JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **154**, 391–402 (1991)

Lyapunov Functions on Product Spaces and Stability Theory of Delay Differential Equations

V. LAKSHMIKANTHAM

Florida Institute of Technology,
Melbourne, Florida 32901

S. LEELA

SUNY College at Geneseo,
Geneseo, New York 14454

AND

S. SIVASUNDARAM

Embry-Riddle Aeronautical University,
Daytona Beach, Florida 32014

Submitted by the Editors

Received August 8, 1989

1. INTRODUCTION

If we examine the Lyapunov functionals constructed for all the examples that have been discussed so far in the literature, we find that the investigators, inadvertently, employ a combination of a Lyapunov function and a functional in such a way that the corresponding derivative can be estimated suitably without demanding minimal classes of functions or the knowledge of solutions as in the case of Lyapunov functions or Lyapunov functionals, respectively [3–6]. The method of Lyapunov functionals, however, demands the knowledge of solutions of delay differential equations considered and consequently the discussion of examples is not really in the spirit of the method. This observation leads us to develop the method of Lyapunov functions on product spaces for studying stability properties of equations with delay, where, except conceptually, the knowledge of solutions is not demanded. We shall also develop stability theory in terms of two measures which unifies several known stability concepts. Our presentation demonstrates the advantage of utilizing Lyapunov functions on product spaces.

0022-247X/91 $3.00

2. PRELIMINARIES

Let $\mathscr{C} = C[[-\tau, 0], R^n]$ and for any $\phi \in \mathscr{C}$, let us use the norm $|\phi|_0 = \max_{-\tau \leqslant s \leqslant 0} |\phi(s)|$. If $x \in C[[t_0 - \tau, \infty), R^n]$, $t_0 \in R_+$, we define $x_t \in \mathscr{C}$ by $x_t(s) = x(t+s)$, $-\tau \leqslant s \leqslant 0$. We consider the initial value problem

$$x'(t) = f(t, x_t), \qquad x_{t_0} = \phi_0 \in \mathscr{C}, \tag{2.1}$$

where $f \in C[R_+ \chi \mathscr{C}, R^n]$. It is known that if f maps bounded sets into bounded sets, then for each (t_0, ϕ_0), there exists a solution $x(t) = x(t_0, \phi_0)(t)$ defined on an interval $[t_0, t_0 + \alpha)$, $\alpha > 0$. We wish to employ Lyapunov functions on the product space $R^n \chi \mathscr{C}$ and develop corresponding theory for studying stability criteria for the system (2.1). If $V \in C[R_+ \chi R^n \chi \mathscr{C}, R_+]$, then we define

$$D^+ V(t, \phi(0), \phi) = \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h, \phi(0) + hf(t, \phi), x_{t+h}(t, \phi)) - V(t, \phi(0), \phi)], \tag{2.2}$$

where it is understood that $x(t, \phi)$ is any solution of (2.1) with the initial function ϕ at time t. To unify several different concepts of stability studied in the literature such as partial stability, conditional stability, eventual stability, it is convenient to introduce stability concepts in terms of two measures [1, 8–11]. We shall therefore discuss stability properties of (2.1) with respect to two measures. We need the following definitions and the classes of functions:

$\mathscr{K} = \{a \in C[[0, \rho), R_+]\ a(0) = 0$ and $a(u)$ is strictly increasing in $u\}$,

$\mathscr{L} = \{\sigma \in C[R_+, R_+]$: $\sigma(t)$ is decreasing with $\lim_{t \to \infty} \sigma(t) = 0\}$,

$\mathscr{C}\mathscr{K} = \{\sigma \in C[R_+ \chi [0, \rho), R_+]$: $\sigma(t, u) \in \mathscr{K}$ for each $t \in R_+\}$,

$\Gamma = \{h \in C[[t_0 - \tau, \infty) \chi R^n, R_+]$: $\inf_x h(t, x) = 0\}$.

