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The behaviour of the $p(x)$-Laplacian eigenvalue problem as $p(x) \to \infty$

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ABSTRACT

In this paper we study the behaviour of the solutions to the eigenvalue problem corresponding to the $p(x)$-Laplacian operator

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \Lambda_{p(x)}|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

as $p(x) \to \infty$. We consider a sequence of functions $p_n(x)$ that goes to infinity uniformly in $\overline{\Omega}$. Under adequate hypotheses on the sequence p_n, namely that the limits

$$\nabla \ln p_n(x) \to \xi(x), \quad \text{and} \quad \frac{p_n}{n}(x) \to q(x)$$

exist, we prove that the corresponding eigenvalues Λ_{p_n} and eigenfunctions u_{p_n} verify that

$$(\Lambda_{p_n})^{1/n} \to \Lambda_\infty, \qquad u_{p_n} \to u_\infty \quad \text{uniformly in } \overline{\Omega},$$

where Λ_∞, u_∞ is a nontrivial viscosity solution of the following problem

$$\begin{cases} \min\{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 \log(|\nabla u_\infty|)\langle \xi, \nabla u_\infty\rangle,\ |\nabla u_\infty|^q - \Lambda_\infty u_\infty^q\} = 0, & \text{in } \Omega, \\ u_\infty = 0, \quad \text{on } \partial\Omega. \end{cases}$$

1. Introduction

In this work we analyze the behaviour of the solutions to the eigenvalue problem corresponding to the $p(x)$-Laplacian operator as $p(x) \to \infty$. More precisely, we consider the following problems

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p_n(x)-2}\nabla u) = \Lambda_{p_n}|u|^{p_n(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$ being a bounded smooth domain, and the sequence of functions $p_n : \overline{\Omega} \to \mathbb{R}$ such that $p_n \in C(\overline{\Omega})$ and $p_n(x) > 1$, for every $n \geqslant 1$ and every $x \in \overline{\Omega}$.

For n fixed, solutions to the eigenvalue problem (1.1) have been analyzed in [10]. Our purpose in this work is to study how the solutions to (1.1) behave when we consider a sequence of functions such that $p_n(x) \to \infty$ for every $x \in \overline{\Omega}$, as $n \to \infty$.

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To give some motivation for this study, let us recall briefly what happens when p is constant in Ω. In this case, the limit of (1.1) as $p \to \infty$ has been studied in [4,14,15], see also the survey [2], and leads naturally to the infinity Laplacian eigenvalue problem

$$\min\{|\nabla u|(x) - \Lambda_\infty u(x), -\Delta_\infty u(x)\} = 0, \tag{1.2}$$

where the infinity Laplacian, Δ_∞, is given by

$$\Delta_\infty u := (D^2 u \nabla u) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j}.$$

In fact, it is proved there that the limit as $p \to \infty$ exists both for the eigenfunctions, $u_p \to u_\infty$ uniformly, and for the eigenvalues $(\Lambda_p)^{1/p} \to \Lambda_\infty$, where the pair u_∞, Λ_∞ is a nontrivial solution to (1.2).

Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) solve the optimal Lipschitz extension problem (see [1] and the survey paper [2]) and are used in several applications, for example, in optimal transportation, image processing and tug-of-war games (see, e.g., [3,20,9,11,5,23,24] and the references therein). On the other hand, problems related to PDEs involving variable exponents became popular recently due to applications in elasticity and the modeling of electrorheological fluids. The functional analytical tools needed for the analysis have been extensively developed, see [17] and [8] and also the recent survey [12] and references therein. Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [21], the authors treat the case of a variable exponent that equals infinity in a subdomain of Ω and in [19,22], the limit of $p(x)$-harmonic functions is studied, that is, the limit as $p(x) \to \infty$ of solutions to $\Delta_{p(x)} u = 0$ with $u = g$ on $\partial\Omega$.

Here we will assume that $p_n(x)$ is a sequence of C^1 functions in Ω such that

$$p_n(x) \to +\infty, \quad \text{uniformly in } \Omega, \tag{1.3}$$

$$\nabla \ln p_n(x) \to \xi(x), \quad \text{uniformly in } \Omega, \tag{1.4}$$

$$\frac{p_n}{n}(x) \to q(x), \quad \text{uniformly in } \Omega. \tag{1.5}$$

For the limit functions ξ and q we assume that $\xi \in C(\Omega : \mathbb{R}^N)$ and that $q \in C(\Omega : \mathbb{R})$ is strictly positive.

Under these assumptions we have the following result.

