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Quaternionic structures ☆

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ABSTRACT

Any oriented 4-dimensional real vector bundle is naturally a line bundle over a bundle of quaternion algebras. In this paper we give an account of modules over bundles of quaternion algebras, discussing Morita equivalence, characteristic classes and K-theory. The results have been used to describe obstructions for the existence of almost quaternionic structures on 8-dimensional Spin^c manifolds in Čadek et al. (2008) [5] and may be of some interest, also, in quaternionic and algebraic geometry.

1. Introduction

In this paper we will consider real and complex vector bundles which are modules over bundles of quaternion algebras. Our interest in such bundles began when we were working on the account [5] of obstruction theory on 8-manifolds. In the last sections of that paper we examined necessary and sufficient conditions for an 8-dimensional real vector bundle over an 8-manifold to have an almost quaternionic structure (which means that its structure group admits a reduction to $\mathrm{Sp}(1)\times_{\mathbb{Z}_2}\mathrm{Sp}(2)$). For this we needed some facts concerning bundles of quaternion algebras and K-theory of their modules. In the present paper we establish not only these facts and their generalizations (Theorems 3.1 and 5.8, Proposition 5.3), but we make a systematic study of algebraic and topological properties of modules over quaternion bundles.

Following classical ring theory we study Morita equivalence of bundles of quaternion algebras. For a given bundle of quaternion algebras we compute the Grothendieck group of left modules as a classical KO-group. For modules over a bundle of quaternion algebras we define characteristic classes and show that they behave as well as the Chern classes for complex vector bundles. We also characterize those complex vector bundles which are bundles over quaternion algebras. In the final section we examine bundles of complexified quaternion algebras and their Morita equivalence.

Topologically, our results extend the results obtained by Atiyah and Rees in [2] on complex quaternionic vector bundles and their K-theory and by Marchiafava and Romani [13–15] concerning characteristic classes. Geometrically, there are close connections to quaternionic geometry. The tangent bundles of Kaehler and almost hyper-Kaehler manifolds are modules over bundles of quaternion algebras (see Remark 1.6). The characteristic classes have been used in [18] to compute, in particular,

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the index of Salamon's elliptic complex of a quaternionic manifold. A possible application in algebraic geometry is outlined in Remark 6.9.

All bundles will be considered over a compact Hausdorff space X. Since the structure group of a bundle of quaternion algebras has to be $\mathrm{Aut}(\mathbb{H}) = \mathrm{SO}(3)$, every such bundle of quaternion algebras over X is of the form $\mathbb{R} \oplus \alpha$ where $\mathbb{R}$ is a trivial real line bundle and α is an oriented 3-dimensional orthogonal vector bundle over X.

Definition 1.1. Let α be an oriented 3-dimensional vector bundle over X with a positive-definite inner product. We write $\mathbb{H}_\alpha$ for the bundle of quaternion algebras $\mathbb{R}1 \oplus \alpha$ with the multiplication given in terms of the inner product $\langle -,- \rangle$ and vector product $\times$ by

$$(s,u)\cdot(t,v) = \bigl(st - \langle u,v\rangle, sv + tu + u \times v\bigr).$$

Alternatively, thinking of the group of automorphisms $\mathrm{Aut}(\mathbb{H})$ of $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\mathrm{i} \oplus \mathbb{R}\mathrm{j} \oplus \mathbb{R}\mathrm{k}$ as the special orthogonal group $\mathrm{SO}(3)$ of $\mathbb{R}^3 = \mathrm{im}\,\mathbb{H}$, we have

$$\mathbb{H}_\alpha = P \times_{\mathrm{Aut}(\mathbb{H})} \mathbb{H} \quad \text{and} \quad \alpha = P \times_{\mathrm{SO}(3)} \mathbb{R}^3,$$

where $P \to X$ is the principal $\mathrm{SO}(3) = \mathrm{Aut}(\mathbb{H})$-bundle given by α.

It is clear that different inner products on α define isomorphic bundles of quaternion algebras. Conjugation $^- : \mathbb{H} \to \mathbb{H}$ gives an isomorphism from $\mathbb{H}$ to the opposite algebra $\mathbb{H}^{\mathrm{o}}$ compatible with the action of $\mathrm{Aut}(\mathbb{H})$. Hence $\mathbb{H}_\alpha$ and $\mathbb{H}_\alpha^{\mathrm{o}}$ are isomorphic as bundles of algebras. Notice that $^- : \alpha \to \alpha$ (that is, -1) is an orientation-reversing isomorphism.

Definition 1.2. Let ξ be a real vector bundle over X. We say that ξ is an $\mathbb{H}_\alpha$*-bundle* if it has a left $\mathbb{H}_\alpha$-module structure, that is, a bundle map $\mathbb{H}_\alpha \otimes_{\mathbb{R}} \xi \to \xi$ that restricts to a module structure in each fibre.

Remark 1.3. An $\mathbb{H}_\alpha$-bundle ξ has a canonical orientation. To orient the fibre ξ_x at $x \in X$ we choose a basis $e_1, \ldots, e_n$ as $\mathbb{H}_{\alpha_x}$-vector space and choose an oriented orthonormal basis i, j, k of α_x. Then e_1, $\mathrm{i}e_1$, $\mathrm{j}e_1$, $\mathrm{k}e_1, \ldots, e_n$, $\mathrm{i}e_n$, $\mathrm{j}e_n$, $\mathrm{k}e_n$ orients ξ_x. In particular, the orientation of $\mathbb{H}_\alpha$ is determined by the orientation of α.

Remark 1.4. If ξ has an $\mathbb{H}_\alpha$-structure, we may (using a partition of unity to glue together local metrics) choose a positive-definite real inner product on ξ such that the structure homomorphism

$$\rho : \mathbb{H}_\alpha \to \mathrm{End}_{\mathbb{R}}(\xi)$$

is a $*$-homomorphism, that is, $\rho(r)^* = \rho(\bar{r})$ for $r \in \mathbb{H}_\alpha$.

Given a $4n$-dimensional real inner product space V with a left $\mathbb{H}$-module structure compatible, as above, with the inner product, we write $\mathrm{O}(V)$ and $\mathrm{Sp}(V)$ for the orthogonal and symplectic groups of V and define

$$\mathrm{TSp}(V) = \bigl\{ g \in \mathrm{O}(V) \bigm| g(rv) = \kappa(r)g(v) \text{ for some } \kappa \in \mathrm{Aut}(\mathbb{H}) \text{ and all } v \in V, r \in \mathbb{H} \bigr\}.$$

(The 'T' is intended to indicate 'twisted'.) It is evidently a subgroup of the special orthogonal group $\mathrm{SO}(V)$ and we have an extension

$$1 \to \mathrm{Sp}(V) \to \mathrm{TSp}(V) \to \mathrm{Aut}(\mathbb{H}) \to 1.$$

Since automorphisms of $\mathbb{H}$ are inner, we may equivalently describe $\mathrm{TSp}(V)$ as the subgroup $\mathrm{Sp}(1)\cdot\mathrm{Sp}(V) = (\mathrm{Sp}(1) \times \mathrm{Sp}(V))/\{\pm(1,1)\}$ of orthogonal maps of the form $v \mapsto a \cdot g(v)$, where $a \in \mathrm{Sp}(1) \subseteq \mathbb{H}$ and $g \in \mathrm{Sp}(V)$. The group $\mathrm{TSp}(V)$ acts (orthogonally) on $\mathbb{H}$ and on V.

Lemma 1.5. *Let ξ be a $4n$-dimensional orthogonal real vector bundle. Then ξ admits an $\mathbb{H}_\alpha$-structure for some α if and only if the structure group of ξ reduces from $\mathrm{O}(\mathbb{H}^n)$ to $\mathrm{TSp}(\mathbb{H}^n)$.*

Proof. If the structure group of ξ reduces from $\mathrm{O}(\mathbb{H}^n)$ to $\mathrm{TSp}(\mathbb{H}^n)$, there is a principal $\mathrm{TSp}(\mathbb{H}^n)$-bundle $P \to X$ such that $\xi = P \times_{\mathrm{TSp}(\mathbb{H}^n)} \mathbb{H}^n$. Then we have an oriented orthogonal 3-dimensional vector bundle $\alpha = P \times_{\mathrm{TSp}(\mathbb{H}^n)} \mathrm{im}\,\mathbb{H}$, with an associated quaternion algebra $\mathbb{H}_\alpha = P \times_{\mathrm{TSp}(\mathbb{H}^n)} \mathbb{H}$ having an obvious left action on ξ. The real vector space $\mathbb{H}^n$ has a canonical orientation as a left $\mathbb{H}$-module. Then the choice of a $\mathrm{TSp}(\mathbb{H}^n)$-principal bundle gives orientations to the bundles ξ and α such that the orientation of ξ is canonical with respect to $\mathbb{H}_\alpha$-structure.

If ξ admits an $\mathbb{H}_\alpha$-structure for some α, then the bundle of frames

$$\begin{aligned}\mathrm{Fr}(\xi) = \bigl\{ f \in \mathrm{Hom}\bigl(\mathbb{H}^n, \xi\bigr) \bigm| & f \text{ is a real isometry, } f(rv) = \rho(r)f(v) \\ & \text{for some } \rho \in \mathrm{Iso}(\mathbb{H}, \mathbb{H}_\alpha) \text{ and all } v \in \mathbb{H}^n, r \in \mathbb{H} \bigr\}\end{aligned}$$

is a principal $\mathrm{TSp}(\mathbb{H}^n)$-bundle. As above, $\xi = \mathrm{Fr}(\xi) \times_{\mathrm{TSp}(\mathbb{H}^n)} \mathbb{H}^n$ and $\mathbb{H}_\alpha = \mathrm{Fr}(\xi) \times_{\mathrm{TSp}(\mathbb{H}^n)} \mathbb{H}$. Hence the structure group of ξ reduces from $\mathrm{O}(\mathbb{H}^n)$ to $\mathrm{TSp}(\mathbb{H}^n)$. □

Remark 1.6. In quaternionic geometry [16,17] a smooth $4n$-manifold M is said to be almost quaternionic if its tangent bundle is associated to a principal $\mathrm{GL}(n,\mathbb{H}) \cdot \mathrm{Sp}(1)$-bundle. After a choice of compatible metric this principal bundle reduces to a principal $\mathrm{TSp}(n)$-bundle P. The sphere bundle $S(\alpha)$ of the vector bundle $\alpha = P \times_{\mathrm{TSp}(n)} \mathrm{im}\,\mathbb{H}$ is called the twistor space of M.

Considering $\mathbb{H}$ as a 2-dimensional complex vector space with multiplication by complex numbers from the right and with the standard left action of $\mathrm{Sp}(1)$, denote by $S^2_{\mathbb{C}}\mathbb{H}$ the second symmetric power with the induced action of $\mathrm{Sp}(1)$. One can show that $\alpha \otimes \mathbb{C}$ is isomorphic to the complex vector bundle $S^2 H = P \times_{\mathrm{TSp}(n)} S^2\mathbb{H}$. The isomorphism is induced by the $\mathrm{Sp}(1)$-invariant homomorphism of real representations $\varphi : \mathrm{im}\,\mathbb{H} \to S^2\mathbb{H}$, $\varphi(u) = \mathrm{j} \otimes u - 1 \otimes u\mathrm{j}$. Formally, $S^2 H$ is the second symmetric power of a vector bundle H. Such a bundle H exists globally if $w_2(\alpha) = 0$. Then α is associated to a principal $\mathrm{Spin}(3) = \mathrm{SU}(2)$-bundle Q, $H = Q \times_{\mathrm{SU}(2)} \mathbb{H}$, and the twistor space is the complex projective bundle $\mathbb{C}P(H)$ [16].