DEFINITION 2.1. Let $h, h^0 \in \Gamma$ and define for $\phi \in \mathscr{C}$,

$$h_0(t, \phi) = \max_{-\tau \leqslant s \leqslant 0} h^0(t+s, \phi(s)), \qquad \tilde{h}(t, \phi) = \max_{-\tau \leqslant s \leqslant 0} h(t+s, \phi(s)). \tag{2.3}$$

Then h_0 is said to be finer than $\tilde{h}$ if there exists a $\rho > 0$ and a $\psi \in \mathscr{K}$ such that

$$h_0(t, \phi) < \rho \qquad \text{implies} \qquad \tilde{h}(t, \phi) \leqslant \psi(h_0(t, \phi)).$$

DEFINITION 2.2. Let $V \in C[R_+ \chi R^n \chi \mathscr{C}, R_+]$. Then V is said to be

(i) h-positive definite if there exists a $\rho > 0$ and a $b \in \mathscr{K}$ such that

$$b(h(t, x)) \leqslant V(t, x, \phi), \qquad \phi \in \mathscr{C} \text{ whenever } h(t, x) < \rho;$$

(ii) weakly h_0-decrescent if there exists a $\rho > 0$ and a $\psi \in C\mathscr{K}$ such that

$$V(t, \phi(0), \phi) \leqslant \psi(t, h_0(t, \phi)) \qquad \text{if} \quad h_0(t, \phi) < \rho;$$

(iii) h_0-decresent if there exists a $\rho > 0$ and a $\psi \in \mathscr{K}$ such that

$$V(t, \phi(0), \phi) \leqslant \psi(h_0(t, \phi)) \qquad \text{if} \quad h_0(t, \phi) < \rho;$$

(iv) asymptotically h_0-decrescent if there exists a $\rho > 0$ and a $\psi \in \mathscr{K}\mathscr{L} = \{\sigma(t, u): \sigma(\cdot, u) \in \mathscr{K} \text{ and } \sigma(t, \cdot) \in \mathscr{L}\}$ such that

$$V(t, \phi(0), \phi) \leqslant \psi(h_0(t, \phi), t) \qquad \text{if} \quad h_0(t, \phi) < \rho;$$

(v) (h^0, h^*)-decresent if there exists a $\rho > 0$ and two functions $a_0, a_1 \in \mathscr{K}$ such that

$$V(t, x, \phi) \leqslant a_0(h^0(t, x)) + a_1(h^*(t, \phi))$$

whenever $h^0(t, x) < \rho$ and $h^*(t, \phi) < \rho$, where $h^* \in C[R_+ \chi \mathscr{C}, R_+]$.

DEFINITION 2.3. Let $\lambda \in C[R_+, R_+]$. Then λ is said to be integrally positive if $\int_I \lambda(s)\, ds = \infty$, whenever $I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ and $\beta_i - \alpha_i > \delta > 0$.

DEFINITION 2.4. The system (2.1) is said to be (h_0, h)-stable if given $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, \phi_0) < \delta \qquad \text{implies} \qquad h(t, x(t)) < \varepsilon,\ t \geqslant t_0.$$

Based on Definition 2.4 and the usual stability notions, it is easy to formulate other kinds of stability concepts in terms of two measures (h_0, h). We shall give below a few choices of (h_0, h) to demonstrate the generality of Definition 2.4. Definition 2.4 reduces to

(1) the well-known stability of the trivial solution of (2.1) if $h(t, x) = |x|$ and $h_0(t, \phi) = |\phi|_0$;

(2) stability of the prescribed solution $y(t) = y(t_0, \psi_0)(t)$ of (2.1) if $h(t, x) = |x - y(t)|$ and $h_0(t, \phi) = |\phi - \psi_0|_0$;

(3) partial stability of the trivial solution of (2.1) if $h(t, x) = |x|_k$, $1 \leq k < n$ and $h_0(t, \phi) = |\phi|_0$;

(4) stability of conditionally invariant set B with respect to A where $A \subset B \subset R^n$ if $h(t, x) = d(x, B)$ and $h_0(t, \phi) = \max_{-\tau \leq s \leq 0} d(\phi(s), A)$, d being the distance function;

(5) eventual stability of (2.1) if $h(t, x) = |x|$ and $h_0(t, \phi) = |\phi|_0 + \sigma(t)$, $\sigma \in L$.

Remark. Note that in case (5), the set of initial times t_0 is to be prescribed as in [6, Defn. 2.4].

3. Main Results

Let us begin by proving a result on non-uniform stability in terms of two measures under weaker assumptions, which includes other interesting special cases and also shows that in situations where the Lyapunov function employed does not satisfy all the desired conditions, it is more fruitful to perturb it by a family of Lyapunov functions than to discard it [7].