Theorem 1.1. *For any sequence $p_n(x)$ satisfying (1.3)–(1.5) let Λ_{p_n} and u_{p_n} be the corresponding first eigenvalues and eigenfunctions of the problem $-\Delta_{p_n(x)} u_{p_n} = \Lambda_{p_n} |u_{p_n}|^{p_n(x)-2} u_{p_n}$ in Ω with Dirichlet boundary conditions, $u_{p_n}|_{\partial\Omega} = 0$, normalized by $\int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)}\,dx = 1$. Then, there is a subsequence such that*

$$u_{p_i} \to u_\infty \quad \text{in } C^\beta(\overline{\Omega}), \text{ for some } 0 < \beta < 1,$$

and

$$(\Lambda_{p_i})^{1/n_i} \to \Lambda_\infty,$$

where u_∞ is nontrivial and u_∞, Λ_∞ verify, in the viscosity sense,

$$\begin{cases} \min\{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 \log(|\nabla u_\infty|)\langle \xi, \nabla u_\infty\rangle,\ |\nabla u_\infty|^q - \Lambda_\infty u_\infty^q\} = 0, & \text{in } \Omega, \\ u_\infty = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Remark 1.1. Note that hypothesis (1.5) can be replaced by $p_n(x)/a_n \to q(x)$ for any sequence $a_n \to \infty$ as $n \to \infty$. The corresponding statements can be rewritten in terms of a_n (instead of n) but we prefer to simplify notation.

Remark 1.2. Comparing the limit problem (1.6) with (1.2), we note the dependence on x of the sequence p_n. In fact, two limits play a role here, $\nabla \ln p_n(x) \to \xi(x)$ and $\frac{p_n}{n}(x) \to q(x)$.

We now present some examples of possible sequences $p_n(x)$. We are specially interested in understanding (1.4) and (1.5) and hope the examples shed some light on the meaning of this assumption.

(1) $p_n(x) = n$; we have $\xi = 0$ and $q = 1$.
(2) $p_n(x) = p(x) + n$; we get again $\xi = 0$ and $q = 1$.
(3) $p_n(x) = np(x)$; now we get a nontrivial vector field $\xi(x) = \nabla(\ln(p(x)))$ and a nontrivial q, $q(x) = p(x)$.
(4) $p_n(x) = n^a p(x/n)$ [scaling in x]; in this case, we have

$$\nabla(\ln p_n(x)) = \frac{\nabla p}{p}(x/n)\frac{1}{n} \to 0$$

and so $\xi = 0$. Moreover, we have $q(x) = p(0)$ if and only if $a = 1$.

These calculations also hold for $p_n(x) = n + p(x/n)$, we have $\xi = 0$ and $q(x) = 1$.

(5) $p_n(x) = n^a p(nx)$; we get

$$\nabla\bigl(\ln p_n(x)\bigr) = \frac{n\nabla p}{p}(nx),$$

which does not have a limit as $n \to \infty$. The same happens with $p_n(x) = n + p(nx)$, for which

$$\nabla\bigl(\ln p_n(x)\bigr) = \frac{n\nabla p(nx)}{n + p(nx)},$$

that does not have a uniform limit (although it is bounded).

(6) We can modify the previous example to get a nontrivial limit. Assume that $r = r(\theta)$ is a function of the angular variable and that $0 \notin \Omega$; then consider $p_n(x) = n + r(nx)$ to obtain

$$\nabla\bigl(\ln p_n(x)\bigr) = \frac{n\nabla r(nx)}{n + r(nx)} \to \nabla r(\theta).$$

In this case we get $q(x) = 1$.

(7) Finally, we can combine examples (3) and (6). Let $p_n(x) = np(x) + r(nx)$, with q and Ω as in (6). We get

$$\nabla\bigl(\ln p_n(x)\bigr) = \frac{n\nabla p(x) + n\nabla r(nx)}{np(x) + r(nx)} \to \frac{\nabla p(x) + \nabla r(\theta)}{p(x)}.$$

In this case $q(x) = p(x)$.

2. Preliminaries

We introduce now some notation and preliminary results. See [7,8,10,17] and the survey [12] for more details. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows

$$L^{p(x)}(\Omega) = \left\{ u \text{ such that } u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \bigl|u(x)\bigr|^{p(x)}\,dx < +\infty \right\},$$

and is endowed with the norm

$$|u|_{p(x)} = \inf\left\{ \tau > 0 \text{ such that } \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leqslant 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is given by

$$W^{1,p(x)}(\Omega) = \bigl\{ u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \bigr\},$$

with the norm

$$\|u\| = \inf\left\{ \tau > 0 \text{ such that } \int_\Omega \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} + \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leqslant 1 \right\}.$$

Let us denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The following result holds.

Proposition 2.1.