As a specific example we have the quaternionic projective space $\mathbb{H}P(\mathbb{H}^{n+1})$ where α is the Lie algebra bundle.

For any $\mathbb{H}_\alpha$-vector bundle ξ one can form the associated projective bundle, which we denote by $\mathbb{H}_\alpha P(\xi)$.

2. Quaternionic line bundles and Morita equivalence

Since $\mathrm{TSp}(\mathbb{H}) = \mathrm{SO}(4)$, every oriented 4-dimensional real vector bundle μ has an $\mathbb{H}_\alpha$-structure for some α. In this section we describe all such structures and their properties and define the notion of Morita equivalence of bundles of quaternion algebras.

Recall the double covers

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4) = \mathrm{SO}(\mathbb{H}) \xrightarrow{(\rho_+,\rho_-)} \mathrm{SO}(3) \times \mathrm{SO}(3)$$

given by mapping $(a,b) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ to the map $g : v \mapsto av\bar{b}$ in $\mathrm{SO}(\mathbb{H})$ and g to $(\rho_+(g), \rho_-(g)) = (\rho(a), \rho(b))$, where $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ maps a to $v \mapsto av\bar{a}$ (the adjoint representation).

This leads to a complete description of the twisted quaternionic line bundles.

Proposition 2.1. *Let μ be an oriented 4-dimensional orthogonal vector bundle over X. Write $\alpha = \rho_+(\mu)$ and $\beta = \rho_-(\mu)$. Then μ is an $\mathbb{H}_\alpha$-line bundle and a right $\mathbb{H}_\beta$-line bundle, and there is a canonical isomorphism (of bundles of algebras)*

$$\mathbb{H}_\alpha \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{o}}_\beta = \mathrm{End}_{\mathbb{R}}(\mu).$$

Conversely, if α is an oriented orthogonal 3-dimensional vector bundle and μ is an orthogonal $\mathbb{H}_\alpha$-line bundle, then μ acquires an orientation under which $\rho_+(\mu)$ is identified with α and $\beta = \rho_-(\mu)$ is characterized by an isomorphism (of bundles of algebras)

$$\mathbb{H}^{\mathrm{o}}_\beta = \mathrm{End}_{\mathbb{H}_\alpha}(\mu).$$

Moreover, we have

$$w_2(\alpha) = w_2(\beta) = w_2(\mu).$$

Proof. Let μ be an oriented 4-dimensional orthogonal vector bundle. There is a principal $\mathrm{SO}(\mathbb{H}) = \mathrm{TSp}(1)$-bundle P such that $\mu = P \times_{\mathrm{SO}(\mathbb{H})} \mathbb{H}$. From the definition, $\mathbb{H}_\alpha = P \times_{\rho_+} \mathbb{H}$ and $\mathbb{H}_\beta = P \times_{\rho_-} \mathbb{H}$, where ρ_+ and ρ_- determine the respective actions of $\mathrm{TSp}(1)$ on $\mathbb{H}$. Then μ is a left $\mathbb{H}_\alpha$-line and a right $\mathbb{H}_\beta$-line. Next

$$\mathrm{End}_{\mathbb{R}}(\mu) = P \times_{\mathrm{SO}(\mathbb{H})} \mathrm{End}_{\mathbb{R}}(\mathbb{H}),$$

where the action of $\mathrm{SO}(\mathbb{H})$ on $\mathrm{End}_{\mathbb{R}}(\mathbb{H})$ is $(a,b) \cdot f : v \mapsto af(\bar{a}vb)\bar{b}$. Similarly,

$$\mathbb{H}_\alpha \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{o}}_\beta = P \times_{\rho_+ \otimes \rho_-} \left(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{o}}\right),$$

where $\rho_+ \otimes \rho_-$ acts on $\mathbb{H} \otimes \mathbb{H}^{\mathrm{o}}$ by $(a,b) \cdot (h_1 \otimes h_2) = ah_1\bar{a} \otimes bh_2\bar{b}$. Since the isomorphism of algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{o}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ given by $h_1 \otimes h_2 \mapsto (v \mapsto h_1 v h_2)$ is invariant under the actions of $\mathrm{SO}(\mathbb{H})$ described above, we get $\mathbb{H}_\alpha \otimes \mathbb{H}^{\mathrm{o}}_\beta = \mathrm{End}_{\mathbb{R}}(\mu)$.

Conversely, suppose that μ is an orthogonal $\mathbb{H}_\alpha$-line. On μ consider the canonical orientation given by the $\mathbb{H}_\alpha$-structure. As in the proof of Lemma 1.5 we can construct a principal $\mathrm{TSp}(1)$-bundle P such that $\mu = P \times_{\mathrm{TSp}(1)} \mathbb{H}$ and $\mathbb{H}_\alpha = P \times_{\rho_+} \mathbb{H}$. (This gives μ the canonical orientation with respect to $\mathbb{H}_\alpha$.) Hence $\alpha = \rho_+(\mu)$.

Further, we have the monomorphism of algebras $\mathbb{H}^{\mathrm{o}}_\beta \hookrightarrow \mathbb{H}_\alpha \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{o}}_\beta = \mathrm{End}_{\mathbb{R}}(\mu)$. Its image is $\mathrm{End}_{\mathbb{H}_\alpha}(\mu)$. Consequently, $\mathbb{H}^{\mathrm{o}}_\beta = \mathrm{End}_{\mathbb{H}_\alpha}(\mu)$.

Without loss of generality we can assume that X is a finite CW-complex. Every 4-dimensional real vector bundle μ over the 3-skeleton of X has a section. If it is an $\mathbb{H}_\alpha$-line and a right $\mathbb{H}_\beta$-line, then over the 3-skeleton $\mu = \mathbb{H}_\alpha = \mathbb{R} \oplus \alpha$ and $\mu = \mathbb{H}^{\mathrm{o}}_\beta = \mathbb{R} \oplus \beta$. Consequently, $w_2(\mu) = w_2(\alpha) = w_2(\beta)$. □

Lemma 2.2. *The vector bundles α and β defined in the previous proposition are isomorphic to the bundles of eigenspaces of the Hodge star operator on $\Lambda^2\mu$ corresponding to the eigenvalues 1 and -1, respectively, and consequently*

$$\alpha \oplus \beta = \Lambda^2\mu.$$

Proof. On $\mathbb{H}$ consider the real inner product $\langle u, v\rangle = \mathrm{Re}(u\bar{v})$. The inner product enables us to define the isomorphism of real vector spaces $\mathbb{H}\otimes\mathbb{H}\cong \mathrm{End}_{\mathbb{R}}(\mathbb{H}) : h_1\otimes h_2 \mapsto (v\mapsto \langle h_1, v\rangle h_2)$. Under this isomorphism the space $\Lambda^2\mathbb{H}$ maps onto the subspace of all skew-symmetric endomorphisms on $\mathbb{H}$. Since the left multiplication L_h and the right multiplication R_h by a pure imaginary quaternion h are skew-symmetric endomorphisms of $\mathbb{H}$, we get two inclusions $\Lambda^+ : \mathrm{im}\,\mathbb{H}\to\Lambda^2\mathbb{H}$ and $\Lambda^- : \mathrm{im}\,\mathbb{H}\to\Lambda^2\mathbb{H}$ which are isometries (up to a multiple). It is easy to show that

$$\Lambda^+(h) = 1\wedge h + \mathrm{i}\wedge h\mathrm{i} + \mathrm{j}\wedge h\mathrm{j} + \mathrm{k}\wedge h\mathrm{k}, \qquad \Lambda^-(h) = 1\wedge h + \mathrm{i}\wedge \mathrm{i}h + \mathrm{j}\wedge \mathrm{j}h + \mathrm{k}\wedge \mathrm{k}h.$$

Put $\omega = 1\wedge\mathrm{i}\wedge\mathrm{j}\wedge\mathrm{k}$. Then one can check that

$$\Lambda^+(h_1)\wedge\Lambda^+(h_2) = 8\langle h_1, h_2\rangle\omega, \qquad \Lambda^-(h_1)\wedge\Lambda^-(h_2) = -8\langle h_1, h_2\rangle\omega, \qquad \Lambda^+(h_1)\wedge\Lambda^-(h_2) = 0,$$

which means that $\Lambda^+(\mathrm{im}\,\mathbb{H})$ and $\Lambda^-(\mathrm{im}\,\mathbb{H})$ correspond to the perpendicular eigenspaces for eigenvalues 1 and -1, respectively, of the Hodge star operator. Hence we get an isomorphism $\Lambda^+(\mathrm{im}\,\mathbb{H})\oplus\Lambda^-(\mathrm{im}\,\mathbb{H}) = \Lambda^2\mathbb{H}$.

We have associated vector bundles $\Lambda^2\mu = P\times_{\mathrm{SO}(\mathbb{H})}\Lambda^2\mathbb{H}$, $\alpha = \rho_+(\mu) = P\times_{\rho_+}\mathrm{im}\,\mathbb{H}$, $\beta = \rho_-(\mu) = P\times_{\rho_-}\mathrm{im}\,\mathbb{H}$. Since the inclusions Λ^+, Λ^- are invariant with respect to actions of $\mathrm{SO}(\mathbb{H})$ given on $\mathrm{im}\,\mathbb{H}$ by ρ_+ and ρ_-, respectively, we may define $\Lambda^+ : \alpha\to\Lambda^2\mu$, $\Lambda^- : \beta\to\Lambda^2\mu$. The considerations above imply that $\Lambda^+(\alpha)\oplus\Lambda^-(\beta) = \Lambda^2\mu$. $\square$

Corollary 2.3. *Every 4-dimensional oriented orthogonal vector bundle μ is a (left) module over just two (up to orientation preserving isomorphism) quaternion algebras, namely $\mathbb{H}_\alpha$ and $\mathbb{H}_\beta$ where $\alpha = \rho_+(\mu)$ and $\beta = \rho_-(\mu)$. The orientation of μ is canonical with respect to the $\mathbb{H}_\alpha$-structure and is not canonical with respect to the $\mathbb{H}_\beta$-structure. If $-\mu$ is the same vector bundle with opposite orientation then*

$$\rho_+(-\mu) = \rho_-(\mu), \qquad \rho_-(-\mu) = \rho_+(\mu).$$

Proof. An orientation of μ is determined by the choice of the $\mathrm{SO}(\mathbb{H})$-principal bundle P such that $\mu = P\times_{\mathrm{SO}(\mathbb{H})}\mathbb{H}$ and by the standard orientation of $\mathbb{H}$. The standard orientation of $\mathrm{im}\,\mathbb{H}$ gives an orientation to $\alpha = P\times_{\rho_+}\mathrm{im}\,\mathbb{H}$ and $\beta = P\times_{\rho_-}\mathrm{im}\,\mathbb{H}$. Hence μ has the canonical orientation as an $\mathbb{H}_\alpha$-module which coincides with the canonical orientation of μ as a right $\mathbb{H}_\beta$-module (given by the multiplication by i, j, k from the right). Using the previous proposition there are no other (up to orientation preserving isomorphisms) left and right quaternion structures on μ with compatible orientations. Since the left action of $\mathbb{H}_\beta$ is given by the right action of $\mathbb{H}_\beta$ and by the orientation reversing conjugation $^- : \mathbb{H}_\beta\to\mathbb{H}_\beta$, the orientation of μ is not canonical with respect to the left $\mathbb{H}_\beta$-structure. This implies the formulas. $\square$

