented as $\mathrm{SO}(2,1) \ltimes \mathfrak{so}(2,1)^* \simeq \mathrm{SO}(2,1) \ltimes \mathbb{R}^3$. To this end we introduce the rescaled translation generators, $\tilde{P}_a \equiv \sqrt{\Lambda}P_a$. Then (1) is replaced by

$$\begin{aligned} &[J_a, J_b] = \epsilon_{ab}{}^c J_c, \qquad [J_a, \tilde{P}_b] = \epsilon_{ab}{}^c \tilde{P}_c, \\ &[\tilde{P}_a, \tilde{P}_b] = -\Lambda\epsilon_{ab}{}^c J_c \end{aligned} \tag{13}$$

and $\langle J_a \tilde{P}_b\rangle = \sqrt{\Lambda}\eta_{ab}$. If we now take the limit $\Lambda \to 0$ then $[\tilde{P}_a, \tilde{P}_b] = 0$, while the remaining commutators are unchanged. Moreover, $\sqrt{\Lambda}$ in the scalar product cancels its inverse in the definition of $k/4\pi$, cf. (5) and thus no divergencies appear in this limit. Furthermore, with the help of Cartan decomposition a gauge group element can be written as a product

$$g = \mathfrak{j}\mathfrak{p} = \big(\iota_3 + \iota^a J_a\big)\big(1 + \xi^a \tilde{P}_a\big), \quad \mathfrak{j} \in \mathrm{SO}(2,1),\ \mathfrak{p} \in \mathfrak{so}(2,1)^*, \tag{14}$$

where the coordinates $\iota$ on $\mathrm{SO}(2,1)$ group satisfy $\iota_3^2 + \frac{1}{4}\iota_a\iota^a = 1$.

Applying (14) to group elements in the Lagrangian (11) we find that the WZW terms cancel out and only the boundary ones remain, giving

$$L = \frac{k}{2\pi}\int_\Gamma \left\langle \mathfrak{j}^{-1}\dot{\mathfrak{j}}\,d\xi - \bar{\mathfrak{j}}^{-1}\dot{\bar{\mathfrak{j}}}\,d\bar\xi + \frac{1}{k}\mathcal{C}_P d\phi\bar{\mathfrak{j}}^{-1}\dot{\bar{\mathfrak{j}}} + \frac{1}{k}\mathcal{C}_J d\phi\big[\bar{\mathfrak{j}}^{-1}\dot{\bar{\mathfrak{j}}}, \bar\xi\big]\right\rangle, \tag{15}$$

where we also neglected the total time derivative $\frac{1}{k}\mathcal{C}d\phi\dot{\bar\xi}$. Meanwhile, the sewing condition (12) factorizes into $\mathfrak{j}^{-1} = \mathfrak{n}e^{\frac{1}{k}\mathcal{C}_J\phi}\bar{\mathfrak{j}}^{-1}$

and $-\xi = \mathrm{Ad}(\mathfrak{n})h - \mathrm{Ad}(\mathfrak{n}e^{\frac{1}{k}\mathcal{C}_J\phi})\bar{\xi}$, where we write the decomposition (14) of $N$ as $N = \mathfrak{n}(1+h)$, $\mathfrak{n} \in \mathrm{SO}(2,1)$, $h \in \mathrm{so}(2,1)^*$. $\mathcal{C}$ can be separated into $\mathcal{C} = \mathcal{C}_J + \mathcal{C}_P$, $\mathcal{C}_J = m/\sqrt{\Lambda}J_0$, $\mathcal{C}_P = s/\sqrt{\Lambda}\tilde{P}_0$. Substituting the above expressions into (15) we get

$$L = \frac{k}{2\pi}\int_\Gamma d\left\langle e^{-\frac{1}{k}\mathcal{C}_J\phi}\dot{\mathfrak{n}}^{-1}\mathfrak{n}e^{\frac{1}{k}\mathcal{C}_J\phi}\bar{\xi} - \mathfrak{n}^{-1}\dot{\mathfrak{n}}\frac{1}{k}\mathcal{C}_P\phi\right\rangle, \tag{16}$$

where $\bar{\xi} \equiv \bar{\xi}^a\tilde{P}_a$. Integrating (16) over $\phi$ from 0 to $2\pi$ and noticing that $\bar{\xi}$ is a single-valued function on $\Gamma$, hence $\bar{\xi}(0) = \bar{\xi}(2\pi)$, we finally obtain [7]