(i) *The spaces* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|)$ *and* $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *Hölder inequality holds, namely*

$$\int_\Omega |uv|\,dx \leqslant 2|u|_{p(x)}|v|_{q(x)}, \quad \forall u \in L^{p(x)}(\Omega),\ \forall v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

(iii) *If* $q \in C(\overline{\Omega})$ *and* $0 < q(x) < p^*(x)$ *for every* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous, where* $p^*(x)$ *is given by*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geqslant N. \end{cases}$$

(iv) *There exists a constant $C > 0$ such that*

$$|u|_{p(x)} \leqslant C|\nabla u|_{p(x)}, \quad \text{for every } u \in W_0^{1,p(x)}(\Omega).$$

Therefore, $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Let us introduce now some results concerning to problem (1.1) for fixed n. Namely, we consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \Lambda_{p(x)}|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Definition 2.1. Let $\Lambda_{p(x)} \in \mathbb{R}$ and $u \in W_0^{1,p(x)}(\Omega)$. We say that $(\Lambda_{p(x)}, u)$ is a solution to the eigenvalue problem (2.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx = \Lambda_{p(x)} \int_\Omega |u|^{p(x)-2}uv\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

As usual, we call $\Lambda_{p(x)}$ an eigenvalue of (2.1) and u an eigenfunction corresponding to $\Lambda_{p(x)}$.

Let us denote $X = W_0^{1,p(x)}(\Omega)$. We define the following functionals $F, G : X \to \mathbb{R}$ by

$$F(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx, \qquad G(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx,$$

and, for $\alpha > 0$, the C^1-submanifold of X,

$$M_\alpha = \{u \in X \text{ such that } G(u) = \alpha\}.$$

It is well known that $(\Lambda_{p(x)}, u)$ solves problem (2.1) if and only if u is a critical point of the functional $\widetilde{F} := F_{|M_\alpha}$: $M_\alpha \to \mathbb{R}$. In order to determine the critical points of this functional let us introduce the following sets

$$\Sigma = \{A \subset X \setminus \{0\} \text{ such that } A \text{ is compact and } A = -A\},$$
$$\Sigma_k = \{A \in \Sigma \text{ such that } \gamma(A) \geqslant k\},$$

where $\gamma(A)$ denotes the genus of A. The values defined by

$$c_{k,\alpha} = \sup_{A \in \Sigma_k,\, A \subset M_\alpha} \inf_{u \in A} F(u), \quad k = 1, 2, \ldots,$$

are critical values of F on M_α verifying $c_{1,\alpha} \geqslant c_{2,\alpha} \geqslant \cdots \geqslant c_{k,\alpha} \geqslant c_{k+1,\alpha} \geqslant \cdots$ and $c_{k,\alpha} \to 0$ as $k \to \infty$. Then, if $u_k \in M_\alpha$ is a critical point of F, its corresponding eigenvalue is given by

$$\Lambda_{p(x),k} = \frac{\int_\Omega |\nabla u_k|^{p(x)}\, dx}{\int_\Omega |u_k|^{p(x)}\, dx} \geqslant \frac{p^-\alpha}{p^+ c_{k,\alpha}},$$

where

$$p^- = \min_{x \in \overline{\Omega}} p(x), \qquad p^+ = \max_{x \in \overline{\Omega}} p(x). \tag{2.2}$$

If we denote $\Lambda = \{\Lambda_{p(x)} \in \mathbb{R}$ such that $\Lambda_{p(x)}$ is an eigenvalue of (2.1)$\}$, we have that Λ is a nonempty infinite set such that $\sup \Lambda = +\infty$. It is also known that in general $\inf \Lambda = 0$, unless the function p is monotone in at least one direction, in which case $\inf \Lambda > 0$, see [10].

3. The limit problem as $p_n(x) \to \infty$

Our interest in this section is to analyze the behaviour of the first eigenvalue (and its corresponding eigenfunctions) of problem (1.1) when $p_n(x) \to +\infty$. To this end, we note that from the previous section we have that the first eigenvalue for $p_n(x)$ is given by

$$\Lambda_{p_n} = \frac{\int_\Omega |\nabla u_{p_n}|^{p_n(x)}\, dx}{\int_\Omega |u_{p_n}|^{p_n(x)}\, dx}. \tag{3.1}$$

The function u_{p_n} is the critical point for

$$c_{1,1}^n = \sup_{A \in \Sigma_1,\, A \subset M_1} \inf_{u \in A} F(u),$$

where we have fixed the parameter $\alpha = 1$. Note that the definition above is equivalent to

$$c_{1,1}^n = \inf_{u\in B} \int_\omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\,dx, \quad \text{with } B = \left\{ u \in X \colon \int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)}\,dx = 1 \right\}. \tag{3.2}$$