Lemma 2.4. *The first Pontryagin classes of α and β are*

$$p_1(\alpha) = p_1(\mu) + 2e(\mu), \qquad p_1(\beta) = p_1(\mu) - 2e(\mu).$$

Proof. Let H be the Hopf $\mathbb{H}_\alpha$-line bundle over the quaternionic projective bundle $\mathbb{H}_\alpha P(\mathbb{H}_\alpha^\infty)$. Since any $\mathbb{H}_\alpha$-line bundle μ is a summand of a trivial $\mathbb{H}_\alpha$-bundle over X, there is a section $s : X\to\mathbb{H}_\alpha P(\mathbb{H}_\alpha^\infty)$ such that $\mu = s^*(H)$. So it is enough to check the formulas for the bundle H. By the Leray–Hirsch theorem, $H^*(\mathbb{H}_\alpha P(\mathbb{H}_\alpha^\infty)) = H^*(X)[e]$, where e is the Euler class of H. Further, the pullback of H to the sphere bundle $S(\mathbb{H}_\alpha^\infty)$ is $\mathbb{H}_\alpha$. So we have $p_1(H) = p_1(\mathbb{H}_\alpha) + ae$, where $a\in\mathbb{Z}$. Restricting H to a point in X we get Hopf bundle over $\mathbb{H}P^\infty$ and determine that $a = -2$. $\square$

Definition 2.5. Two bundles $\mathbb{H}_\alpha$ and $\mathbb{H}_\beta$ are said to be *Morita equivalent* if there is a 4-dimensional real vector bundle μ and an isomorphism of bundles of algebras

$$\mathbb{H}_\alpha\otimes_{\mathbb{R}}\mathbb{H}_\beta^{\mathrm{o}}\cong\mathrm{End}_{\mathbb{R}}(\mu).$$

Such an isomorphism defines a morphism in the following category $\mathcal{M}(X)$. The objects of $\mathcal{M}(X)$ are the oriented 3-dimensional vector bundles over X. A morphism $\beta\to\alpha$ is represented by an isomorphism $\mathbb{H}_\alpha\otimes_{\mathbb{R}}\mathbb{H}_\beta^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu)$, two such isomorphisms $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu)$ and $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu')$ being regarded as the same if μ and μ' are isomorphic as $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}$-modules. The identity $\alpha\to\alpha$ is given by $\mu = \mathbb{H}_\alpha$. Composition is the tensor product: $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu)$ and $\mathbb{H}_\beta\otimes\mathbb{H}_\gamma^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\nu)$ compose to $\mathbb{H}_\alpha\otimes\mathbb{H}_\gamma^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu\otimes_{\mathbb{H}_\beta}\nu)$. The category $\mathcal{M}(X)$ is a groupoid with the inverse of $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu)$ given by the dual $\mathbb{H}_\beta\otimes\mathbb{H}_\alpha^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu^*)$ (or by $\mathbb{H}_\beta\otimes\mathbb{H}_\alpha^{\mathrm{o}}\to\mathrm{End}_{\mathbb{R}}(\mu)$, using the isomorphisms $\mathbb{H}_\alpha\to\mathbb{H}_\alpha^{\mathrm{o}}$ and $\mathbb{H}_\beta\to\mathbb{H}_\beta^{\mathrm{o}}$ and the inner product on μ).

Morita equivalence is usually defined as an equivalence of categories of left modules. In our case both definitions are equivalent (in Proposition 2.6 we prove one direction) and we have chosen the one which is more suited to our purposes.

Proposition 2.6. *An isomorphism $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}_\mathbb{R}(\mu)$ determines an equivalence from the category of $\mathbb{H}_\beta$-modules to the category of $\mathbb{H}_\alpha$-modules given by $\eta \mapsto \mu \otimes_{\mathbb{H}_\beta} \eta$.*

Proof. This is a consequence of the fact that $\mathcal{M}(X)$ is a groupoid. □

There is a topological criterion for Morita equivalence:

Theorem 2.7. *Two bundles $\mathbb{H}_\alpha$ and $\mathbb{H}_\beta$ are Morita equivalent if and only if $w_2(\alpha) = w_2(\beta)$.*

Proof. The commutative diagram

$$\begin{tikzcd}
\mathrm{SO}(4) \arrow[r] \arrow[d, "(\rho_+,\rho_-)"'] \arrow[drr, "\Lambda^2"] & \mathrm{SU}(4) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Spin}(6) \arrow[d] \\
\mathrm{SO}(3)\times\mathrm{SO}(3) \arrow[rr] & & \mathrm{SO}(6)
\end{tikzcd}$$

implies that the double cover $\mathrm{Spin}(6) \to \mathrm{SO}(6)$ pulls back, under the inclusion of $\mathrm{SO}(3)\times\mathrm{SO}(3)$ in $\mathrm{SO}(6)$, to $(\rho_+, \rho_-): \mathrm{SO}(4) \to \mathrm{SO}(3)\times\mathrm{SO}(3)$. Hence the obstruction to lifting from $\mathrm{SO}(3)\times\mathrm{SO}(3)$ to $\mathrm{SO}(4)$ is given by the sum of the second Stiefel–Whitney classes. □

Remark 2.8. In [9] the orthogonal Brauer group of a space X was defined as the quotient of the monoid of all bundles of simple central $\mathbb{R}$-algebras over X (with a multiplication induced by the tensor product) by the submonoid of all bundles of the form $\mathrm{End}_\mathbb{R}(\mu)$ where μ is a real vector bundle over X. The inverse is given by the opposite algebra. Two bundles $\mathbb{H}_\alpha$ and $\mathbb{H}_\beta$ are Morita equivalent if and only if they determine the same element of the Brauer group $\mathrm{BrO}(X)$. There is a group isomorphism

$$\mathrm{BrO}(X) \to H^0(X;\mathbb{Z}/2) \oplus H^2(X;\mathbb{Z}/2),$$

in which $\mathbb{H}_\alpha$ corresponds to $(1, w_2(\alpha))$.

If α and β are Morita equivalent, we can describe all the morphisms from α to β in $\mathcal{M}(X)$ from the knowledge of one of them.

Proposition 2.9. *Given an isomorphism $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}_\mathbb{R}(\mu)$ and a real line bundle δ we use the isomorphism $\mathrm{End}_\mathbb{R}(\mu) \to \mathrm{End}_\mathbb{R}(\delta \otimes \mu)$ defined by the tensor product with the identity on δ to get $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}_\mathbb{R}(\delta \otimes \mu)$. This defines a bijection from $H^1(X;\mathbb{Z}/2)$ to $\mathrm{Hom}_{\mathcal{M}(X)}(\beta,\alpha)$. In particular, the automorphism group $\mathrm{Aut}_{\mathcal{M}(X)}(\alpha)$ is isomorphic to $H^1(X;\mathbb{Z}/2)$.*

Proof. Isomorphisms $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}_\mathbb{R}(\mu)$ and $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}_\mathbb{R}(\mu')$ determine an isomorphism $f: \mathrm{End}_\mathbb{R}(\mu) \to \mathrm{End}_\mathbb{R}(\mu')$. Since every automorphism of $\mathrm{End}_\mathbb{R}(\mathbb{R}^4)$ is inner, we can define δ as the line bundle with the fibre at x generated by an isomorphism $g: \mu_x \to \mu'_x$ such that $f_x(a) = gag^{-1}$. Then the map $g \otimes v \mapsto g(v)$ gives an isomorphism $\delta \otimes \mu \to \mu'$. □

Corollary 2.10. *For a given 4-dimensional oriented orthogonal vector bundle μ the bundles $\alpha = \rho_+(\mu)$ and $\beta = \rho_-(\mu)$ are isomorphic if and only if μ has a 1-dimensional (not necessarily trivial) summand.*

With respect to the canonical orientations of α and β given by μ, this isomorphism can only be orientation reversing. The existence of a 1-dimensional summand is equivalent to the existence of an orientation reversing involution of μ.

Proof of Corollary 2.10. If $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\alpha \cong \mathrm{End}_\mathbb{R}(\mu)$, then by the previous proposition $\mu \cong \delta \otimes \mathbb{H}_\alpha$ for some δ which is thus a subbundle of μ. Conversely, if an $\mathbb{H}_\alpha$-bundle μ has a subbundle δ, then the multiplication gives an isomorphism $\mu \to \delta \otimes \mathbb{H}_\alpha$ which means that $\rho_+(\mu) = \rho_+(\mathbb{H}_\alpha) = \rho_-(\mathbb{H}_\alpha) = \rho_-(\mu)$. □

In Section 4 we will need

Lemma 2.11. *Let α, β, γ be oriented 3-dimensional vector bundles. Suppose that $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\beta \cong \mathrm{End}(\mu)$, $\mathbb{H}_\beta \otimes \mathbb{H}^o_\gamma \cong \mathrm{End}(\nu)$ so that $\mathbb{H}_\alpha \otimes \mathbb{H}^o_\gamma \cong \mathrm{End}(\mu \otimes_{\mathbb{H}_\beta} \nu)$. Give μ, ν and $\mu \otimes_{\mathbb{H}_\beta} \nu$ the canonical orientations. Then the Euler class of $\mu \otimes_{\mathbb{H}_\beta} \nu$ is*

$$e(\mu \otimes_{\mathbb{H}_\beta} \nu) = e(\mu) + e(\nu).$$

In other words, the Euler class gives a functor from $\mathcal{M}(X)$ to the group $H^4(X;\mathbb{Z})$.

Proof. As in the proof of Lemma 2.4 we use fibrewise classifying spaces. Let $P\mathbb{H}_\beta(\mathbb{H}_\beta^\infty)$ be the right quaternion projective bundle with the Hopf bundle H_1 (a right $\mathbb{H}_\beta$-line bundle) and let $\mathbb{H}_\beta P(\mathbb{H}_\beta^\infty)$ be the left quaternion projective bundle with the left Hopf bundle H_2. Then there are sections $s_1 : X \to P\mathbb{H}_\beta(\mathbb{H}_\beta^\infty)$ and $s_2 : X \to \mathbb{H}_\beta P(\mathbb{H}_\beta^\infty)$ such that $\mu = s_1^*(H_1)$ and $\nu = s_2^*(H_2)$. So it is enough to look at the fibre product $P\mathbb{H}_\beta(\mathbb{H}_\beta^\infty) \times_X \mathbb{H}_\beta P(\mathbb{H}_\beta^\infty)$ and to show that $e(H_1 \otimes_{\mathbb{H}_\beta} H_2) = e(H_1) + e(H_2)$.