Let for any $h \in \Gamma$, $S(h, \rho) = \{(t, x): h(t, x) < \rho\}$ and for any $h^* \in C[R_+ \chi \mathscr{C}, R_+]$, $S(h^*, \rho) = \{(t, \phi): h^*(t, \phi) < \rho\}$.

THEOREM 3.1. *Assume that*

(A_0) *$h, h^0 \in \Gamma$ and h_0 is finer than $\tilde{h}$, where $h_0, \tilde{h}$ are defined by* (2.3);

(A_1) *$V_1 \in C[S(h, \rho) \chi S(\tilde{h}, \rho), R_+]$, $V_1(t, x, \phi)$ is locally Lipschitzian in x and weakly h_0-decrescent*;

(A_2) *for every $0 < \eta < \rho$, there exists a $V_{2\eta} \in C[\Omega, R_+]$, $V_{2\eta}(t, x, \phi)$ is locally Lipschitzian in x and for $(t, \phi(0), \phi) \in \Omega$,*

$$b(h(t, \phi(0))) \leq V_{2\eta}(t, \phi(0)), \phi) \leq a(h_0(t, \phi)),$$

where $a, b \in \mathscr{K}$ and $\Omega = S(h, \rho) \cap S^c(h^0, \eta) \chi S(\tilde{h}, \rho) \cap S^c(h_0, \eta)$, $S^c(h^0, \eta)$ and $S^c(h_0, \eta)$ being the complements of $S(h^0, \eta)$ and $S(h_0, \eta)$, respectively;

(A_3) *for $(t, \phi(0), \phi) \in \Omega$.*

$$D^+ V_1(t, \phi(0), \phi) + D^+ V_{2\eta}(t, \phi(0), \phi)$$
$$\leq g_2(t, V_1(t, \phi(0), \phi) + V_{2\eta}(t, \phi(0), \phi))),$$

where $g_2 \in C[R_+ \chi R_+, R_+]$, $g_2(t, 0) \equiv 0$ *and the trivial solution of*

$$w' = g_2(t, w), \qquad w(t_0) = w_0 \geqslant 0 \tag{3.1}$$

is uniformly stable;

(A$_4$) $D^+ V_1(t, \phi(0)\phi) \leqslant g_1(t, V_1(t, \phi(0), \phi))$ *on* $S(h, \rho) \chi S(\tilde{h}, \rho)$, *where* $g_1 \in C[R_+ \chi R_+, R_+]$, $g_1(t, 0) \equiv 0$ *and the trivial solution of*

$$u' = g_1(t, u), \qquad u(t_0) = u_0 \geqslant 0 \tag{3.2}$$

is stable.

Then, the delay differential system (2.1) *is* (h_0, h)*-stable.*

Proof. Since V_1 is weakly h_0-decrescent, there exists a $0 < \rho_1 \leqslant \rho$ and a $\psi_0 \in C\mathscr{K}$ such that

$$V_1(t, \phi(0), \phi) \leqslant \psi_0(t, h_0(t, \phi)) \qquad \text{if} \quad h_0(t, \phi) < \rho_1. \tag{3.3}$$

Also, h_0 is finer than $\tilde{h}$ implies that there exists a $0 < \rho_0 \leqslant \rho_1$ and a $\psi \in \mathscr{K}$ such that

$$\tilde{h}(t, \phi) \leqslant \psi(h_0(t, \phi)) \qquad \text{if} \quad h_0(t, \phi) < \rho_0, \tag{3.4}$$

where ρ_0 is such that $\psi(\rho_0) \leqslant \rho$. Hence, by (2.3), we have

$$h(t, \phi(0)) \leqslant \tilde{h}(t, \phi) \leqslant \psi(h_0(t, \phi)) \qquad \text{if} \quad h_0(t, \phi) < \rho_0. \tag{3.5}$$

Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given. Since the trivial solution of (3.1) is uniformly stable, given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_0 = \delta_0(\varepsilon)$ such that

$$w(t, t_0, w_0) < b(\varepsilon), \qquad t \geqslant t_0 \qquad \text{if} \quad w_0 < \delta_0, \tag{3.6}$$

where $w(t, t_0, w_0)$ is any solution of (3.1). Since a and ψ belong to class $\mathscr{K}$, we can find a $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$a(\delta_1) < \delta_0/2 \qquad \text{and} \qquad \psi(\delta_1) < \varepsilon. \tag{3.7}$$