$$L = \kappa x^a\big(\dot{\Pi}\Pi^{-1}\big)_a - s\big(\mathfrak{n}^{-1}\dot{\mathfrak{n}}\big)_0, \tag{17}$$

with the new variables of particle's position $x = x^a\tilde{P}_a \equiv \mathfrak{n}\bar{\xi}(0)\mathfrak{n}^{-1}$ and "group valued momentum" $\Pi$, defined as

$$\Pi \equiv \mathfrak{n}e^{-\frac{2\pi}{k}\mathcal{C}_J}\mathfrak{n}^{-1} = e^{-\frac{m}{\kappa}\mathfrak{n}J_0\mathfrak{n}^{-1}}. \tag{18}$$

Thus the Lagrangian (17) describes a deformed particle, whose momentum is now given by the group element $\Pi$ defined above instead of the algebra element $mJ_0$.

The group valued momentum $\Pi$ is not arbitrary, but is given by the conjugation of $e^{-\frac{2\pi}{k}\mathcal{C}_J}$ by the Lorentz group element $\mathfrak{n}$. If we parametrize $m\mathfrak{n}J_0\mathfrak{n}^{-1} = q^aJ_a$ then from (18) we find

$$\begin{aligned} &\Pi = p_3 - \frac{1}{\kappa}p^aJ_a, \qquad p_3^2 + \frac{1}{4\kappa^2}p_ap^a = 1, \\ &p_3 = \cos\left(\frac{|q|}{2\kappa}\right), \qquad p^a = 2\kappa\frac{q^a}{|q|}\sin\left(\frac{|q|}{2\kappa}\right). \end{aligned} \tag{19}$$

It follows that the momenta $p_a$ are coordinates on the three dimensional anti-de Sitter space constrained by the deformed mass shell condition. Introducing the Lagrange multiplier $\lambda$, the final action for the spinless case $\mathcal{C}_P = 0$ can be written in the components as

$$\begin{aligned} S = \int dt\bigg(&p_3\dot{p}_ax^a + \frac{1}{2\kappa}\epsilon_{abc}\dot{p}^ax^bp^c - \dot{p}_3p_ax^a\bigg) \\ &+ \lambda\left(p_ap^a - 4\kappa^2\sin^2\frac{m}{2\kappa}\right), \end{aligned} \tag{20}$$

where $p_3 \equiv \sqrt{1 - \frac{1}{4\kappa^2}p_ap^a}$. The detailed discussion of the properties of this action can be found in [6]. In the spinning case there is an additional term of the form $-s(n_3\dot{n}_0 - n_0\dot{n}_3 + \frac{1}{2}(n^1\dot{n}^2 - n^2\dot{n}^1))$.

Let us now turn to the main result of this paper. The contraction of the de Sitter group SO(3,1) to Poincaré group is pretty well known and the resulting Lagrangian (17) has been derived and thoroughly analyzed e.g., in [6,7]. It turns out, however, that there exist another contraction of the de Sitter group that to our knowledge has not been discussed in the literature. Contrary to the case considered above, where the translation sector of SO(3,1) was "flattened", we consider "flattening" of the Lorentz sector of SO(3,1).

To describe this new contraction let us return to the original algebra (1) and consider its Iwasawa decomposition into SO(2,1) and AN(2). The generators of the latter are defined as a linear combination of the original Lorentz and translation generators

$$S_a = P_a + \epsilon_{a0b}J^b, \tag{21}$$

so that we have

$$\begin{aligned} &[J_a, J_b] = \epsilon_{ab}{}^cJ_c, \qquad [J_a, S_b] = \epsilon_{ab}{}^cS_c - \eta_{ab}J_0 + \eta_{b0}J_a, \\ &[S_a, S_b] = \eta_{a0}S_b - \eta_{b0}S_a. \end{aligned} \tag{22}$$