It is known (see [10] for details) that for each n fixed $u_{p_n}(x) > 0$ for every $x \in \Omega$ or $u_{p_n}(x) < 0$ for every $x \in \Omega$. In the sequel we will consider for each n the positive solution

$$u_{p_n}(x) > 0, \quad \text{for every } x \in \Omega. \tag{3.3}$$

Our purpose is to study the pair (u_{p_n}, Λ_{p_n}), given by (3.1) and (3.2), as the function $p_n(x)$ goes to infinity as $n \to \infty$. Next, we introduce the following notation: we define

$$p_n^- = \min_{x\in\overline{\Omega}} p_n(x), \qquad p_n^+ = \max_{x\in\overline{\Omega}} p_n(x). \tag{3.4}$$

By (1.5) it is clear that there exist the limits

$$\lim_{n\to\infty} \frac{p_n^-}{n} = q^-, \qquad \lim_{n\to\infty} \frac{p_n^+}{n} = q^+, \tag{3.5}$$

for some q^-, q^+.

Our next aim is to find an upper bound for $(\Lambda_{p_n})^{1/n}$.

Lemma 3.1. *Let Λ_{p_n} be the first eigenvalue of problem* (1.1) *given in* (3.1)*. There exists a positive constant K, independent of n, such that*

$$(\Lambda_{p_n})^{1/n} \leqslant K. \tag{3.6}$$

Proof. We begin with a uniform bound for $(c_{1,1}^n)^{1/n}$. Let us consider the function $u(x) = a\delta(x)$, with $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and the constant $a > 0$ such that $u \in B$, that is, we chose a verifying

$$\int_\Omega \frac{(a\delta(x))^{p_n(x)}}{p_n(x)}\,dx = 1.$$

Let us show that a is uniformly bounded. Let us denote $\Omega_1 = \{x \in \Omega \colon \varepsilon \leqslant \delta(x) \leqslant 1\}$ and $\Omega_2 = \{x \in \Omega \colon \delta(x) > 1\}$. Then, taking into account the definitions (3.4) and (3.5) we have

$$\begin{aligned} 1 &\geqslant \left(\int_{\Omega_1\cup\Omega_2} \frac{(a\delta(x))^{p_n(x)}}{p_n(x)}\,dx \right)^{1/n} \geqslant \left(\max\{a^{p_n^+}, a^{p_n^-}\}\mu(\Omega) \frac{\varepsilon^{p_n^+}+1}{p_n^+} \right)^{1/n} \\ &\geqslant \max\{a^{q^+-\varepsilon}, a^{q^-+\varepsilon}\} \left(\frac{1}{p_n^+} \right)^{1/n} \geqslant \frac{1}{2} \max\{a^{q^+-\varepsilon}, a^{q^-+\varepsilon}\}, \end{aligned}$$

for n sufficiently large and $\varepsilon > 0$ small, and the uniform bound on a follows.

Using u as test function in

$$c_{1,1}^n = \inf_{u\in B} \int_\omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\,dx, \quad \text{with } B = \left\{ u \in X \colon \int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)}\,dx = 1 \right\}$$

we get that

$$\left(c_{1,1}^n\right)^{1/n} \leqslant \left(\int_\Omega \frac{a^{p_n(x)}}{p_n(x)}\,dx \right)^{1/n} \leqslant \left(\frac{\max\{a^{p_n^+}, a^{p_n^-}\}}{p_n^-} \mu(\Omega) \right)^{1/n} \leqslant \max\{a^{q^++\varepsilon}, a^{q^--\varepsilon}\} \left(\frac{\mu(\Omega)}{p_n^-} \right)^{1/n}.$$

Since $(\frac{\mu(\Omega)}{p_n^-})^{1/n} \to 1$ as $n \to \infty$, it holds that $(c_{1,1}^n)^{1/n} \leqslant C$ for n large.

We proceed now with the bound on the first eigenvalue. Let u_{p_n} be the point at which $c_{1,1}^n$ reaches its infimum. We observe that

$$\left(\int_\Omega |\nabla u_{p_n}|^{p_n(x)}\,dx \right)^{1/n} \leqslant \left(p_n^+ \int_\Omega \frac{|\nabla u_{p_n}|^{p_n(x)}}{p_n(x)}\,dx \right)^{1/n} \leqslant 2\left(c_{1,1}^n\right)^{1/n} \leqslant 2C. \tag{3.7}$$

On the other hand

$$\left(\int_\Omega |\nabla u_{p_n}|^{p_n(x)}\,dx\right)^{1/n} = (\Lambda_{p_n})^{1/n}\left(\int_\Omega |u_{p_n}|^{p_n(x)}\,dx\right)^{1/n}$$
$$\geqslant (\Lambda_{p_n})^{1/n}\left(p_n^-\int_\Omega \frac{|u_{p_n}|^{p_n(x)}}{p_n(x)}\,dx\right)^{1/n} \geqslant c(\Lambda_{p_n})^{1/n},$$

which together with (3.7) gives the uniform bound on the first eigenvalue (3.6). □

The previous result allows us to consider a subsequence $n_i \to \infty$ such that $(\Lambda_{p_{n_i}})^{1/n_i} \to \Lambda_\infty$ and, as we see in the next lemma, we can also extract a subsequence $u_{p_{n_i}} \to u_\infty$ in $C^\beta(\Omega)$.