The stability of the trivial solution of (3.2) yields that given $\delta_0/2 > 0$ and $t_0 \in R_+$, there exists a $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$u_0 < \delta^* \qquad \text{implies} \qquad u(t, t_0, u_0) < \delta_0/2, \qquad t \geqslant t_0, \tag{3.8}$$

where $u(t, t_0, u_0)$ is any solution of (3.2). Choose $V_1(t_0, \phi_0(0), \phi_0) = u_0$. Since $\psi_0 \in C\mathscr{K}$ and (3.3) holds, there exists a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that $\delta_2 \in (0, \min(\delta_1, \rho_1))$ and

$$h_0(t_0, \phi_0) < \delta_2 \qquad \text{implies} \qquad V_1(t_0, \phi_0(0), \phi_0) \leqslant \psi_0(t_0, h_0(t_0, \phi_0)) < \delta^*. \tag{3.9}$$

We set $\delta = \min(\delta_1, \delta_2)$ and suppose that $h_0(t_0, \phi_0) < \delta$. We note that we obtain

$$\left.\begin{aligned} &h(t_0, \phi_0(0)) \leqslant \tilde{h}(t_0, \phi_0) \leqslant \psi(h_0(t_0, \phi_0)) < \psi(\delta_1) < \varepsilon, \\ &h^0(t_0, \phi_0(0)) \leqslant h_0(t_0, \phi_0) < \delta_1 \end{aligned}\right\} \tag{3.10}$$

in view of (3.4) and (3.7).

We now claim that $h_0(t_0, \phi_0) < \delta$ implies that $h(t, x(t)) < \varepsilon$, $t \geqslant t_0$, where $x(t) = x(t_0, \phi_0)(t)$ is any solution of (2.1). If this is not true, because of (3.10), there exists a solution $x(t)$ of (2.1) with $h_0(t_0, \phi_0) < \delta$ and $t_2 > t_1 > t_0$ such that

$$\left.\begin{aligned} &h^0(t_1, x(t_1)) = \delta_1, \qquad h(t_2, x(t_2)) = \varepsilon, \\ &x(t) \in S(h, \varepsilon) \cap S^c(h^0, \delta_1) \qquad \text{for} \quad t \in [t_1, t_2]. \end{aligned}\right\} \tag{3.11}$$

This also implies that $\tilde{h}(t_2, x_{t_2}) = \varepsilon$, $h_0(t_1, x_{t_1}) = \delta_1$ and $x_t \in S(\tilde{h}, \varepsilon) \cap S^c(h_0, \delta_1)$. Setting $\eta = \delta_1$, we see by (A_2) that there exists a $V_{2\eta}$ and for $t \in [t_1, t_2]$, we have

$$D^+ m(t) \leqslant g_2(t, m(t)),$$

where $m(t) = V_1(t, x(t), x_t) + V_2(t, x(t), x_t)$. Hence, by comparison theorem [6], we obtain

$$m(t) \leqslant r_2(t, t_1, m(t_1)), \qquad t \in [t_1, t_2], \tag{3.12}$$

where $r_2(t, t_1, m(t_1))$ is the maximal solution of (3.1). We can obtain similarly the estimate

$$V_1(t, x(t), x_t) \leqslant r_1(t, t_0, V_1(t_0, \phi_0(0)\, \phi_0)), \qquad t \in [t_0, t_1], \tag{3.13}$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (3.2). Hence, by (3.8), (3.9) and (3.13), we have

$$V_1(t_1, x(t_1), x_{t_1}) < \delta_0/2.$$

Also, by (A_2) and (3.7), we obtain

$$V_{2\eta}(t_1, x(t_1), x_{t_1}) \leqslant a(\delta_1) < \delta_0/2.$$

Hence, it follows that $m(t_1) < \delta_0$ and therefore (3.6) and (3.12) yield that

$$m(t_2) \leqslant r_2(t_2, t_1, m(t_1)) < b(\varepsilon). \tag{3.14}$$

But $m(t_2) \geqslant V_{2\eta}(t_2, x(t_2), x_{t_2}) \geqslant b(h(t_2, x(t_2))) = b(\varepsilon)$, which is a contradiction to (3.14). Hence, the proof is complete.