Now $H^*(P\mathbb{H}_\beta(\mathbb{H}_\beta^\infty) \times_X \mathbb{H}_\beta P(\mathbb{H}_\beta^\infty)) = H^*(X)[e_1, e_2]$, where $e_i = e(H_i)$. But $e(H_1 \otimes H_2) = ae_1 + be_2$ with $a, b \in \mathbb{Z}$: it does not involve any element of $H^*(X)$ since the restriction of $H_1 \otimes_{\mathbb{H}_\beta} H_2$ to X is $\mathbb{H}_\beta$ with the Euler class equal to zero. So one can calculate by restricting to the two factors $P\mathbb{H}_\beta(\mathbb{H}_\beta^\infty)$ and $\mathbb{H}_\beta P(\mathbb{H}_\beta^\infty)$ (using the inclusion of X as $P\mathbb{H}_\beta(\mathbb{H}_\beta^1)$ or $\mathbb{H}_\beta P(\mathbb{H}_\beta^1)$). The class restricts to e_1 and to e_2. Hence $a = b = 1$. $\square$

3. *K*-theory

We define $K\mathrm{Sp}_\alpha^0(X)$ to be the Grothendieck group of (left) $\mathbb{H}_\alpha$-bundles over the compact Hausdorff space X. The aim of this section is to compute $K\mathrm{Sp}_\alpha^0(X)$ as a classical $K\mathrm{O}$-group. Let X^α stand for the Thom space of α.

Theorem 3.1. *There is an isomorphism*

$$K\mathrm{Sp}_\alpha^0(X) \to K\mathrm{O}^0(\mathbb{H}_\alpha) = \tilde{K}\mathrm{O}^{-1}\big(X^\alpha\big)$$

given by mapping the class $[\xi]$ to the element represented in $K\mathrm{O}$-theory with compact supports by the linear map

$$x \mapsto vx : \pi^*\xi \to \pi^*\xi$$

over $v \in \mathbb{H}_\alpha$, where $\pi : \mathbb{H}_\alpha \to X$ is the projection.

Remark 3.2. The K-groups give a functor $\alpha \mapsto K\mathrm{Sp}_\alpha^0(X)$ from $\mathcal{M}(X)$ to the category of abelian groups arising from the functor $\mu \otimes_{\mathbb{H}_\beta}$ described in Proposition 2.6. So a Morita equivalence $\mathbb{H}_\alpha \otimes \mathbb{H}_\beta^o \to \mathrm{End}_{\mathbb{R}}(\mu)$ determines an isomorphism $K\mathrm{Sp}_\beta^0(X) \to K\mathrm{Sp}_\alpha^0(X)$, which translates into the Bott isomorphism

$$K\mathrm{O}^0(\mathbb{H}_\beta) \to K\mathrm{O}^0(\mathbb{H}_\alpha)$$

given by an associated spin structure for the virtual real vector bundle $\mathbb{H}_\alpha - \mathbb{H}_\beta = \alpha - \beta$.

We shall derive Theorem 3.1 from the following two propositions. The first is the Karoubi–Segal periodicity theorem as given in [6, Theorem 6.1]. See also [1, Theorem 3.3] and [12, pages 193–194]. In the statement L is the representation $\mathbb{R}$ of $\mathbb{Z}/2$ with the action of the generator as multiplication by -1.

Proposition 3.3. *Let ζ be a real vector bundle over X. Then there is an isomorphism from the Grothendieck group $K\mathrm{O}_{C(\zeta)}(X)$ of graded $C(\zeta)$-modules, over the Clifford algebra $C(\zeta)$ of a positive-definite inner product on ζ, to the $\mathbb{Z}/2$-equivariant $K\mathrm{O}$-group of the total space of the $\mathbb{Z}/2$-equivariant vector bundle $L \otimes \zeta$:*

$$K\mathrm{O}_{C(\zeta)}(X) \to K\mathrm{O}^0_{\mathbb{Z}/2}(L \otimes \zeta),$$

given by mapping the class of a graded (left) $C(\zeta)$-module $\mu = \mu_0 \oplus \mu_1$ to the element represented by the linear map

$$x \mapsto vx : \pi^*\mu_0 \to L \otimes \pi^*\mu_1$$

over $v \in L \otimes \zeta$, where $\pi : \zeta \to X$ is the projection.

Suppose that an orthogonal vector bundle ζ has dimension $4k$ and is oriented. Then we may define a central involution $\omega_x \in C_0(\zeta_x)$ by $\omega_x = e_1 \cdots e_{4k}$, where $e_1, \ldots, e_{4k}$ is any positively oriented orthonormal basis of the fibre ζ_x at $x \in X$. Then any graded $C(\zeta)$-module $\mu = \mu_0 \oplus \mu_1$ splits as a direct sum of two graded submodules $\mu^+ = \mu_0^+ \oplus \mu_1^+$ and $\mu^- = \mu_0^- \oplus \mu_1^-$ such that ω_x acts as identity in the fibres of μ_0^+ and μ_1^-, and as multiplication by -1 in the fibres of μ_0^- and μ_1^+. We will say that μ is *positive* if $\mu = \mu^+$, and *negative*, if $\mu = \mu^-$. Then the Grothendieck group of graded $C(\zeta)$-modules splits as a sum of the Grothendieck groups of positive and negative modules $KO^+_{C(\zeta)}(X) \oplus KO^-_{C(\zeta)}(X)$. The periodicity theorem ([6, Proposition 6.3] or [7, Proposition 3.1]) gives an isomorphism between $KO^0_{\mathbb{Z}/2}(L \otimes \zeta)$ and $KO^0_{\mathbb{Z}/2}(\zeta)$. This, combined with the Karoubi–Segal theorem, establishes

Proposition 3.4. *Let ζ be an oriented real vector bundle over X with dimension a multiple of 4. Then there is an isomorphism from the Grothendieck group of positive $C(\zeta)$-modules to the $K\mathrm{O}$-theory of the total space of ζ*

$$KO^+_{C(\zeta)}(X) \to KO^0(\zeta),$$

given by mapping the positive $C(\zeta)$-module μ to the linear map

$$x \mapsto vx : \pi^*\mu_0 \to \pi^*\mu_1$$

over $v \in \zeta$.

Proof of Theorem 3.1. We apply the previous proposition with $\zeta = \mathbb{R} \oplus \alpha$. It will be convenient to name the generator of the first summand and write $\zeta = \mathbb{R}e \oplus \alpha$. We show that positive graded $C(\zeta)$-modules correspond to (ungraded) $\mathbb{H}_\alpha$-modules.

Indeed, let $\mu = \mu_0 \oplus \mu_1$ be a positive graded $C(\mathbb{R}e \oplus \alpha)$-module. Put $\xi = \mu_0$. The inclusion of $\mathbb{H}_\alpha = \mathbb{R}1 \oplus \alpha$ in $C_0(\mathbb{R}e \oplus \alpha)$ as $\mathbb{R}1 \oplus e\omega\alpha$ gives ξ an $\mathbb{H}_\alpha$-structure. Now $C_0(\mathbb{R}e \oplus \alpha) = \mathbb{H}_\alpha \oplus \mathbb{H}_\alpha\omega$, which, as a ring, is the product $\mathbb{H}_\alpha \times \mathbb{H}_\alpha$ (with the factors corresponding to the idempotents $(1 \pm \omega)/2$). In the opposite direction, given an $\mathbb{H}_\alpha$-bundle ξ, put $\mu_0 = \xi$. The inclusion $\mathbb{H}_\alpha \subseteq C_0(\mathbb{R}e \oplus \alpha)$ described above and the action of ω as the identity give μ_0 the structure of a $C_0(\zeta)$-module, which extends uniquely to a positive graded $C(\zeta)$-module μ. □

4. Characteristic classes

In this section we introduce characteristic classes for $\mathbb{H}_\alpha$-bundles, then describe their properties and relation to the Stiefel–Whitney, Euler and Pontryagin characteristic classes.

Given an $\mathbb{H}_\alpha$-module ξ of dimension n over X, we have an associated projective bundle $\mathbb{H}_\alpha P(\xi)$ over X and a Hopf $\mathbb{H}_\alpha$-line bundle H (which we endow with the canonical orientation). The following proposition constructs characteristic classes $d_i^\alpha(\xi)$.

Theorem 4.1. *For every $\mathbb{H}_\alpha$-bundle ξ of dimension n there are uniquely determined classes $d_i^\alpha(\xi) \in H^{4i}(X;\mathbb{Z})$, $1 \leqslant i \leqslant n$ such that the integral cohomology ring of $\mathbb{H}_\alpha P(\xi)$ is given by*

$$H^*\big(\mathbb{H}_\alpha P(\xi);\mathbb{Z}\big) = H^*(X)[t]/\big(t^n - d_1^\alpha(\xi)t^{n-1} + \cdots + (-1)^n d_n^\alpha(\xi)\big),$$

where $t = e(H) \in H^4(\mathbb{H}_\alpha P(\xi);\mathbb{Z})$ is the Euler class of the Hopf bundle H.

Proof. This follows at once from the Leray–Hirsch theorem, because the cohomology is freely generated as an $H^*(X;\mathbb{Z})$-module by $1, t, \dots, t^{n-1}$. □

To derive the properties of the characteristic classes d_i^α we will use a splitting principle for $\mathbb{H}_\alpha$-bundles, which follows from Proposition 4.1 by induction, see [13].

Proposition 4.2. *For each $\mathbb{H}_\alpha$-bundle ξ over X let $p : F(\xi) \to X$ be the bundle whose fibre at x is the flag manifold of orthogonal splittings of ξ_x as a sum $L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of $\mathbb{H}_{\alpha_x}$-lines. Then $p^*(\xi)$ splits as a direct sum of $\mathbb{H}_\alpha$-line bundles and $p^* : H^*(X;\mathbb{Z}) \to H^*(F(\xi);\mathbb{Z})$ is injective.*

To shorten our notation put $d^\alpha(\xi) = 1 + d_1^\alpha(\xi) + d_2^\alpha(\xi) + \cdots + d_n^\alpha(\xi)$. We also write $d_0^\alpha(\xi) = 1$. The classes $d^\alpha(\xi)$ have the properties that one would expect and determine the other characteristic classes of ξ as a real vector bundle.

Theorem 4.3.

(a) *The classes d^α are multiplicative, i.e. $d^\alpha(\xi \oplus \xi') = d^\alpha(\xi)d^\alpha(\xi')$ for $\mathbb{H}_\alpha$-vector bundles ξ and ξ'.*
(b) *If ξ is an $\mathbb{H}_\alpha$-bundle of dimension n with the canonical orientation, then its Euler class is $e(\xi) = d_n^\alpha(\xi)$.*
(c) *The Stiefel–Whitney classes of ξ are*

$$1 + w_1(\xi) + w_2(\xi) + \cdots + w_{4n}(\xi) = \sum_{i=0}^{n} \big(1 + w_2(\alpha) + w_3(\alpha)\big)^{n-i} d_i^\alpha(\xi).$$

In particular, $w_2(\xi) = nw_2(\alpha)$.
(d) *The Chern classes of the complexification of ξ (and so the Pontryagin classes $p_i(\xi) = (-1)^i c_{2i}(\mathbb{C} \otimes \xi)$) are given by*

$$\sum_{k=0}^{4n} c_k(\mathbb{C} \otimes \xi) = \sum_{i,j=0}^{n} q_{n-i,n-j}(\alpha) d_i^\alpha(\xi) d_j^\alpha(\xi),$$

where the classes $q_{i,j}(\alpha)$ are determined by the generating function

$$\sum_{i,j\geqslant 0} q_{i,j}(\alpha)s^i t^j = \frac{(1-s)(1-t)+(e(\alpha)^2-p_1(\alpha))st}{((1-s)^2+(e(\alpha)^2-p_1(\alpha))s^2)((1-t)^2+(e(\alpha)^2-p_1(\alpha))t^2)}.$$

Proof. (a) The proof is the same as in the case of the Stiefel–Whitney and Chern classes.