Lemma 3.2. *There exists a subsequence $\{u_{p_{n_i}}\}$ converging to some nontrivial function u_∞ in $C^\beta(\Omega)$, for some $0<\beta<1$.*

Proof. Let us take $m<n$. Then by (3.7) we get

$$\int_\Omega \left(|\nabla u_{p_n}|^{\frac{p_n(x)}{n}}\right)^m dx \leqslant \left(\int_\Omega |\nabla u_{p_n}|^{p_n(x)}\,dx\right)^{m/n}\left(\mu(\Omega)\right)^{1-m/n} \leqslant K,$$

thus $|\nabla u_{p_n}|^{\frac{p_n(x)}{n}}$ is uniformly bounded in $L^m(\Omega)$, which implies that $|\nabla u_{p_n}|$ is uniformly bounded in $L^{\frac{mp_n(x)}{n}}(\Omega) \subset L^{m(q^-(x)-\varepsilon)}(\Omega)$, by Hölder inequality (we take ε such that $q^-(x)-\varepsilon>1$, $\forall x\in\Omega$). If we take now m such that $m(q^-(x)-\varepsilon)\geqslant N$, then by the continuous embedding in (iii) of Proposition 2.1 we have that $W_0^{1,m(q^-(x)-\varepsilon)}(\Omega)\subset C^\beta(\Omega)$, $0<\beta<1$. Therefore, there exists a subsequence $\{u_{p_{n_i}(x)}\}$ such that

$$u_{p_{n_i}(x)} \rightharpoonup u_\infty, \quad \text{weakly in } W^{1,m(q^-(x)-\varepsilon)}(\Omega) \quad \text{and} \quad u_{p_{n_i}(x)} \to u_\infty, \quad \text{strongly in } C^\beta(\Omega). \tag{3.8}$$

Note that we have the normalization

$$\left(\int_\Omega \frac{1}{p_n(x)}|u_{p_n}|^{p_n(x)}\,dx\right)^{1/n} = 1,$$

hence

$$\left(\frac{1}{p_n^-}\right)^{1/n}\left(\int_\Omega |u_{p_n}|^{p_n(x)}\,dx\right)^{1/n} \geqslant 1,$$

and then we have that

$$\left(\frac{\mu(\Omega)}{p_n^-}\right)^{1/n}\max\left\{\left(\|u_{p_n}\|_\infty\right)^{p_n^+},\left(\|u_{p_n}\|_\infty\right)^{p_n^-}\right\}^{1/n} \geqslant 1.$$

If we pass to the limit as $n\to\infty$ in the previous estimate, taking into account (1.5) and (3.8) we get that

$$\max\left\{\left(\|u_\infty\|_\infty\right)^{q^+},\left(\|u_\infty\|_\infty\right)^{q^-}\right\} \geqslant q^+,$$

and thus u_∞ is nontrivial. □

In order to identify the limit problem satisfied by any cluster point u_∞ we introduce the concept of viscosity solutions to problem (1.1). Assuming that u_{p_n} are smooth enough to differentiate (1.1), we get

$$-|\nabla u_{p_n}|^{p_n(x)-2}\left(\Delta u_{p_n} + \log\left(|\nabla u_{p_n}|\right)\sum_{i=1}^N \frac{\partial u_{p_n}}{\partial x_i}\frac{\partial p_n(x)}{\partial x_i}\right)$$
$$-\left(p_n(x)-2\right)|\nabla u_{p_n}|^{p_n(x)-4}\sum_{i,j=1}^N \frac{\partial u_{p_n}}{\partial x_i}\frac{\partial u_{p_n}}{\partial x_j}\frac{\partial^2 u_{p_n}}{\partial x_i\partial x_j} = \Lambda_{p_n}|u_{p_n}|^{p_n(x)-2}u_{p_n}. \tag{3.9}$$

We recall that the last operator involving the second derivatives is denoted as Δ_∞, that is

$$\Delta_\infty u = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}.$$

Eq. (3.9) is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it. Let $y \in \mathbb{R}$, $z, \theta \in \mathbb{R}^N$, and S be a real symmetric matrix. We define the following continuous function

$$H_{p_n(x)}(y,z,\theta,S) = -|z|^{p_n(x)-2}\big(\operatorname{trace}(S) + \log(|z|)\langle z,\theta\rangle\big) \\ - \big(p_n(x)-2\big)|z|^{p_n(x)-4}\langle S\cdot z, z\rangle - \Lambda_{p_n}|y|^{p_n(x)-2}y. \tag{3.10}$$