Remark. Let $V_1 \equiv 0$ and $g_1 \equiv 0$ in Theorem 3.1. Then, we obtain (h_0, h)-uniform stability showing the advantage of using a family of Lyapunov functions in proving uniform stability. If, on the other hand, $V_{2\eta} \equiv 0$ and $g_2 \equiv 0$, assuming V_1 to be h-positive definite guarantees (h_0, h)-stability by Theorem 3.1. As it is, Theorem 3.1 is an extension of [7, Theorem 2; and 8, Theorem 3.1].

We shall next consider a result on asymptotic stability in the same spirit as that of Theorem 3.1 which is an extension of [8, Theorem 3.2].

THEOREM 3.2. *Assume that* (A_0)–(A_3) *of Theorem* 3.1 *hold. Suppose further that*

(A_4^*) *there exist* $V_3, V_4 \in C[S(h, \rho)\, \chi S(\tilde{h}, \rho), R_+]$ *such that* $V_1 = V_3 + V_4$, V_3 *is h-positive definite and on* $S(h, \rho)\, \chi S(\tilde{h}, \rho)$

$$D^+ V_1(t, \phi(0), \phi) \leqslant -\lambda(t)\, \gamma(V_3(t, \phi(0), \phi))$$

holds, where λ *is integrally positive and* $\gamma \in \mathscr{K}$;

(A_5) *for every* $y \in C[[t_0 - \tau, \infty), R^n]$ *such that* $h(t, y(t)) < \rho$, $t \in [t_0 - \tau, \infty)$, *the function* $\int_{t_0}^{t} [D^+ V_4(s, y(s), y_s)]_{\pm}\, ds$ *is uniformly continuous on* $[t_0, \infty)$, *where* $[\cdot]_{\pm}$ *means that either the positive or the negative part is considered for all* $s \in [t_0, \infty)$.

Then, the delay differential system (2.1) *is* (h_0, h)*-asymptotically stable and* $\lim_{t \to \infty} V_4(t, x(t), x_t)$ *exist and is finite for any solution* $x(t)$ *of* (2.1).

Proof. Since (A_4^*) implies that $D^+ V_1(t, \phi(0), \phi) \leqslant 0$ *on* $S(h, \rho)\, \chi S(\tilde{h}, \rho)$, by Theorem 3.1, the assumptions (A_0)–(A_3) yield (h_0, h)-stability for the system (2.1). Choosing $\varepsilon = \rho$ and designating $\delta_0 = \delta_0(t_0, \rho)$, it is clear that

$$h_0(t_0, \phi_0) < \delta_0 \qquad \text{implies} \qquad h(t, x(t)) < \rho, \qquad t \geqslant t_0, \tag{3.15}$$

where $x(t) = x(t_0, \phi_0)(t)$ is any solution of (2.1). Set $m_1(t) = V_1(t, x(t), x_t)$, $m_3(t) = V_3(t, x(t), x_t)$ and $m_4(t) = V_4(t, x(t), x_t)$ so that $m_1(t) = m_3(t) + m_4(t)$. Assumptions (A_1) and (A_4^*) yield that $m_1(t)$ is non-increasing and bounded from below and thus, $\lim_{t \to \infty} m_1(t) = \sigma < \infty$.

We show that $\lim_{t \to \infty} m_3(t) = 0$. Clearly, $\lim_{t \to \infty} \inf m_3(t) = 0$. If not, we arrive at a contradiction because of (A_4^*). Now, suppose that $\lim_{t \to \infty} \sup m_3(t) > 0$. Then there exists a $\mu > 0$ such that $\lim_{t \to \infty} \sup m_3(t) > 3\mu$. Since $\lim_{t \to \infty} m_1(t) = \sigma$ and $m_1(t)$ is non-increasing, there exists a $T > 0$ such that

$$\sigma \leqslant m_1(t) \leqslant \sigma + \mu, \qquad t \geqslant t_0 + T. \tag{3.16}$$