The virtue of this decomposition is that the generators $J_a$ and $S_a$ form subalgebras of the algebra so(3,1); the price to pay, however, is that the cross commutators become quite complicated. Let us now rescale $\tilde{J}_a \equiv \sqrt{\Lambda}J_a$ to obtain

$$\begin{aligned} &[\tilde{J}_a, \tilde{J}_b] = \sqrt{\Lambda}\epsilon_{abc}\tilde{J}^c, \\ &[\tilde{J}_a, S_b] = \sqrt{\Lambda}\epsilon_{abc}S^c + (\eta_{b0}\tilde{J}_a - \eta_{ab}\tilde{J}_0), \\ &[S_a, S_b] = \eta_{a0}S_b - \eta_{b0}S_a, \end{aligned} \tag{23}$$

which after contraction $\Lambda \to 0$ takes the form

$$\begin{aligned} &[\tilde{J}_a, \tilde{J}_b] = 0, \qquad [\tilde{J}_a, S_b] = (\eta_{b0}\tilde{J}_a - \eta_{ab}\tilde{J}_0), \\ &[S_a, S_b] = \eta_{a0}S_b - \eta_{b0}S_a. \end{aligned} \tag{24}$$

It is worth mentioning that, as it was in the case of the Poincaré algebra above, the algebra (24) is a Lie algebra of the group $G \ltimes \mathfrak{g}^*$, where $G$ is now the group AN(2) generated by the last commutator in (24).

In terms of the new generators the scalar products read

$$\langle\tilde{J}_aS_b\rangle = \sqrt{\Lambda}\eta_{ab}, \qquad \langle\tilde{J}_a\tilde{J}_b\rangle = \langle S_aS_b\rangle = 0. \tag{25}$$

In spite of the fact that this scalar product becomes degenerate in the limit $\Lambda \to 0$, in the effective particle action $\Lambda$ is cancelled out and the contraction limit is not singular.

A gauge group element can now be decomposed into

$$\gamma = \mathfrak{j}\mathfrak{s} = \big(1 + \iota^a\tilde{J}_a\big)e^{\sigma^iS_i}e^{\sigma^0S_0}, \quad i = 1, 2, \tag{26}$$

where for $\mathfrak{s}$ we use the parametrization that proved convenient in the context $\kappa$-Poincaré theories [21] and is related to the other parametrization $\mathfrak{s} = \xi_3 + \xi^aS_a$ via $\sigma^0 = 2\log(\xi_3 + \frac{1}{2}\xi^0)$, $\sigma^i = (\xi_3 + \frac{1}{2}\xi^0)\xi^i$.

Since it is our goal to obtain a curved momentum space after the $\Lambda \to 0$ limit is taken, we must change the form of $\mathcal{C} = \mathcal{C}_J + \mathcal{C}_S$, describing the particle at rest, so as to have the mass in the $S$ sector. Adjusting dimensions properly we get $\mathcal{C}_J = s/\sqrt{\Lambda}\tilde{J}_0$, $\mathcal{C}_S = m/\sqrt{\Lambda}S_0$.

After these preparations we can return to the formulae (11), (12). Writing $N = (1+n)\mathfrak{h}$, with $\mathfrak{h} \in \mathrm{AN}(2)$, $n \in \mathrm{so}(2,1)$ and using the factorization (26) one first finds that

$$\mathfrak{s}^{-1} = \mathfrak{h}\exp\left(\frac{1}{k}\mathcal{C}_S\phi\right)\bar{\mathfrak{s}}^{-1}. \tag{27}$$

Next, from the commutation relations (24), for an arbitrary $\mathfrak{s}$ we have

$$\mathfrak{s}\exp\left(\frac{1}{k}\mathcal{C}_J\phi\right)\mathfrak{s}^{-1} = \exp\left(\frac{1}{k}\mathcal{C}_J\phi\right). \tag{28}$$

As a result one obtains the second condition

$$\mathfrak{u} = \exp\left(\frac{1}{k}\mathcal{C}_J\phi\right)\mathfrak{s}(1-n)\mathfrak{s}^{-1}, \tag{29}$$

where we denote $\mathfrak{u} \equiv \bar{\mathfrak{j}}^{-1}\mathfrak{j}$.

We now plug the factorization (26) into our starting Lagrangian (11), finding

$$L = \frac{k}{2\pi}\int_\Gamma\left\langle\dot{\mathfrak{u}}\mathfrak{u}^{-1}\left(d\bar{\mathfrak{s}}\bar{\mathfrak{s}}^{-1} - \bar{\mathfrak{s}}\frac{1}{k}\mathcal{C}d\phi\bar{\mathfrak{s}}^{-1}\right) + \frac{1}{k}\mathcal{C}d\phi\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}}\right\rangle. \tag{30}$$