(b) If μ is an $\mathbb{H}_\alpha$-line, then $d_1^\alpha(\mu)=e(\mu)$ by definition. Now using the multiplicativity and the splitting principle, we get $d_n^\alpha(\xi)=e(\xi)$.

(c) There is a quotient map $\pi:\mathbb{R}P(\xi)\to\mathbb{H}_\alpha P(\xi)$ from the real projective bundle to the $\mathbb{H}_\alpha$-projective bundle. The Hopf $\mathbb{H}_\alpha$-bundle H lifts to the tensor product $\mathbb{H}_\alpha\otimes H_\mathbb{R}$ with the real Hopf bundle $H_\mathbb{R}$. So

$$\pi^*(t)=x^4+x^2w_2(\alpha)+xw_3(\alpha),$$

where $x=w_1(H_\mathbb{R})$. Comparing our definition of d^α with the corresponding definition of the Stiefel–Whitney classes (or using the splitting principle), we get our formula.

(d) For an $\mathbb{H}_\alpha$-line bundle μ we have

$$c_1(\mathbb{C}\otimes\mu)=0,\qquad c_2(\mathbb{C}\otimes\mu)=2e(\mu)-p_1(\alpha),\qquad c_3(\mathbb{C}\otimes\mu)=e(\alpha)^2,\qquad c_4(\mathbb{C}\otimes\mu)=e(\mu)^2.$$

From the splitting principle and multiplicativity, we can express the total Chern class of $\mathbb{C}\otimes\xi$ as

$$\prod_{j=1}^n\big(y_j^2+2ay_j+b\big),$$

where $d_i^\alpha(\xi)$ is the i-th elementary symmetric polynomial in $y_1,y_2,\dots,y_n$, $a=1$ and $b=1+e(\alpha)^2-p_1(\alpha)$. The stated formula is obtained by computing in the polynomial ring $\mathbb{Z}[a,b][y_1,u_2,\dots,y_n]$ on formal variables a, b, y_j by embedding $\mathbb{Z}[a,b]$ as a subring of the polynomial ring $\mathbb{Q}[r,s]$, where $a=r+s$ and $b=rs$. $\square$

Corollary 4.4. *If n is even and ξ admits an $\mathbb{H}_\alpha$-structure for some α, then ξ is spin. If n is odd and ξ admits an $\mathbb{H}_\alpha$-structure, then $w_2(\xi)=w_2(\alpha)$.* $\square$

Now we examine the relation between the characteristic classes and Morita equivalence.

Proposition 4.5. *Let $\mathbb{H}_\alpha\otimes\mathbb{H}_\beta^{\mathrm{o}}\cong\mathrm{End}_\mathbb{R}(\mu)$ and let η be an $\mathbb{H}_\beta$-bundle. Then the characteristic classes of the $\mathbb{H}_\alpha$-bundle $\xi=\mu\otimes_{\mathbb{H}_\beta}\eta$ are*

$$\sum_{i=0}^n d_i^\alpha(\xi)=\sum_{i=0}^n\big(1+e(\mu)\big)^{n-i}d_i^\beta(\eta).$$

Proof. For an $\mathbb{H}_\beta$-line bundle η this is Lemma 2.11. Applying multiplicativity (or by using the equivalence between $\mathbb{H}_\alpha P(\xi)$ and $\mathbb{H}_\alpha P(\eta)$ under which the Hopf bundles correspond by tensoring with μ), we get the formula. $\square$

Using the Gysin exact sequence as in [3] or in [4] one can describe the cohomology rings of $B\,\mathrm{TSp}(n)$. Denote by ρ_2 the reduction mod 2 and by Δ the corresponding Bockstein homomorphism. Let α and ξ be the associated vector bundles with fibres $\mathrm{im}\,\mathbb{H}$ and $\mathbb{H}^n$, respectively, to the classifying $\mathrm{TSp}(n)$-principal bundle $E\,\mathrm{TSp}(n)$ over $B\,\mathrm{TSp}(n)$. Put $w_2=w_2(\alpha)$, $p_1=p_1(\alpha)$ and $d_i=d_i^\alpha(\xi)$.

Theorem 4.6. *The cohomology rings of $B\,\mathrm{TSp}(n)$ are given by*

$$H^*\big(B\,\mathrm{TSp}(n);\mathbb{Z}/2\big)=\mathbb{Z}/2\big[w_2,Sq^1w_2,\rho_2d_1,\rho_2d_2,\dots,\rho_2d_n\big],$$
$$H^*\big(B\,\mathrm{TSp}(n);\mathbb{Z}\big)=\mathbb{Z}[\Delta w_2,p_1,d_1,d_2,\dots,d_n]/(2\Delta w_2).$$

The cohomology ring with $\mathbb{Z}/2$ coefficients was described in [14] and [15].

5. Complex quaternionic bundles

In this section we will deal with bundles of quaternion algebras which admit as a subbundle a bundle of fields of complex numbers. Modules over them are complex vector bundles where the complex structure extends to a quaternionic structure. We will characterize bundles of quaternion algebras which are Morita equivalent to such bundles of algebras (Theorem 5.8). At the end we will derive a relation between the Chern classes and the classes d_i^α introduced above (Theorem 5.10).

We start with the description of bundles of fields of complex numbers. See [8].

Definition 5.1. A principal $\mathrm{Aut}(\mathbb{C}) = \mathrm{O}(1)$-bundle over X determines an orthogonal real line bundle δ and a bundle of complex algebras $\mathbb{C}_\delta = \mathbb{R} \oplus \delta$. An element of the fibre δ_x with length 1 is a square-root of -1. We say that a real vector bundle ξ over X is a $\mathbb{C}_\delta$-bundle if it has a $\mathbb{C}_\delta$-module structure.

If ξ has a $\mathbb{C}_\delta$-structure, then as in Remark 1.4 there is a real inner product g on ξ such that the structure map $\mathbb{C}_\delta \to \mathrm{End}(\xi)$ is a $*$-homomorphism, i.e. $g(au, v) = g(u, \bar{a}v)$. Let i be an element of δ_x of the length 1 and put $f(u, v) = -\mathrm{i}g(\mathrm{i}u, v)$. The real bilinear form $f : \xi \times \xi \to \delta$ is non-singular skew-symmetric and $\langle u, v\rangle = g(u, v) + f(u, v)$ is a $\mathbb{C}_\delta$-inner product which is $\mathbb{C}_\delta$-linear in the first variable and $\mathbb{C}_\delta$-conjugate-linear in the second.

Conversely, if a $2n$-dimensional real vector bundle ξ is equipped with a non-singular skew-symmetric real bilinear form $\xi \times \xi \to \delta$, one can prove that there is a $\mathbb{C}_\delta$-structure on ξ. (See [8, Remark 5.5] or the proof of the similar Proposition 5.3 below.)

Given a $2n$-dimensional vector space V with a complex structure, we can choose a compatible real inner product on it. It enables us to introduce a subgroup

$$\mathrm{TU}(V) = \bigl\{ g \in \mathrm{O}(V) \bigm| g(rv) = \kappa(r) g(v) \text{ for some } \kappa \in \mathrm{Aut}(\mathbb{C}) \text{ and all } v \in V \bigr\}$$

of $\mathrm{O}(V)$. Thus, $\mathrm{TU}(V)$ consists of the $\mathbb{C}$-linear and the conjugate-linear isometries.

As in the case of quaternionic structures, a $2n$-dimensional orthogonal real vector bundle ξ admits a $\mathbb{C}_\delta$-structure for some δ if and only if its structure group $\mathrm{O}(\mathbb{C}^n)$ can be reduced to $\mathrm{TU}(\mathbb{C}^n)$.

Any 2-dimensional real vector bundle λ has a canonical structure as a $\mathbb{C}_\delta$-line bundle, where $\delta = \det \lambda$. Moreover, $\mathbb{C}_\delta$-line bundles over X are classified by their Euler class in the cohomology group $H^2(X; \mathbb{Z}(\delta))$ with integral coefficients twisted by δ. This also follows from the fact that $\mathrm{TU}(\mathbb{C}) = \mathrm{O}(2)$.

An n-dimensional $\mathbb{C}_\delta$-bundle has twisted Chern classes $c_j^\delta(\xi) \in H^{2j}(X; \mathbb{Z}(\delta^{\otimes j}))$ defined in the same way as the classes d_j^α for an $\mathbb{H}_\alpha$-bundle. And these determine the Stiefel–Whitney classes of ξ: in particular, $w_1(\xi) = n w_1(\delta)$ and $e(\xi) = c_n^\delta(\xi)$.

As in Section 3 one can show that the Grothendieck group $K_\delta^0(X)$ of $\mathbb{C}_\delta$-vector bundles is isomorphic to $K\mathrm{O}_{C(\mathbb{R}\oplus\delta)}(X)$ and, hence, to $K\mathrm{O}^0_{\mathbb{Z}/2}(L \otimes \mathbb{C}_\delta)$, and then define $K\mathrm{O}^i_\delta(X) = K\mathrm{O}^i_{\mathbb{Z}/2}(L \otimes \mathbb{C}_\delta)$ for $i \in \mathbb{Z}$. See [8].

We now consider the situation in which a bundle of quaternions $\mathbb{H}_\alpha$ admits a bundle of fields $\mathbb{C}_\delta$ as a subalgebra.

Proposition 5.2. *Let α be an* $\mathrm{SO}(3)$*-bundle. Then $\mathbb{H}_\alpha$ admits a subbundle of the form $\mathbb{C}_\delta$ if and only if $\alpha = \delta \oplus \lambda$, where λ is a 2-dimensional orthogonal real vector bundle and $\delta = \Lambda^2\lambda$ (or, in other words, if the structure group of α can be reduced to the subgroup* $\mathrm{O}(2) \subset \mathrm{SO}(3)$*).*

Proof. On $\mathbb{H}_\alpha$ consider a real inner product in which the multiplication by a given element from $\mathbb{H}_\alpha$ is a $*$-homomorphism. If $\mathbb{H}_\alpha$ admits a $\mathbb{C}_\delta$ as a sub-algebra then the bundle λ perpendicular to $\mathbb{C}_\delta$ is a $\mathbb{C}_\delta$-line bundle. Hence $\delta = \Lambda^2\lambda$ and $\alpha = \delta \oplus \lambda$. $\square$

We will denote the bundle of quaternions $\mathbb{H}_{\Lambda^2\lambda\oplus\lambda}$ determined by an $\mathrm{O}(2)$-bundle λ by simply $\mathbb{H}_\lambda$. The following propositions give two necessary and sufficient conditions for a vector bundle to have $\mathbb{H}_\lambda$-structure.

