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A degree bound for codimension two lattice ideals

Leah H. Gold

Department of Mathematics, Texas A & M University, College Station, TX 77843-3368, USA

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Abstract

Herzog and Srinivasan have conjectured that for any homogeneous k-algebra, the degree is bounded above by a function of the maximal degrees of the syzygies. Combining the syzygy quadrangle decomposition of Peeva and Sturmfels and a delicate case analysis, we prove that this conjectured bound holds for codimension 2 lattice ideals.

MSC: Primary: 13H15; secondary: 13D02; 13F20; 14M25

1. Introduction

Let $R=k[x_1,\dots,x_n]$ be a polynomial ring in n variables over a field k, let $\deg(x_i)=1$, and let $I\subset R$ be a homogeneous ideal. If the Hilbert polynomial of R/I is $\sum_{i=0}^{m} a_it^i$, then the *degree* of the ideal I, written $\deg(I)$, is simply $a_m m!$.

In this section we briefly describe the progress to date on bounding the degree of an ideal. In particular, we recall several conjectures which were made about the degree and discuss what is known about the conjectures. In Section 2 we define codimension 2 lattice ideals and explain Peeva and Sturmfels' decomposition of the resolution of any such ideal. Finally, in Section 3 we use the decomposition and a careful case analysis of the possible syzygies to prove that the conjectured bound on the degree holds for codimension 2 lattice ideals.

E-mail address: lgold@math.tamu.edu (L.H. Gold).

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A resolution is called *pure* if at each step there is only a single degree. That is, the resolution looks like

$$0 \to R(-d_p)^{b_p} \to R(-d_{p-1})^{b_{p-1}} \to \cdots \to R(-d_2)^{b_2} \to R(-d_1)^{b_1} \to R.$$

Huneke and Miller proved the following formula for the degree of a Cohen–Macaulay algebra with a pure resolution [5].

Theorem 1 (Huneke and Miller). *Let R/I be a Cohen–Macaulay algebra with a pure resolution as displayed above. Then* $\deg(I) = (\prod_{i=1}^{p} d_i)/p!$.

One might hope that when the resolution is not pure that it is possible to write a similar closed formula for the degree in terms of the degrees of the syzygies. This does not appear to be the case, however, Huneke and Srinivasan made a conjecture using similar formulas to bound the degree [4].

Conjecture 2 (Huneke and Srinivasan). *Let R/I be a Cohen–Macaulay algebra with resolution of the form*

$$0 \to \bigoplus_{j\in J_p} R(-d_{p,j}) \to \cdots \to \bigoplus_{j\in J_2} R(-d_{2,j}) \to \bigoplus_{j\in J_1} R(-d_{1,j}) \to R.$$

Let $m_i = \min\{d_{i,j} \in J_i\}$ be the minimum degree shift at the ith step and let $M_i = \max\{d_{i,j} \in J_i\}$ *be the maximum degree shift at the ith step. Then*

$$\frac{\prod_{i=1}^{p} m_i}{p!} \leqslant \deg(I) \leqslant \frac{\prod_{i=1}^{p} M_i}{p!}.$$

Notice that since R/I is Cohen–Macaulay, p is the codimension of I.

Due to work by Herzog and Srinivasan [4], Conjecture 2 is known to be true for the following types of ideals.

- complete intersections
- perfect ideals with quasipure resolutions ($d_{i,j} \leqslant d_{i+1,j}$ for all i, j)
- perfect ideals of codimension 2
- Gorenstein ideals of codimension 3 generated by 5 elements (the upper bound holds for all codimension 3 Gorenstein ideals)
- perfect stable monomial ideals (as defined by Eliahou and Kervaire [3])
- perfect squarefree strongly stable monomial ideals (see Aramova et al. [1])

Generalizing even further, one might want to omit the Cohen–Macaulay restriction. Consider $I = (x^2, xy) \subset k[x, y]$. Then $\deg(I) = 1$, $m_1 = 2$ and $m_2 = 3$, but $(2)(3)/2! \geqslant 1$. So we know that the lower bound does not hold for non-Cohen–Macaulay algebras and therefore we consider just the upper bound in the non-Cohen–Macaulay case.