For the sake of clarity, suppose that assumption (A_5) holds with $[\cdot]_+$. Since $m_3(t)$ is continuous, we can choose a sequence

$$t_0 + T < t_1^{(1)} < t_1^{(2)} < \cdots < t_i^{(1)} < t_i^{(2)} < \cdots$$

such that for $i = 1, 2, \ldots,$

$$\left.\begin{array}{ll} m_3(t_i^{(1)}) = 3\mu, & m_{(3)}(t_i^{(2)}) = \mu \\ \mu \leqslant m_3(t) \leqslant 3\mu, & t \in [t_i^{(1)} \cdot t_i^{(2)}]. \end{array}\right\} \tag{3.17}$$

From (3.16) and (3.17), it is easy to see that

$$\left.\begin{array}{l} m_1(t_i^{(1)}) - m_3(t_i^{(1)}) \leqslant \sigma - 2\mu, \\ m_1(t_i^{(2)}) - m_3(t_i^{(2)}) \geqslant \sigma - \mu. \end{array}\right\} \tag{3.18}$$

Since $m_1 = m_3 + m_4$, it follows from (3.18) that

$$0 < \mu \leqslant m_4(t_i^{(2)}) - m_4(t_i^{(1)}) \leqslant \int_{t_i^{(1)}}^{t_i^{(2)}} [D^+ m_4(s)]_+ \, ds,$$

which shows by (A_5) that there exists a $d > 0$ such that

$$t_i^{(2)} - t_i^{(1)} \geqslant \delta, \qquad i = 1, 2, \ldots . \tag{3.19}$$

By (3.17), (3.19), and (A_5), we obtain

$$\lim_{t \to \infty} m_1(t) \leqslant m_1(t_0 + T) - \int_{t_0 + T}^{\infty} \gamma(\mu) \lambda(s) \, ds$$

$$\leqslant m_1(t_0 + T) - \gamma(\mu) \int_I \lambda(s) \, ds = -\infty,$$

where $I = \bigcup_{i=1}^{\infty} [t_i^{(1)}, t_i^{(2)}]$. This contradiction proves that $\lim_{t \to \infty} m_3(t) = 0$ and since V_3 is h-positive definite, we obtain $\lim_{t \to \infty} h(t, x(t)) = 0$. Thus, we conclude that the system (2.1) is (h_0, h)-asymptotically stable. To prove the last assertion of the theorem, note that $\lim_{t \to \infty} m_1(t) = \sigma$ and $\lim_{t \to \infty} m_3(t) = 0$ and consequently, $\lim_{t \to \infty} m_4(t) = \sigma$. The proof is therefore complete.

The following corollaries of Theorem 3.2 are interesting.

COROLLARY 3.1. *Let (A_0), (A_1), (A_4^*), and (A_5) of Theorem 3.2 hold. If V_1 is h-positive definite, then the conclusion of Theorem 3.2 remains valid.*

COROLLARY 3.2. *Let (A_0), (A_1), and (A_4^*) of Theorem 3.2 hold. Assume further that V_1 is h-positive definite, $V_4(t, \phi(0), \phi) = \int_{-\tau}^{0} h^*(t + s, \phi(0), \phi) \, ds$*

and $\tilde{h}$ is finer than h^*, where $h^* \in C[S(h, \rho) \chi S(\tilde{h}, \rho), R_+]$. Then, the conclusion of Theorem 3.2 remains valid.

The next corollary uses the facts $\lim_{t\to\infty} V_1(t, x(t), x_t) = \sigma < \infty$ and V_1 is (h^0, V_3)-decresent.

COROLLARY 3.3. *Let the assumptions of Corollary 3.2 hold except that V_3 is not h-positive definite. Assume that V_1 is (h^0, V_3)-decrescent and $\lim_{t\to\infty} V_3(t, x(t), x_t) = 0$ implies $\lim_{t\to\infty} \inf h^0(tx(t)) = 0$. Then the conclusion of Theorem 3.2 remains valid.*

Remark. We observe that the special form of V_4 in Corollaries 3.2 and 3.3 immediately shows that condition (A_5) is satisfied in view of the property of h^*. This observation makes the direct proofs given in [2–4] redundant.

The next result shows that when D^+V_1 is not h-negative definite, it is beneficial to find another Lyapunov function relative to which suitable conditions yield (h_0, h)-asymptotic stability.