To define the notion of viscosity solution we are interested in viscosity super- and subsolutions of the partial differential equation

$$\begin{cases} H_{p_n(x)}(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2 u_{p_n}) = 0, & \text{in } \Omega, \\ u_{p_n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Definition 3.1. An upper semicontinuous function u defined in Ω is a *viscosity subsolution* of (3.11) if, $u|_{\partial\Omega} \leqslant 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

(i) $u(x_0) = \phi(x_0)$,
(ii) $u(x) < \phi(x)$, if $x \neq x_0$,

then

$$H_{p_n(x)}\big(\phi(x_0), \nabla\phi(x_0), \nabla p_n(x_0), D^2\phi(x_0)\big) \leqslant 0.$$

Definition 3.2. A lower semicontinuous function u defined in Ω is a *viscosity supersolution* of (3.11) if, $u|_{\partial\Omega} \geqslant 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

(i) $u(x_0) = \phi(x_0)$,
(ii) $u(x) > \phi(x)$, if $x \neq x_0$,

then

$$H_{p_n(x)}\big(\phi(x_0), \nabla\phi(x_0), \nabla p_n(x_0), D^2\phi(x_0)\big) \geqslant 0.$$

We observe that in both of the above definitions the second condition is required just in a neighbourhood of x_0 and the strict inequality can be relaxed. We refer to [6] for more details about general theory of viscosity solutions, and [13,16,18] for viscosity solutions related to the ∞-Laplacian and the p-Laplacian operators. The following result can be shown as in [15], we include the proof for convenience of the reader.

Lemma 3.3. *A continuous weak solution to Eq.* (1.1) *is a viscosity solution to* (3.11)*.*

Proof. The proof is analogous to this one of Proposition 2.4 in [21]. We reproduce it here for the sake of completeness and readability.

We omit the subscript n in this proof. Let us show that if u is continuous weak supersolution, then it is a viscosity supersolution. Let $x_0 \in \Omega$ and let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$-\Delta_{p(x_0)}\phi(x_0) = -\big|\nabla\phi(x_0)\big|^{p(x_0)-2}\Delta\phi(x_0) - \big(p(x_0)-2\big)\big|\nabla\phi(x_0)\big|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ - \big|\nabla\phi(x_0)\big|^{p(x_0)-2}\ln\big(|\nabla\phi|\big)(x_0)\big\langle\nabla\phi(x_0), \nabla p(x_0)\big\rangle \\ \geqslant \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(x_0).$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\Delta_{p(x)}\phi(x) = -\big|\nabla\phi(x)\big|^{p(x)-2}\Delta\phi(x) - \big(p(x)-2\big)\big|\nabla\phi(x)\big|^{p(x)-4}\Delta_\infty\phi(x) \\ - \big|\nabla\phi(x)\big|^{p(x)-2}\ln\big(|\nabla\phi|\big)(x)\big\langle\nabla\phi(x), \nabla p(x)\big\rangle \\ < \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(x),$$

for every $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r}(u-\phi)(x)$$

and let $\Phi(x) = \phi(x) + m/2$.

This function Φ verifies $\Phi(x_0) > u(x_0)$, $\Phi < u$ on $\partial B(x_0, r)$ and

$$-\Delta_{p(x)}\Phi = -\operatorname{div}\big(|\nabla\Phi|^{p(x)-2}\nabla\Phi\big) < \Lambda_{p(x)}|\phi|^{p(x)-2}\phi, \quad \text{in } B(x_0, r). \tag{3.12}$$

Multiplying (3.12) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int\limits_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx < \int\limits_{B(x_0,r)\cap\{\Phi>u\}} \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(\Phi-u)\,dx.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of the eigenvalue problem, we obtain

$$\int\limits_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)\,dx = \int\limits_{B(x_0,r)\cap\{\Phi>u\}} \Lambda_{p(x)}|u|^{p(x)-2}u(\Phi-u)\,dx.$$

Upon subtraction and using a well-know inequality, we conclude

$$\begin{aligned} 0 &> \int\limits_{B(x_0,r)\cap\{\Phi>u\}} \big(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\big)\cdot\nabla(\Phi-u)\,dx \\ &\geqslant c \int\limits_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi - \nabla u|^{p(x)}\,dx, \end{aligned}$$

a contradiction.

This proves that u is a viscosity supersolution. The proof that u is a viscosity subsolution runs as above and we omit the details. □

We have all the ingredients to compute the limit of the equation

$$H_{p_n(x)}\big(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2u_{p_n}\big) = 0$$

as $p_n(x) \to \infty$ in the viscosity sense, that is to identify the limit equation verified by any u_∞ as in (3.8).