Conjecture 3 (Herzog and Srinivasan). *Let I be a homogeneous ideal of codimension d and M_i as defined above*, *then* $\deg(I) \leqslant (\prod_{i=1}^{d} M_i)/d!$.

Herzog and Srinivasan showed that Conjecture 3 is true in three cases.

– stable monomial ideals (as defined by Eliahou and Kervaire [3])
– squarefree strongly stable monomial ideals (see Aramova et al. [1])
– ideals with a q-linear resolution (all the generators are in degree q and all the syzygies are linear)

Prior to the result presented here, the above cases formed a complete list of all known cases where the conjectures are true.

2. Codimension 2 lattice ideals

Lattice ideals are a slight generalization of toric ideals. Codimension 2 lattice ideals were studied by Peeva and Sturmfels in their paper [6]. We briefly describe here the relevant results from their paper, namely, the construction of an explicit resolution of any such ideal.

We begin by defining a lattice ideal. Let $R = k[x_1,\dots,x_n]$ be a polynomial ring and for any nonnegative integer vector $\mathbf{a} = (a_1,\dots,a_n)$, let $x^{\mathbf{a}} = x_1^{a_1}\cdots x_n^{a_n}$. For any lattice $\mathscr{L} \subset \mathbb{Z}^n$ we define

$$I_{\mathscr{L}} = (\mathbf{x}^{\mathbf{a}_+} - \mathbf{x}^{\mathbf{a}_-} \mid \mathbf{a} \in \mathscr{L}),$$

where $\mathbf{a}_+$ is the positive part of the vector $\mathbf{a}$ and $\mathbf{a}_-$ is the negative part of $\mathbf{a}$. That is, in the ith component, $(\mathbf{a}_+)_i = \mathbf{a}_i$ if $\mathbf{a}_i \geqslant 0$ and zero otherwise. We define $\mathbf{a}_-$ in a similar manner. We consider only lattices with no nonnegative vectors in order to ensure the lattice ideal is homogeneous with respect to some positive grading.

We may define a multigrading on R, and also on $I_{\mathscr{L}}$, by the group $\mathbb{Z}^n/\mathscr{L}$. We will move back and forth between this grading and the standard grading. It should be clear from context which one is meant.

The codimension of $I_{\mathscr{L}}$ is the minimal number of generators of the lattice $\mathscr{L}$. When $I_{\mathscr{L}}$ has codimension 2, Peeva and Sturmfels constructed a resolution for $I_{\mathscr{L}}$ in the following way.

Let $\mathbf{c}$ be a multidegree and let $\mathbf{x}^{\mathbf{a}}$ be a monomial of degree $\mathbf{c}$. Then there is a correspondence between monomials of degree $\mathbf{c}$ and vectors $\mathbf{u} \in \mathbb{Z}^2$ such that $\mathrm{B}\mathbf{u} \leqslant \mathbf{a}$. The monomial $\mathbf{x}^{\mathbf{a}} - \mathrm{B}\mathbf{u}$ corresponds to the vector $\mathbf{u}$. Define the polytope $P_{\mathbf{a}} = \mathrm{conv}(\{\mathbf{u} \in \mathbb{Z}^2 \mid \mathrm{B}\mathbf{u} \leqslant \mathbf{a}\})$. Notice that $P_{\mathbf{a}}$ and $P_{\mathbf{b}}$ are lattice translates of each other if and only if $\mathbf{a} - \mathbf{b} \in \mathscr{L}$. So, we generally write $P_{\mathbf{c}}$ instead of $P_{\mathbf{a}}$.

Peeva and Sturmfels showed that each multidegree in which there is a minimal syzygy corresponds to a primitive polytope. In particular, first syzygies correspond to line segments, second syzygies correspond to triangles and third syzygies correspond to quadrangles. Further, the syzygy triangles consist of three syzygy line segments and the syzygy quadrangles consist of four syzygy triangles. For details on this correspondence, see Peeva and Sturmfels paper [6]. A resolution of the ideal generated by the binomials corresponding to the four segments is found by the following method.