THEOREM 3.3. *Assume that (A_0) and (A_1) of Theorem 3.1 hold. Suppose further that*

(i) *$D^+V_1(t, \phi(0), \phi) \leqslant -\lambda(t)\ \gamma(V_3(t, \phi(0), \phi))$ on $S(h, \rho)\ \chi S(\tilde{h}, \rho)$, where λ is integrally positive, $\gamma \in \mathscr{K}$, $V_3 \in C[S(h, \rho) \chi S(\tilde{h}, \rho), R_+]$ and V_3 either satisfies (A_5) or $V_3(t, \phi(0), \phi) = \int_{-\tau}^{0} h^*(t+s, \phi(0), \phi)\, ds$, where $h^* \in C[S(h, \rho) \chi S(\tilde{h}, \rho), R_+]$ and h is finer than h^*;*

(ii) *$V_2 \in C[S(h, \rho) \chi S(\tilde{h}, \rho), R_+]$, $V_2(t, x, \phi)$ is locally Lipschitzian in x, $V_2 + V_3$ is h-positive definite and on $S(h, \rho) \chi S(\tilde{h}, \rho)$,*

$$D^+V_2(t, \phi(0), \phi) \leqslant -c_1(V_2(t, \phi(0), \phi)) + c_2(V_3(t, \phi(0), \phi)),$$

where $c_1, c_2 \in \mathscr{K}$.

Then the system (2.1) is (h_0, h)-asymptotically stable.

Proof. Following the proof of Theorem 3.2, we can prove that $\lim_{t\to\infty} m_3(t) = 0$, where $m_3(t) = V_3(t, x(t), x_t)$, for any solution $x(t) = x(t_0, \phi_0)(t)$ for which we have

$$h_0(t_0, \phi_0) < \delta_0 \qquad \text{implies} \qquad h(t, x(t)) < \rho, \qquad t \geqslant t_0.$$

We shall next prove that $\lim_{t\to\infty} V_2(t, x(t), x_t) = 0$. Letting $m_2(t) = V_2(t, x(t), x_t)$, we see that condition (ii) yields

$$D^+m_2(t) \leqslant -c_1(m_2(t)) + c_2(m_3(t)), \qquad t \geqslant t_0. \tag{3.20}$$

We claim that $\lim_{t\to\infty}\inf m_2(t)=0$. If it is not true, then there exists a $d>0$ and a $T_1>0$ such that

$$m_2(t)\geqslant d \qquad \text{for} \quad t\geqslant t_0+T_1. \tag{3.21}$$

Since $\lim_{t\to\infty} m_3(t)=0$, there exists a $T_2>0$ such that

$$c_2(m_3(t))\leqslant \frac{c_1(d)}{2} \qquad \text{for} \quad t\geqslant t_0+T_2. \tag{3.22}$$

Choose $T=\max(T_1, T_2)$ so that for $t\geqslant t_0+T$, we have by (3.20) and (3.22),

$$D^+m_2(t)\leqslant -\frac{c_1(d)}{2}. \tag{3.23}$$

The differential inequality (3.23) yields

$$m_2(t)\leqslant m_2(t_0+T)-\frac{c_1(d)}{2}(t-T-t_0), \qquad t\geqslant t_0+T$$

which implies a contradiction. Suppose that $\lim_{t\to\infty}\sup m_2(t)\neq 0$. Then there exists a $\mu>0$ and a sequence

$$t_0<\alpha_1<\beta_1\cdots<\alpha_i<\beta_i<\cdots, \qquad \alpha_i\to\infty \text{ as } i\to\infty,$$

such that

$$\begin{gathered} m_2(\alpha_i)=\mu,\ m_2(\beta_i)=2\mu, \text{ and } \mu\leqslant m_2(t)\leqslant 2\mu, \qquad t\in[\alpha_i,\beta_i], \\ i=1,2,\ldots. \end{gathered} \tag{3.24}$$

Since $\lim_{t\to\infty} m_3(t)=0$, there exists a $T_3>0$ such that

$$c_2(m_3(t))\leqslant\frac{c_1(\mu)}{2}, \qquad t\geqslant t_0+T_3. \tag{3.25}$$

From (3.20), (3.24), and (3.25), we obtain for sufficiently large i,

$$D^+m_2(t)\leqslant -\frac{c_1(\mu)}{2}, \qquad t\in[\alpha_i,\beta_i],$$

which shows that $m_2(t)$ is non-increasing on $[\alpha_i,\beta_i]$. Hence $m_2(\beta_i)\leqslant m_2(\alpha_i)$ which contradicts (3.24). Thus, $\lim_{t\to\infty} m_2(t)=0$. Since V_2+V_3 is h-positive definite, it follows that $\lim_{t\to\infty} h(t,x(t))=0$, proving (h_0,h)-asymptotic stability. The proof is complete.