In the sequel we assume that we have a subsequence $p_{n_i}(x) \to \infty$ with the assumptions stated in the introduction such that

$$\lim_{i\to\infty} u_{p_{n_i}} = u_\infty$$

uniformly in Ω and $(\Lambda_{p_{n_i}})^{1/n_i} \to \Lambda_\infty$. We denote as u_{p_n} and Λ_{p_n} such subsequences for readable reasons.

We define for $y \in \mathbb{R}$, $z, \theta \in \mathbb{R}^N$ and S a symmetric real matrix,

$$H_\infty(y, z, q, \theta, S) = \min\big\{-\langle S\cdot z, z\rangle - \log\big(|z|\big)\langle\theta, z\rangle, |z|^q - \Lambda_\infty y^q\big\}. \tag{3.13}$$

Note that $H_\infty(u, \nabla u, q, \xi, D^2u) = 0$ is the equation that appears in (1.6).

Theorem 3.1. *A function u_∞ obtained as a limit of a subsequence of $\{u_{p_n}\}$ is a viscosity solution of the equation*

$$H_\infty\big(u, \nabla u, q, \xi, D^2u\big) = 0,$$

with H_∞ defined in (3.13)*, and ξ and q given by* (1.4) *and* (1.5) *respectively.*

Proof. Consider $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$ for every $x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and verifying that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ it holds that

$$\inf\big\{u_\infty - \phi \text{ in } B(x_0, R) \setminus B(x_0, r)\big\} > 0.$$

Since $u_{p_n} \to u_\infty$ uniformly in $\overline{B(x_0, R)}$, for $n \geqslant n_0$ the function $u_{p_n} - \phi$ attains its minimum value in $B(x_0, r)$. Let us denote by $x_n \in B(x_0, r)$ such a point. By letting $r \to 0$ we get a subsequence such that $x_{n_r} \to x_0$ as $n_r \to \infty$. To simplify we denote such subindexes by x_n and u_{p_n}.

On the other hand we have that u_{p_n} is a viscosity supersolution of (3.11). Then,

$$\begin{aligned} &-\big|\nabla\phi(x_n)\big|^{p_n(x_n)-2}\big(\Delta\phi(x_n) + \log\big(\big|\nabla\phi(x_n)\big|\big)\big\langle\nabla p_n(x_n), \nabla\phi(x_n)\big\rangle\big) \\ &\quad - \big(p_n(x_n) - 2\big)\big|\nabla\phi(x_n)\big|^{p_n(x_n)-4}\big\langle\nabla\phi(x_n)D^2\phi(x_n), \nabla\phi(x_n)^t\big\rangle \\ &\geqslant \Lambda_{p_n}\big|\phi(x_n)\big|^{p_n(x_n)-2}\phi(x_n). \end{aligned} \tag{3.14}$$

We observe that, at the point x_n

$$\Lambda_{p_n}\left|\phi(x_n)\right|^{p_n(x_n)-2}\phi(x_n)=\Lambda_{p_n}\left|u_{p_n}(x_n)\right|^{p_n(x_n)-2}u_{p_n}(x_n)>0,$$

if we assume that $u_\infty(x_0)>0$. In consequence, by (3.14) we deduce that $|\nabla\phi(x_n)|>0$ and we can multiply this inequality by $(p_n(x_n)-2)^{-1}|\nabla\phi(x_n)|^{-(p_n(x_n)-4)}$, to obtain that

$$\begin{aligned}&\frac{-|\nabla\phi(x_n)|^2(\Delta\phi(x_n)+\log(|\nabla\phi(x_n)|)\langle\nabla p_n(x_n),\nabla\phi(x_n)\rangle)}{p_n(x_n)-2}-\left\langle\nabla\phi(x_n)D^2\phi(x_n),\nabla\phi(x_n)^t\right\rangle\\&\quad\geqslant\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}.\end{aligned}$$

If we take limit as $n\to\infty$ in the previous inequality, taking into account (1.4) we have that

$$\begin{aligned}&-\Delta_\infty\phi(x_0)-|\nabla\phi(x_0)|^2\log\left(\left|\nabla\phi(x_0)\right|\right)\left\langle\xi(x_0),\nabla\phi(x_0)\right\rangle\\&\quad\geqslant\lim_{n\to\infty}\left[\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}\right].\end{aligned}\tag{3.15}$$

For any ϕ,

$$\lim_{n\to\infty}\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}=0.$$

By (1.5) it also holds that

$$\lim_{n\to\infty}\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\to\frac{\Lambda_\infty\phi(x_0)^{q(x_0)}}{|\nabla\phi(x_0)|^{q(x_0)}}.\tag{3.16}$$