Substituting the continuity conditions for $\mathfrak{u}$ and $\mathfrak{s}$ into (30) and keeping in mind the limit $\Lambda \to 0$ we obtain

$$L=\frac{k}{2\pi}\int\Big\langle\partial_0\big(\bar{\mathfrak{s}}e^{-\frac{1}{k}\mathcal{C}_S\phi}\mathfrak{h}^{-1}n\mathfrak{h}e^{\frac{1}{k}\mathcal{C}_S\phi}\bar{\mathfrak{s}}^{-1}\big)\times\left(-d\bar{\mathfrak{s}}\bar{\mathfrak{s}}^{-1}+\bar{\mathfrak{s}}\frac{1}{k}\mathcal{C}_S d\phi\bar{\mathfrak{s}}^{-1}\right)+\frac{1}{k}\mathcal{C}_J d\phi\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}}\Big\rangle. \tag{31}$$

In the last step, neglecting the total time derivative and integrating over the angular variable we eventually obtain the expression very similar to (17)

$$L=\frac{k}{2\pi}\langle\Pi\dot{\Pi}^{-1}x\rangle+\langle\mathcal{C}_J\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}}\rangle, \tag{32}$$

but with the deformed momenta $\Pi$ being now the elements of the AN(2) group, instead of the Lorentz group SO(2, 1), to wit

$$\Pi\equiv\bar{\mathfrak{s}}e^{\frac{2\pi}{k}\mathcal{C}_S}\bar{\mathfrak{s}}^{-1},\qquad x\equiv\bar{\mathfrak{s}}\mathfrak{h}^{-1}(0)n(0)\mathfrak{h}(0)\bar{\mathfrak{s}}^{-1}. \tag{33}$$

The Lagrangian (32) is the main result of our paper.

Since the Lorentz group SO(2, 1) is, as a manifold, the three dimensional anti-de Sitter space, while AN(2) is a submanifold of the three dimensional de Sitter space, we managed to obtain the momentum space of positive, instead of the negative, constant curvature. Moreover, contrary to (17), which is defined only in $2+1$ dimension, the expression (32) can be readily generalized to any spacetime dimension.

Let us now turn to the detailed discussion of the properties of Lagrangian (32). The first thing to notice is that the first equation in (33) puts severe restrictions on the form of momentum $\Pi$. Indeed if we write

$$\Pi=e^{p^i/\kappa S_i}e^{p^0/\kappa S_0}, \tag{34}$$

and take $\bar{\mathfrak{s}}$ to have the form

$$\bar{\mathfrak{s}}=e^{\bar{\sigma}^i S_i}e^{\bar{\sigma}^0 S_0}, \tag{35}$$

we immediately find that

$$p^0=m,\qquad p^i=\kappa\big(1-e^{\frac{m}{\kappa}}\big)\bar{\sigma}^i. \tag{36}$$

As it was in the case considered above these equations play a role of the mass shell relation, and force the energy to be constant, independently of the particle dynamics. In the undeformed case (which can be obtained in the limit $\kappa\to\infty$) such mass shell condition makes the particle effectively frozen, it cannot move, and for that reason, following [14] we call it the "Carroll particle".

In the following discussion we will consider only the spinless case. It turns out that this is in fact the general case, because the spin term does not contribute nontrivially to the equations of motion. Indeed an arbitrary variation $\delta\bar{\mathfrak{s}}=\varpi\bar{\mathfrak{s}}$, $\delta\bar{\mathfrak{s}}^{-1}=-\bar{\mathfrak{s}}^{-1}\varpi$ of this term results in a total time derivative

$$\delta\langle\mathcal{C}_J\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}}\rangle=\langle\bar{\mathfrak{s}}\mathcal{C}_J\bar{\mathfrak{s}}^{-1}\dot{\varpi}\rangle=\frac{d}{dt}\langle\mathcal{C}_J\varpi\rangle, \tag{37}$$

because $\tilde{J}_0$ commutes with all $S$ generators (cf. (24)).