As in [10], one can show that Theorem 3.3 includes several interesting cases.

So far, we have considered the estimation of D^+V covering the following cases:

(i) $D^+V(t, \phi(0), \phi) \leqslant g(t, V(t, \phi(0), \phi))$,

(ii) $D^+V(t, \phi(0), \phi) \leqslant w(t, \phi(0), \phi)$.

In general, we may have the estimation

(iii) $D^+V(t, \phi(0), \phi) \leqslant g(t, V(t, (0), \phi), \phi(0), \phi)$,

and hence, we shall next discuss this general case. For this purpose, we need to define stability concepts relative to the comparison equation

$$u' = g(t, u, x(t), x_t), \qquad u(t_0) = u_0 \geqslant 0, \tag{3.26}$$

where $g \in C[R_+ \chi R_+ \chi R^n \chi \mathscr{C}, R]$, $g(t, 0, x, \phi) \equiv 0$ and $x(t) = x(t_0, \phi_0)(t)$ is any solution of (2.1).

Definition 3.1. The trivial solution of (3.26) is said to be h_0-conditionally stable if given $\varepsilon > 0$ and $t_0 \in R_+$, there exist $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ and $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, \phi_0) < \delta_2 \text{ and } u_0 < \delta_1 \text{ implies } u(t, t_0, \phi_0, u_0) < \varepsilon, \qquad t \geqslant t_0,$$

where $u(t, t_0, \phi_0, u_0)$ is any solution of (3.26).

We shall now state the following result whose proof may be contructed based on the proof of Theorem 3.1 and the proofs of corresponding results for ODE in [8].

Theorem 3.4. *Assume that* (A_0) *holds. Suppose further that* $V \in C[S(h, \rho) \chi S(h, \rho), R_+]$, $V(t, x, \phi)$ *is locally Lipschitzian in* x, *h-positive definite*, h_0*-decrescent and on* $S(h, \rho) \chi S(\tilde{h}, \rho)$,

$$D^+V(t, \phi(0), \phi) \leqslant g(t, V(t, \phi(0), \phi), \phi(0), \phi)$$

holds, where g is defined as in (3.26). *Then the stability properties of the trivial solution of* (3.26) *imply the corresponding* (h_0, h)*-stability properties of* (2.1).

References

1. S. R. Bernfeld and L. Salvadori, Stability and bifurcation, to appear.
2. T. A. Burton, Uniform asymptotic stability in functional differential equations, *Proc. Amer. Math. Soc.* **68** (1978), 195–199.

3. T. A. BURTON AND L. HATVANI, Stability theorems for nonautonomous functional differential equations, *Tohoku Math. J.* **41** (1989) 65–104.
4. T. A. BURTON AND L. HATVANI, On nonuniform asymptotic stability for nonautonomous functional differential equations, *J. Differential Integral Equations* **3** (1990), 285–293.
5. N. N. KRASOVSKII, "Stability of Motion," Stanford Univ. Press, Stanford, CA, 1963.
6. V. LAKSHMIKANTHAM AND S. LEELA, "Differential and Integral Inequalities," Vols. I and II, Academic Press, New York, 1969.
7. V. LAKSHMIKANTHAM AND S. LEELA, On perturbing Lyapunov functions, *Math. Systems Theory* **10** (1976), 85–90.
8. V. LAKSHMIKANTHAM, S. LEELA, AND A. A. MARTYNYUK, "Stability Analysis of Nonlinear Systems," Dekker, New York, 1989.
9. V. LAKSMIKANTHAM AND LIU XINZHI, Perturbing families of Lyapunov functions and stability theory in terms of two measures, *J. Math. Anal. Appl.* **140** (1989), 107–114.
10. V. LAKSHMIKANTHAM AND LIA XINZHI, On asymptotic stability for nonautonomous differential equations, *Nonlinear Anal.* **13** (1989), 1181–1189.
11. LIU XINZHI, Stability in terms of two measures for functional differential equations, to appear.