Proposition 5.3. *Fix a $\mathbb{C}_\delta$-line bundle λ. Let ξ be a $\mathbb{C}_\delta$-vector bundle. Then there is a natural correspondence, up to homotopy, between $\mathbb{H}_\lambda$-structures on ξ and non-singular skew-symmetric $\mathbb{C}_\delta$-bilinear forms $\xi \otimes_{\mathbb{C}_\delta} \xi \to \lambda$.*

Proof. We will carry out all the constructions in fibres at a given point $x \in X$. Suppose that ξ has an $\mathbb{H}_\lambda$-structure. Then ξ can be equipped with a real inner product from Remark 1.4 which is the first component of a $\mathbb{C}_\delta$-inner product as shown in the remark following Definition 5.1. For every $a \in \lambda_x$ the multiplication $\varphi_a(v) = a \cdot v$ defines a $\mathbb{C}_\delta$-conjugate linear map $\xi \to \xi$ such that $\varphi_a^2(v) = -|a|^2 v$, where $|a|$ is the norm given by the real inner product on $\mathbb{H}_\lambda$. For the adjoint we get $\varphi_a^* = -\varphi_a$. In the first place $*$ means the adjoint with respect to the real inner product, but this translates into the adjoint for the $\mathbb{C}_\delta$-inner product. (For a conjugate-linear map φ, the adjoint is defined so that $\langle \varphi(u), v\rangle = \langle \varphi^*(v), u\rangle = \overline{\langle u, \varphi^*(v)\rangle}$.) So we can define a skew-symmetric bilinear form $f : \xi \otimes_{\mathbb{C}_\delta} \xi \to \lambda$ by

$$\bigl\langle f(u, v), a\bigr\rangle = \bigl\langle u, \varphi_a(v)\bigr\rangle$$

using the $\mathbb{C}_\delta$-inner product. f is non-singular since φ_a is non-singular for $a \neq 0$.

Conversely, given f, we define a $\mathbb{C}_\delta$-conjugate linear map $\psi_a : \xi_x \to \xi_x$ by $\langle u, \psi_a(v)\rangle = \langle f(u, v), a\rangle$. Then $\psi_a^* = -\psi_a$ and $\psi_a \circ \psi_a^*$ is $\mathbb{C}_\delta$-linear and positive definite for $a \neq 0$. So we can define

$$\varphi_a = |a| \bigl(\psi_a \psi_a^*\bigr)^{-1/2} \psi_a$$

with $\varphi_a^2 = -|a|^2$ and $\varphi_a^* = -\varphi_a$.

These two constructions define maps from $\mathbb{H}_\lambda$-structures to non-singular skew-symmetric bilinear forms and back. Composing in one direction we get the identity on $\mathbb{H}_\lambda$-structures. In the other direction we get a homotopic form, through the homotopy $(|a|(\psi_a\psi_a^*)^{-1/2})^t$, $0 \leqslant t \leqslant 1$. $\square$

Proposition 5.4. *Let ξ be a $4n$-dimensional orthogonal real vector bundle. Then ξ admits an $\mathbb{H}_\lambda$-structure for some complex line bundle λ if and only if the structure group of ξ can be reduced from $\mathrm{SO}(\mathbb{H}^n)$ to $\mathrm{TSp}(\mathbb{H}^n) \cap \mathrm{U}(\mathbb{H}^n)$, and an $\mathbb{H}_\lambda$-structure for some $\mathbb{C}_\delta$-line bundle λ and some δ if and only if the structure group reduces to $\mathrm{TSp}(\mathbb{H}^n) \cap \mathrm{TU}(\mathbb{H}^n)$.*

Proof. Use Lemma 1.5 and the analogous statement for $\mathbb{C}_\delta$-structures. □

It is advantageous to express the intersections of the groups as quotients of products.

Lemma 5.5. *For any $4n$-dimensional real vector space V with a left $\mathbb{H}$-module structure*

$$\mathrm{TSp}(V) \cap \mathrm{U}(V) = \mathrm{U}(1) \cdot \mathrm{Sp}(V) = \big(\mathrm{U}(1) \times \mathrm{Sp}(V)\big)/(\mathbb{Z}/2),$$
$$\mathrm{TSp}(V) \cap \mathrm{TU}(V) = \mathrm{TU}(1) \cdot \mathrm{Sp}(V) = \big(\mathrm{TU}(1) \times \mathrm{Sp}(V)\big)/(\mathbb{Z}/2).$$

Proof. Let us prove only the second formula. Consider an element of $\mathrm{TSp}(V)$ given by an isomorphism $v \mapsto ag(v)$ where $a \in \mathrm{Sp}(1)$ and $g \in \mathrm{Sp}(V)$. This element lies in $\mathrm{TU}(V)$ if and only if it is a $\mathbb{C}$-linear or conjugate linear isometry. Since g is $\mathbb{C}$-linear, this means that for all $z \in \mathbb{C}$ either $az = za$ or $az = \bar{z}a$. Hence $a \in \mathrm{U}(1) \cup \mathrm{j}\mathrm{U}(1) = \mathrm{TU}(1) \subset \mathrm{Sp}(1)$. □

Now we return to the construction from the beginning of Section 2. A 2-dimensional complex bundle μ or, more generally, a 2-dimensional $\mathbb{C}_\delta$-vector bundle has a natural orientation as a 4-dimensional real vector bundle. The double cover $\mathrm{SO}(\mathbb{H}) \xrightarrow{(\rho_+,\rho_-)} \mathrm{SO}(3) \times \mathrm{SO}(3)$ considered in Proposition 2.1 restricts to maps

$$\mathrm{U}(2) = \mathrm{U}(\mathbb{H}) \to \mathrm{SO}(3) \times \mathrm{SO}(3), \qquad \mathrm{TU}(2) = \mathrm{TU}(\mathbb{H}) \to \mathrm{SO}(3) \times \mathrm{SO}(3).$$

Since $\mathrm{TSp}(\mathbb{H}) = \mathrm{SO}(\mathbb{H})$, we can apply Lemma 5.5 to get

$$\mathrm{U}(2) \cong \mathrm{U}(1) \cdot \mathrm{Sp}(1) = \mathrm{Spin}^c(3), \qquad \mathrm{TU}(2) \cong \mathrm{TU}(1) \cdot \mathrm{Sp}(1).$$

Lemma 5.6. *Let μ be a 2-dimensional $\mathbb{C}_\delta$-vector bundle. Then μ is an $\mathbb{H}_\lambda$-line bundle, where $\lambda = \Lambda^2_{\mathbb{C}_\delta}\mu$, and also an $\mathrm{End}_{\mathbb{H}_\lambda}(\mu)$-line bundle.*

Proof. It follows from Proposition 5.3 that μ has an $\mathbb{H}_\lambda$-structure. Then $\mathbb{H}^{\mathrm{o}}_{\rho_-(\mu)} = \mathrm{End}_{\mathbb{H}_\lambda}(\mu)$ by Proposition 2.1 and $\rho_-(\mu)$ is the bundle of skew-Hermitian endomorphisms of μ. □

The homomorphism ρ_- restricted to $\mathrm{TU}(2) \cong \mathrm{TU}(1) \cdot \mathrm{Sp}(1)$ is the projection onto $\mathrm{SO}(3)$. We prove that the structure group $\mathrm{SO}(3)$ of an oriented 3-dimensional real vector bundle β can be lifted to $\mathrm{TU}(2)$ if and only if there are a real line bundle δ and an element $l \in H^2(X; \mathbb{Z}(\delta))$ such that $w_2(\beta) = \rho_2 l + w_1^2(\delta)$.

The condition is necessary. According to Lemma 5.6 for a 2-dimensional $\mathbb{C}_\delta$-vector bundle μ with $\rho_-(\mu) = \beta$ we have $\rho_+(\mu) = \delta \oplus \lambda$ and Proposition 2.1 implies that $w_2(\beta) = w_2(\rho_+(\mu)) = \rho_2 e(\lambda) + w_1^2(\delta)$.

To show that the condition is sufficient take the $\mathbb{C}_\delta$-line bundle λ with the Euler class l. Then by Proposition 2.7 the quaternion bundles $\mathbb{H}_\lambda$ and $\mathbb{H}_\beta$ are Morita equivalent and the vector bundle μ from the definition of the equivalence has the structure group $\mathrm{TU}(2)$. So we obtain

Lemma 5.7. *Let β be an oriented 3-dimensional vector bundle with $w_2(\beta) = \rho_2(l) + w_1^2(\delta)$ for an element $l \in H^2(X; \mathbb{Z}(\delta))$ and a real line bundle δ. Then there is a 2-dimensional $\mathbb{C}_\delta$-vector bundle μ such that $\det_{\mathbb{C}_\delta} \mu = \lambda$, where $e(\lambda) = l$, and*

$$\mathbb{H}_\beta = \mathrm{End}_{\mathbb{H}_\lambda}(\mu)^{\mathrm{o}}.$$

For δ trivial, if β is a 3-dimensional Spin^c real vector bundle, then $\mathbb{H}_\beta$ is Morita equivalent to an $\mathbb{H}_\lambda$ for some complex line bundle λ.

Applying Proposition 2.6 we obtain:

Theorem 5.8. *Let η be a $4n$-dimensional real vector bundle. Then η admits an $\mathbb{H}_\beta$-structure with $w_2(\beta) = \rho_2(l) + w_1^2(\delta)$ for an $l \in H^2(X; \mathbb{Z}(\delta))$ and a real line bundle δ if and only if for the $\mathbb{C}_\delta$-line bundle λ with $e(\lambda) = l$ there exist an $\mathbb{H}_\lambda$-line bundle μ and an n-dimensional $\mathbb{H}_\lambda$-vector bundle ξ such that*

$$\eta \cong \mu^* \otimes_{\mathbb{H}_\lambda} \xi = \mathrm{Hom}_{\mathbb{H}_\lambda}(\mu, \xi).$$

The twisted quaternionic $\mathbb{H}_\beta$-structure is given by the action of the bundle $\mathrm{End}_{\mathbb{H}_\lambda}(\mu)^{\mathrm{o}}$.

The following statement is a complement to Proposition 2.1.

Lemma 5.9. *Suppose that the oriented orthogonal 4-dimensional bundle μ is an orthogonal direct sum $\mu_0 \oplus \mu_1$ of two 2-dimensional subbundles. Write $\delta = \det \mu_0 = \det \mu_1$, so that μ_0 and μ_1 become $\mathbb{C}_\delta$-bundles. Then $\rho_+(\mu) = \delta \oplus (\mu_0 \otimes_{\mathbb{C}_\delta} \mu_1)$ and $\rho_-(\mu) = \delta \oplus (\mu_0 \otimes_{\mathbb{C}_\delta} \overline{\mu_1})$, where the conjugate is given by conjugation on $\mathbb{C}_\delta$.*

Proof. The vector bundle $\mu = \mu_0 \oplus \mu_1$ has the structure group $\mathrm{O}(2) \cdot \mathrm{O}(2) = \mathrm{TU}(1) \cdot \mathrm{TU}(1) \subset \mathrm{TU}(2)$. So it follows from Lemma 5.6 that $\rho_+(\mu)$ is given by the projection onto the first factor $\mathrm{TU}(1)$ and is equal to $\delta \oplus \Lambda^2_{\mathbb{C}_\delta}(\mu_0 \oplus \mu_1) = \delta \oplus (\mu_0 \otimes_{\mathbb{C}_\delta} \mu_1)$.

Since the orientation of $\mu_0 \oplus \overline{\mu_1}$ is opposite to that of μ,

$$\rho_-(\mu) = \rho_+(\mu_0 \oplus \overline{\mu_1}) = \delta \oplus (\mu_0 \otimes_{\mathbb{C}_\delta} \overline{\mu_1}). \qquad \square$$

In the case that λ is a $\mathbb{C}_\delta$-line bundle, and $\alpha = \delta \oplus \lambda$ we can express the Chern classes c_i^δ in terms of the classes d_j^α.