Now, we claim that the previous limit is smaller than one, namely,

$$\left|\nabla\phi(x_0)\right|^{q(x_0)}-\Lambda_\infty\phi(x_0)^{q(x_0)}\geqslant 0.\tag{3.17}$$

To prove this claim we argue by contradiction. Assume that

$$\frac{\Lambda_\infty\phi(x_0)^{q(x_0)}}{|\nabla\phi(x_0)|^{q(x_0)}}>1.$$

Then, from (3.16) we conclude that there exists $\theta>1$ such that

$$\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\geqslant\theta>1$$

for n large. Therefore,

$$\lim_{n\to\infty}\left[\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}\right]\geqslant\lim_{n\to\infty}\frac{\theta^n}{n}\left[\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{\frac{(p_n(x_n)-2)}{n}|\phi(x_n)|^2}\right]=\infty.$$

Hence the limit in (3.15) diverges, but the left hand side is bounded, so we reach a contradiction.

Now, if $u_\infty(x_0)=0$ and $\nabla\phi(x_0)\neq 0$ we can use the same arguments to conclude that (3.17) holds, and if $\nabla\phi(x_0)=0$, then (3.17) holds trivially.

On the other hand, it always holds that

$$-\Delta_\infty\phi(x_0)-|\nabla\phi(x_0)|^2\log\left(\left|\nabla\phi(x_0)\right|\right)\left\langle\xi(x_0),\nabla\phi(x_0)\right\rangle\geqslant 0.\tag{3.18}$$

Thus, we can combine the two equations (3.17) and (3.18) into the following

$$\min\left\{-\Delta_\infty\phi(x_0)-|\nabla\phi(x_0)|^2\log\left(\left|\nabla\phi(x_0)\right|\right)\left\langle\xi(x_0),\nabla\phi(x_0)\right\rangle,\left|\nabla\phi(x_0)\right|^{q(x_0)}-\Lambda_\infty\phi(x_0)^{q(x_0)}\right\}\geqslant 0.\tag{3.19}$$

To complete the proof it just remains to see that u_∞ is a viscosity subsolution. Let us consider a point $x_0\in\Omega$ and a function $\phi\in C^2(\Omega)$ such that $u_\infty(x_0)=\phi(x_0)$ and $u_\infty(x)<\phi(x)$ for every x in a neighbourhood of x_0. We want to show that

$$H_\infty\left(\phi(x_0),\nabla\phi(x_0),q(x_0),\xi(x_0),D^2\phi(x_0)\right)\leqslant 0.$$

We first observe that if $\nabla\phi(x_0) = 0$ the previous inequality trivially holds. Hence, let us assume that $\nabla\phi(x_0) \neq 0$. Now, we argue as follows: assuming that

$$\left|\nabla\phi(x_0)\right|^{q(x_0)} - \Lambda_\infty \phi(x_0)^{q(x_0)} > 0, \tag{3.20}$$

we will show that

$$-\Delta_\infty \phi(x_0) - |\nabla\phi(x_0)|^2 \log\left(\left|\nabla\phi(x_0)\right|\right)\left\langle \xi(x_0), \nabla\phi(x_0)\right\rangle \leqslant 0. \tag{3.21}$$

As before, we get a sequence of points $x_n \to x_0$ such that

$$\frac{-|\nabla\phi(x_n)|^2(\Delta\phi(x_n) + \log(|\nabla\phi(x_n)|)\langle\nabla p_n(x_n), \nabla\phi(x_n)\rangle)}{p_n(x_n) - 2} - \left\langle \nabla\phi(x_n) D^2\phi(x_n), \nabla\phi(x_n)^t \right\rangle$$
$$\leqslant \left(\frac{\Lambda_{p_n}^{1/n} \phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}} \right)^n \frac{|\nabla\phi(x_n)|^4 \phi(x_n)}{(p_n(x_n) - 2)|\phi(x_n)|^2}.$$

Taking limit as $n \to \infty$ in the above inequality we get an equation similar to (3.15), namely

$$-\Delta_\infty \phi(x_0) - |\nabla\phi(x_0)|^2 \log\left(\left|\nabla\phi(x_0)\right|\right)\left\langle \xi(x_0), \nabla\phi(x_0)\right\rangle$$
$$\geqslant \lim_{n\to\infty} \left[\left(\frac{\Lambda_{p_n}^{1/n} \phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}} \right)^n \frac{|\nabla\phi(x_n)|^4 \phi(x_n)}{(p_n(x_n) - 2)|\phi(x_n)|^2} \right].$$

Now, we observe that the limit above is equal to zero, since we are assuming (3.20). Thus (3.21) holds and the proof is complete. □

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