Let us discuss the deformed Lagrangian in more details. Expressed in components of momentum $p_a$ it reads

$$L=x^0\dot{p}_0+x^i\dot{p}_i-\kappa^{-1}x^i p_i\dot{p}_0+\lambda\big(p_0^2-m^2\big), \tag{38}$$

where, as before, we introduce the Lagrange multiplier $\lambda$ to enforce the mass shell constraint. The equations of motion following from variations over $x$ are momentum conservations $\dot{p}_a=0$, while the ones resulting from the variation over momenta give

$$\dot{x}^0=2\lambda p_0=2\lambda m,\qquad \dot{x}^i=0, \tag{39}$$

so that indeed the Carroll particle is always at rest. Furthermore, from (33) we may also find the explicit expressions for the components $x^0=n^0-(e^{\zeta_0-\bar{\sigma}_0}\bar{\sigma}^i-\zeta^i)n_i$, $x^i=e^{\zeta_0-\bar{\sigma}_0}n^i$. Then (39) will give some conditions for coordinates of $n$, $\mathfrak{h}=e^{\zeta^i S_i}e^{\zeta^0 S_0}$ and $\bar{\mathfrak{s}}$.

The presence of the nonlinear, deformed term in the Lagrangian (38) results in the nontrivial Poisson bracket algebra of the $\kappa$-deformed phase space [22]

$$\{x^i,p_j\}=\delta^i_j,\qquad\{x^0,p_0\}=1,\qquad\{x^0,p_i\}=-\frac{1}{\kappa}p_i,\qquad \{x^0,x^i\}=\frac{1}{\kappa}x^i. \tag{40}$$

Meanwhile, symmetries of the action obtained from the Lagrangian (38) form the algebra of infinitesimal deformed Carroll transformations, containing

- rotations

$$\delta x^i=\rho\epsilon^i{}_j x^j,\qquad\delta p_i=\rho\epsilon_i{}^j p_j,\qquad\delta x^0=\delta p_0=0; \tag{41}$$

- deformed boosts

$$\delta x^0=\big(1+\kappa^{-1}p_0\big)\lambda_i x^i,\qquad\delta p_i=-\lambda_i p_0,\qquad \delta x^i=\delta p_0=0; \tag{42}$$

- deformed translations

$$\delta x^0=a^0,\qquad\delta x^i=e^{p_0/\kappa}a^i,\qquad\delta p_a=0; \tag{43}$$

- the spatial conformal transformation

$$\delta x^i=\eta x^i,\qquad\delta p_i=-\eta p_i,\qquad\delta x^0=\delta p_0=0, \tag{44}$$

where $\rho$, $\lambda_i$, $a^a$, $\eta$ are parameters of the respective transformations.

Actually, as noted in [14] the undeformed Carroll particle has an infinite dimensional symmetry. This property holds in the deformed case as well, and the generator of the infinitesimal transformations $\delta\phi^a=\{\phi^a,G\}$, where $\phi$ is an arbitrary function on phase space, is given by

$$G=f(p_0/\kappa)p_0\xi^0\big(x^i\big)+p_i\xi^i\big(x^i\big),\qquad f(0)=1, \tag{45}$$

where $f(p_0/\kappa)$ is an arbitrary function of energy, while $\xi^i(x^i)$, $\xi^0(x^i)$ are arbitrary functions of position.

Let us complete this paper with some comments.

First it should be stressed that the particle model we derived (32) can be extended to any number of spacetime dimensions $D+1$, simply by replacing the group AN(2) with AN($D$). This makes it potentially much more relevant for real physical systems than the model based on Poincaré group (17), whose application is strictly restricted to $2+1$ dimensions.

In our view, the most significant result of this paper is the derivation of a particle model with $\kappa$-deformed phase space (40) from the first principles as a deformation of the free particle model resulting from the interaction of the particle with its own gravitational field. However, it turns out that the model we obtained is not the $\kappa$-Poincaré particle discussed in the context of Doubly Special Relativity [10] or Relative Locality [23,21], but the deformed Carroll particle, with completely frozen dynamics. It is therefore still an open problem if the $\kappa$-Poincaré particle can be derived from the particle-gravity system as an effective deformed particle theory. We will revisit this issue in the forthcoming paper.

There is a curious similarity between Carrollian relativity, being the relativistic theory obtained in the limit $c\to 0$, in which no local interactions are possible and the very similar feature of the

$2+1$ gravity and the topological limit of gravity in $3+1$ dimensions [24–26]. Especially in the $3+1$ case it would be of interest to find out if gravity is described by a topological field theory in the Carrollian limit (for some discussion of this issue see [27]).

As already mentioned, the Carrollian limit appears in many distinct areas of theoretical physics, the common feature of whose is the presence of gravity in one form or another. The free and interacting particle models are extremely useful in that they help to grasp the underlying physics. It seems that the deformed model presented here, which already takes into account self-gravitational interactions might be of great interest, especially in the context of cosmological investigations [19].

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