Theorem 5.10. *Let $\alpha = \delta \oplus \lambda$, where λ is a $\mathbb{C}_\delta$-line bundle. Let ξ be an $\mathbb{H}_\alpha$-bundle of dimension n. Then*

$$1 + c_1^\delta(\xi) + \cdots + c_{2n}^\delta(\xi) = \sum_{i=0}^{n} \bigl(1 + c_1^\delta(\lambda)\bigr)^{n-i} d_i^\alpha(\xi).$$

In particular, $c_1^\delta(\xi) = nc_1^\delta(\lambda)$.

Proof. We use the splitting principle for $\mathbb{H}_\alpha$-vector bundles and multiplicativity of the Chern classes $c^\delta = 1 + c_1^\delta + c_2^\delta + \cdots$ and the characteristic classes d^α. Thus it is sufficient to carry out the proof only for $n = 1$. Let μ be an $\mathbb{H}_\lambda$-line, where λ is a $\mathbb{C}_\delta$-line. Then $c_1^\delta(\mu) = c_1^\delta(\lambda)$, since we may assume that X is a 3-dimensional CW-complex, over which $\mu = \mathbb{C}_\delta \oplus \lambda$. Further, $c_2^\delta(\mu) = e(\mu) = d_1^\alpha(\mu)$. Consequently,

$$1 + c_1^\delta(\mu) + c_2^\delta(\mu) = 1 + c_1^\delta(\lambda) + d_1^\alpha(\mu). \qquad \square$$

Remark 5.11. Consider an $\mathrm{SO}(3)$-bundle α and an $\mathbb{H}_\alpha$-bundle ξ over X. We can lift to the sphere bundle of α by $\pi : S(\alpha) \to X$. Over $S(\alpha)$ we have a complex line bundle λ such that $\pi^*\alpha = \mathbb{R} \oplus \lambda$. So $\pi^*\xi$ is an $\mathbb{H}_\lambda$-bundle. In particular, $\pi^*\xi$ is complex and has Chern classes in $H^*(S(\alpha))$. Since the Euler class of α is 2-torsion, we have a rational splitting: $H^{2i}(S(\alpha); \mathbb{Q}) = H^{2i}(X; \mathbb{Q}) \oplus H^{2(i-1)}(X; \mathbb{Q})$. More precisely, $H^*(S(\alpha); \mathbb{Q})$ is an $H^*(X; \mathbb{Q})$-module with a generator $s \in H^2(S(\alpha); \mathbb{Q})$ subject to a relation $s^2 + as + b = 0$ for some elements $a \in H^2(X; \mathbb{Q})$ and $b \in H^4(X; \mathbb{Q})$. If $e(\alpha) = 0$, this is true also over $\mathbb{Z}$.

Now using Theorem 5.10 and naturality of the classes d_i, the Chern classes of $\pi^*\xi$ are

$$1 + c_1\bigl(\pi^*\xi\bigr) + \cdots + c_{2n}\bigl(\pi^*\xi\bigr) = \sum_{i=0}^{n} \bigl(1 + c_1(\lambda)\bigr)^{n-1} \pi_* d_i^\alpha(\xi).$$

If X is an almost quaternionic smooth manifold as in Remark 1.6, then the tangent bundle of $S(\alpha)$ being isomorphic to $\lambda \oplus \pi^*\tau X$ has a complex structure, i.e. the twistor space $S(\alpha)$ is almost complex, which is well known in quaternionic geometry.

6. Complexified quaternionic bundles

Given a real line bundle δ and an oriented 3-dimensional vector bundle α (with inner product) we may consider the bundle of algebras $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$. It depends only on the 3-dimensional vector bundle $\delta \otimes \alpha$. Indeed, it may be identified with the (ungraded) Clifford algebra bundle $C(\alpha \otimes \delta)$ with the positive-definite quadratic form. Locally, if e_1, e_2, e_3 is an orthonormal basis of $\alpha \otimes \delta$ then the corresponding fibre of $C(\alpha \otimes \delta)$ is generated by e_1, e_2, e_3 with $e_i^2 = 1$ and $e_ie_j = -e_je_i$ for $i \neq j$, the fibre of $\mathbb{C}_\delta$ is $\mathbb{R}1 \oplus \mathbb{R}(e_1e_2e_3)$, and the fibre of $\mathbb{H}_\alpha$ is $\mathbb{R}1 \oplus \mathbb{R}(e_1e_2, e_2e_3, e_1e_3)$. The centre (formed in each fibre) is $\mathbb{C}_\delta$.

There is an $\mathbb{R}$-algebra isomorphism $\mathbb{H} \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{H})$, that is $M_2(\mathbb{C})$, under which the group of automorphisms $\mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) \times \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ maps to the retract $\mathrm{SU}(2)/\{\pm 1\} \rtimes \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ of the full automorphism group $\mathrm{GL}(2)/\mathbb{C}^* \rtimes \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ of $M_2(\mathbb{C})$.

Consequently, we can describe bundles of algebras $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$ as just those $\mathbb{R}$-algebra bundles with fibres of type $\mathbb{H} \otimes \mathbb{C}$ which have the structure group $\mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) \times \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}) = \mathrm{SO}(3) \times \mathrm{O}(1) = \mathrm{O}(3)$.

Consider a 4-dimensional $\mathbb{C}_\delta$-vector bundle μ which is a left $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-bundle. This means that the fibre is isomorphic to the complex vector space $\mathbb{H} \otimes \mathbb{C}$ with the obvious action of the algebra $\mathbb{H} \otimes \mathbb{C}$. On μ we can choose a real inner product such that the action $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta \to \mathrm{End}_{\mathbb{C}_\delta}(\mu)$ is a $*$-homomorphism. We will show that μ is associated to a principal bundle with the structure group $\mathrm{TSp}(\mathbb{H}) \cdot \mathrm{TU}(\mathbb{C}) = \mathrm{SO}(\mathbb{H}) \cdot \mathrm{TU}(\mathbb{C}) \subset \mathrm{TU}(\mathbb{H} \otimes \mathbb{C})$. In traditional terminology this is the group $(\mathrm{SO}(4) \times \mathrm{TU}(1))/\{\pm 1\} = \mathrm{SO}(4) \cdot \mathrm{TU}(1) \subset \mathrm{TU}(4)$. Put

$$\begin{aligned}\mathrm{Fr}(\mu) = \big\{ f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}\otimes\mathbb{C}, \mu) \bigm| & f \text{ preserves the inner product, } f(rv) = \kappa(r) f(v) \\ & \text{for some } \kappa \in \mathrm{Iso}(\mathbb{H}, \mathbb{H}_\alpha) \times \mathrm{Iso}(\mathbb{C}, \mathbb{C}_\delta) \text{ and all } v \in \mathbb{H}\otimes\mathbb{C}, r \in \mathbb{H}\otimes\mathbb{C}\big\}.\end{aligned}$$

Then $P = \mathrm{Fr}(\mu)$ is a principal bundle with the structure group $\mathrm{TSp}(\mathbb{H}) \cdot \mathrm{TU}(\mathbb{C})$ such that $\mu = P \times_{\mathrm{TSp}(\mathbb{H})\cdot\mathrm{TU}(\mathbb{C})} (\mathbb{H}\otimes\mathbb{C})$ and $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta = P \times_p (\mathbb{H}\otimes\mathbb{C})$, where $p : \mathrm{TSp}(\mathbb{H}) \cdot \mathrm{TU}(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) \times \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ is the projection. This proves

Proposition 6.1. *A* 4*-dimensional $\mathbb{C}_\delta$-vector bundle has an $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-module structure if and only if its structure group* $\mathrm{TU}(4)$ *can be reduced to* $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$.

Now we can proceed as in Section 2. In many of the arguments $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$ plays the role taken there by $\mathrm{SO}(4)$.

Since $\mathrm{TU}(1)$ is a semidirect product of groups $\mathrm{U}(1)$ and $\mathbb{Z}/2$, its elements can be described by pairs $(c, s) \in \mathrm{U}(1) \times \mathbb{Z}/2$ having the action on $\mathbb{C}$ given by $(c, 1)z = cz$ and $(c, -1)z = c\bar{z}$. Consider the covers

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{TU}(1) \to \mathrm{SO}(4) \cdot \mathrm{TU}(1) \xrightarrow{(\rho_+, \rho_-, \rho_0)} \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{TU}(1)$$

given by mapping $(a, b, (c, s)) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{TU}(1)$ to the map $g : \mathbb{H}\otimes\mathbb{C} \to \mathbb{H}\otimes\mathbb{C} : v \otimes z \mapsto av\bar{b} \otimes (c, s)z$ and g to $(\rho_+(g), \rho_-(g), \rho_0(g)) = (\rho(a), \rho(b), z \mapsto (c^2, s)z)$. Further, let $\tau : \mathrm{SO}(4) \cdot \mathrm{TU}(1) \to \mathrm{Aut}(\mathbb{C}) = \mathbb{Z}/2$ be the map $\tau(g) = s$.

Proposition 6.2. *Let μ be a* 4*-dimensional $\mathbb{C}_\delta$-vector bundle over X the structure group of which can be reduced to* $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$. *Put $\alpha = \rho_+(\mu)$, $\beta = \rho_-(\mu)$, $\sigma = \rho_0(\mu)$. Then $\mathbb{C}_\delta = \tau(\mu)$, the vector bundle μ is a left $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-module and a right $\mathbb{H}_\beta \otimes \mathbb{C}_\delta$-module, σ is a $\mathbb{C}_\delta$-line bundle and there is a canonical isomorphism of bundles of $\mathbb{C}_\delta$-algebras*

$$(\mathbb{H}_\alpha \otimes \mathbb{C}_\delta) \otimes_{\mathbb{C}_\delta} (\mathbb{H}_\beta \otimes \mathbb{C}_\delta)^{\mathrm{o}} = \big(\mathbb{H}_\alpha \otimes \mathbb{H}_\beta^{\mathrm{o}}\big) \otimes \mathbb{C}_\delta \cong \mathrm{End}_{\mathbb{C}_\delta}(\mu).$$

Moreover, there are canonical isomorphisms of vector bundles

$$(\alpha \oplus \beta) \otimes \sigma \cong \Lambda^2_{\mathbb{C}_\delta} \mu \quad \text{and} \quad \overline{\sigma}^{\otimes 2} \otimes_{\mathbb{C}_\delta} \Lambda^4 \mu \cong \mathbb{C}_\delta$$

where $\overline{\sigma}$ is the conjugate vector bundle to σ.

Conversely, if α is an oriented orthogonal 3*-dimensional vector bundle and μ is an $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-bundle of dimension* 4 *over $\mathbb{C}_\delta$, then the real form of $\overline{\sigma}^{\otimes 2} \otimes_{\mathbb{C}_\delta} \Lambda^4 \mu$ acquires an orientation under which $\rho_+(\mu)$ is identified with α, $\tau(\mu)$ is identified with $\mathbb{C}_\delta$ and $\beta = \rho_-(\mu)$ is characterized by an isomorphism* (*of bundles of algebras*)

$$(\mathbb{H}_\beta \otimes \mathbb{C}_\delta)^{\mathrm{o}} = \mathrm{End}_{\mathbb{H}_\alpha \otimes \mathbb{C}_\delta}(\mu).$$

Proof. The 4-dimensional $\mathbb{C}$-vector space $\mathbb{H}\otimes\mathbb{C}$ is the complexification of the real vector space $\mathbb{H}$. Now we can carry out the proof by complexifying all the vector spaces $\mathrm{im}\,\mathbb{H}$, $\mathrm{End}_{\mathbb{R}}(\mathbb{H})$, $\mathbb{H}\otimes\mathbb{H}^{\mathrm{o}}$, $\Lambda^2\mathbb{H}$, $\Lambda^4\mathbb{H}$ and the homomorphisms used in the proof of Proposition 2.1 and Lemma 2.2, by checking that these complexified homomorphisms are invariant with respect to appropriate actions of $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$ and by writing α, β, $\mathbb{C}_\delta$, σ, $\Lambda^2_{\mathbb{C}_\delta}\mu$ and $\Lambda^4_{\mathbb{C}_\delta}$ as vector bundles associated to an $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$-principal bundle determined by μ. The details are left to the reader. $\square$

Remark 6.3. The isomorphism $\Lambda^4_{\mathbb{C}_\delta}\mu \cong \sigma^{\otimes 2}$ implies that $c_1^\delta(\mu) = 2c_1^\delta(\sigma)$, and from the isomorphism $(\alpha \oplus \beta) \otimes \sigma \cong \Lambda^2_{\mathbb{C}_\delta}\mu$ we get $2c_2^\delta(\mu) = -p_1(\alpha) - p_1(\beta) + 3(c_1^\delta(\sigma))^2$.

Proposition 6.4. *Given* 3*-dimensional oriented vector bundles α, β and a $\mathbb{C}_\delta$-vector bundle σ, there is a* 4*-dimensional vector bundle μ such that $\rho_+(\mu) = \alpha$, $\rho_-(\mu) = \beta$ and $\rho_0(\mu) = \sigma$ if and only if $w_2(\sigma) = w_2(\alpha) + w_2(\beta)$.*

Proof. The commutative diagram

$$\begin{tikzcd}
\mathrm{SO}(4) \cdot \mathrm{TU}(1) \arrow[r] \arrow[d, "(\rho_+, \rho_-, \rho_0)"'] \arrow[drr] & \mathrm{SU}(4) \cdot \mathrm{TU}(1) \arrow[r, "\cong"] \arrow[dr, "\Lambda^2_{\mathbb{C}}"] & \mathrm{Spin}(6) \cdot \mathrm{TU}(1) \arrow[d] \\
\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{TU}(1) \arrow[rr] & & \mathrm{SO}(6) \cdot \mathrm{TU}(1)
\end{tikzcd}$$

implies that the double cover $\mathrm{Spin}(6) \cdot \mathrm{TU}(1) \to \mathrm{SO}(6) \cdot \mathrm{TU}(1)$ pulls back, under the homomorphism $(a, b, c) \mapsto (a + b) \otimes c$, to (ρ_+, ρ_-, ρ_0). Hence the obstruction to lifting from $\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{TU}(1)$ to $\mathrm{SO}(4) \cdot \mathrm{TU}(1)$ is given by $w_2(\sigma) = w_2(\alpha) + w_2(\beta)$. $\square$

Definition 6.5. For each δ we define a Morita category $\mathcal{M}_\delta(X)$ with objects the oriented 3-dimensional orthogonal vector bundles α, β and morphisms $\beta \to \alpha$ given by a $\mathbb{C}_\delta$-isomorphism of algebras

$$(\mathbb{H}_\alpha \otimes \mathbb{C}_\delta) \otimes_{\mathbb{C}_\delta} (\mathbb{H}_\beta \otimes \mathbb{C}_\delta)^{\mathrm{o}} = \big(\mathbb{H}_\alpha \otimes \mathbb{H}_\beta^{\mathrm{o}}\big) \otimes \mathbb{C}_\delta \to \mathrm{End}_{\mathbb{C}_\delta}(\mu),$$

where μ is a 4-dimensional $\mathbb{C}_\delta$-vector bundle, up to isomorphisms of μ.

Given any $\mathbb{C}_\delta$-line bundle λ, we use the isomorphism $\mathrm{End}_{\mathbb{C}_\delta}(\mu) = \mathrm{End}_{\mathbb{C}_\delta}(\lambda \otimes_{\mathbb{C}_\delta} \mu)$ to get an action of the Piccard group $\mathrm{Pic}_\delta(X) = H^2(X; \mathbb{Z}(\delta))$ on $\mathrm{Hom}_{\mathcal{M}_\delta(X)}(\beta, \alpha)$, and then, as in Proposition 2.9 we have $\mathrm{Aut}_{\mathcal{M}_\delta(X)}(\alpha) = \mathrm{Pic}_\delta(X)$.

Proposition 6.4 implies immediately

Theorem 6.6. *There is a morphism from β to α in $\mathcal{M}_\delta(X)$ if and only if $e(\alpha \otimes \delta) = e(\beta \otimes \delta)$.*

Proof. Denote the Bockstein homomorphism corresponding to the exact sequence $0 \to \mathbb{Z}(\delta) \to \mathbb{Z}(\delta) \to \mathbb{Z}/2 \to 0$ by Δ_δ. A $\mathbb{C}_\delta$-line bundle σ such that $w_2(\sigma) = w_2(\alpha) + w_2(\beta)$ exists if and only if $\Delta_\delta w_2(\alpha) = \Delta_\delta w_2(\beta)$ and this is equivalent to our condition, since $e(\alpha \otimes \delta) = \Delta_\delta(w_2(\alpha) + w_1^2(\delta))$. □

Proposition 6.7. *Any isomorphism $\mathbb{H}_\alpha \otimes \mathbb{H}_\beta^{\mathrm{o}} \otimes \mathbb{C}_\delta \cong \mathrm{End}_{\mathbb{C}_\delta}(\mu)$ determines an equivalence from the category of $\mathbb{H}_\beta \otimes \mathbb{C}_\delta$-modules to the category of $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-modules given by*

$$\eta \mapsto \mu \otimes_{\mathbb{H}_\beta \otimes \mathbb{C}_\delta} \eta.$$

As a consequence we get

Corollary 6.8. *Let $e(\alpha \otimes \delta) = 0$. Then there is an $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-bundle ω of $\mathbb{C}_\delta$-dimension 2 such that any $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-bundle of $\mathbb{C}_\delta$-dimension $2n$ is of the form $\omega \otimes_{\mathbb{C}_\delta} \zeta$ for a $\mathbb{C}_\delta$-bundle ζ of dimension n.*

Proof. Just as $\mathbb{H} \otimes \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\mathbb{H})$, we have $\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta = \mathrm{End}_{\mathbb{C}_\delta}(\mathbb{H}_{\mathbb{C}_\delta})$. Every $\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta$-bundle η is therefore of the form $\mathbb{H}_{\mathbb{C}_\delta} \otimes_{\mathbb{C}_\delta} \zeta$, where $\zeta = \mathrm{Hom}_{\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta}(\mathbb{H}_{\mathbb{C}_\delta}, \eta)$.

Since $e((\delta \oplus \mathbb{C}_\delta) \otimes \delta) = 0 = e(\alpha \otimes \delta)$, there is an isomorphism from $\delta \oplus \mathbb{C}_\delta$ to α in $\mathcal{M}_\delta(X)$ represented by a 4-dimensional vector bundle μ. Now any $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-bundle is of the form

$$\mu \otimes_{\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta} (\mathbb{H}_{\mathbb{C}_\delta} \otimes_{\mathbb{C}_\delta} \zeta) = (\mu \otimes_{\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta} \mathbb{H}_{\mathbb{C}_\delta}) \otimes_{\mathbb{C}_\delta} \zeta.$$

Putting $\omega = \mu \otimes_{\mathbb{H}_{\mathbb{C}_\delta} \otimes \mathbb{C}_\delta} \mathbb{H}_{\mathbb{C}_\delta}$, we get the assertion. □

Remark 6.9. Consider a compact Hausdorff space Y with an involution. Let E be a complex vector bundle over Y and let $J : \xi \to \xi$ be a conjugate-linear map lifting the involution on Y such that $J^2 = -1$. The pair (E, J) is variously called a quaternionic or symplectic bundle over Y [11,19,10]. We relate this notion to complexified quaternionic bundles in our sense.

Let $\mathbb{C}P(\mathbb{H})$ be the complex projective space modelled on the 2-dimensional complex vector space $\mathbb{H}$ with the involution given by multiplication by j. Define X to be the quotient of $Y \times \mathbb{C}P(\mathbb{H})$ by the free involution. Let δ be the real line bundle over X given by the double covering $p : Y \times \mathbb{C}P(\mathbb{H}) \to X$. We associate with E a vector bundle η such that the fibre over $x \in X$ is $\eta_x = E_y \oplus E_{y'}$ where $p^{-1}(x) = \{y, y'\}$. This bundle has an $\mathbb{H} \otimes \mathbb{C}_\delta$-structure. The multiplication by $\mathrm{j} \in \mathbb{H}$ is given by $\mathrm{j}(u, v) = (Jv, Ju)$. The $\mathbb{C}_\delta$-structure is defined as follows. Let t be the involution $(-1, 1) : E_y \oplus E_{y'} = \eta_x \to \eta_x$, and $t' = -t$. Then $t\mathrm{i} = \mathrm{i}t$ and $t\mathrm{j} = -\mathrm{j}t$. So $(\mathrm{i}t)\mathrm{i} = \mathrm{i}(\mathrm{i}t)$ and $(\mathrm{i}t)\mathrm{j} = \mathrm{j}(\mathrm{i}t)$, and $(\mathrm{i}t)^2 = -1$. So we can use $\mathrm{i}t = -\mathrm{i}t'$ to define the $\mathbb{C}_\delta$-structure commuting with the $\mathbb{H}$-multiplication.

In the same way the complex Hopf bundle H over $\mathbb{C}P(\mathbb{H})$ determines an $\mathbb{H} \otimes \mathbb{C}_\delta$-vector bundle ω over X of complex dimension 2 (where $\omega_x = H_y \oplus H_{y'}$). As in Corollary 6.8 there is a $\mathbb{C}_\delta$-bundle $\zeta = \mathrm{Hom}_{\mathbb{H} \otimes \mathbb{C}_\delta}(\omega, \eta)$ over X such that $\eta = \omega \otimes_{\mathbb{C}_\delta} \zeta$. Its lift to $Y \times \mathbb{C}P(\mathbb{H})$ is $E \otimes_{\mathbb{C}} H$.

Remark 6.10. One can define a Brauer group $\mathrm{BrU}_\delta(X)$ of central simple $\mathbb{C}_\delta$-algebras and show that it is isomorphic to $\mathrm{Tor}\, H^3(X; \mathbb{Z}(\delta))$. The class of $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta = C(\alpha \otimes \delta)$ is $e(\alpha \otimes \delta)$.

There is a complex K-theory of $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-vector bundles modelled on the real K-theory of Section 3.

Proposition 6.11. *There is an isomorphism from the Grothendieck group of $\mathbb{H}_\alpha \otimes \mathbb{C}_\delta$-modules over X to $K_\delta^0(\mathbb{H}_\alpha)$, the K_δ-theory with compact supports, given by mapping the class $[\xi]$ to the element represented by the linear map*

$$x \mapsto vx : \pi^*\xi \to \pi^*\xi$$

over $x \in \mathbb{H}_\alpha$, where $\pi : \mathbb{H}_\alpha \to X$ is the projection.

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