Let $P_{\mathbf{c}}$ be a polytope corresponding to a syzygy quadrangle. We start by writing the two generators of $I_{\mathscr{L}}$ corresponding to the sides of the quadrangle as $\alpha = \alpha' - \alpha''$

and $\beta = \beta' - \beta''$. Then we determine vectors $\mathbf{p}$, $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ by taking the greatest common divisors of a term of α and a term of β. For example, choose $\mathbf{p}$ such that $x^{\mathbf{p}} = \gcd(\alpha', \beta')$. We set the remaining factors to be $\mathbf{x}^{\mathbf{u}_+}$, $\mathbf{x}^{\mathbf{u}_-}$, $\mathbf{x}^{\mathbf{v}_+}$, and $\mathbf{x}^{\mathbf{v}_-}$ and so we have $\alpha = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{t}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{s}}$ and $\beta = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{s}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{t}}$.

A diagonal vector of the quadrangle is a sum or difference of the two edge vectors. Hence we can derive representations for the generators which correspond to the diagonals from the generators for α and β by taking the sum or difference of the exponent vectors of the binomials α and β. This procedure gives $\gamma = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}}$ and $\delta = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{t}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{s}}$. Notice that in order to generate an ideal of codimension 2, the four generators of the ideal $I_{\mathscr{L}}$ cannot share a common factor.

Putting all of this into a sequence, the resolution of the four generators derived from $P_{\mathbf{c}}$ has the form

$$0 \to R \xrightarrow{\begin{pmatrix} -\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{t}} \\ \mathbf{x}^{\mathbf{r}} \\ -\mathbf{x}^{\mathbf{p}} \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}} & -\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}} \\ -\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{s}} & 0 & 0 \\ 0 & 0 & \mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{r}} \end{pmatrix}} R^4 \xrightarrow{(\alpha \quad \beta \quad \gamma \quad \delta)} R,$$

where

$$\alpha = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}, \quad \beta = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}},$$

$$\gamma = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}} \quad \text{and} \quad \delta = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{t}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{s}}.$$

The resolutions corresponding to the syzygy quadrangles may then be used to build a resolution for $R/I_{\mathscr{L}}$.

Theorem 4 (Peeva and Sturmfels). *If $R/I_{\mathscr{L}}$ is not Cohen–Macaulay, then the sum of the complexes corresponding to syzygy quadrangles is a minimal free resolution of $R/I_{\mathscr{L}}$.*

3. Bounding the degree for codimension 2 lattice ideals

Using the decomposition of the resolution in terms of the syzygy quadrangles given in the previous section, we can now show that Conjecture 3 is true for codimension 2 lattice ideals.

Theorem 5. *If $I_{\mathscr{L}}$ is a homogeneous codimension 2 lattice ideal, then Conjecture 2 holds. That is, for a homogeneous codimension 2 lattice ideal $I_{\mathscr{L}}$, if M_1 is the maximal degree of the generators of $I_{\mathscr{L}}$ and M_2 is the maximal degree of the syzygies on the generators, then*

$$\deg(I_{\mathscr{L}}) \leqslant \frac{M_1 M_2}{2}.$$

In order to prove this theorem, we first prove a special case.

Lemma 6. *Let $I_{\mathscr{L}}$ be a lattice ideal and let J be an ideal whose four generators are associated to a single syzygy quadrangle of $I_{\mathscr{L}}$. Then Theorem 5 holds for J.*

Proof. Let J be the ideal whose four generators are associated to a single syzygy quadrangle of $I_{\mathscr{L}}$. Using the resolution described in the previous section, we can write down the Hilbert series for R/J. That is,

$$H_{R/J}(y) = \frac{f(y)}{(1-y)^n} \quad \text{where } f(y) = \sum_{i=1}^{n} \sum_{j \in J_i} (-1)^i y^{d_{i,j}}.$$

Canceling powers of $(1-y)$, we obtain

$$H_{R/J}(y) = \frac{g(y)}{(1-y)^{n-2}}.$$

So $\deg(J) = g(1) = \frac{1}{2} f''(1)$.

For any vector $\mathbf{v} = |v| = (v_1, v_2, \ldots, v_n)$, let $v = v_1 + v_2 + \cdots + v_n$. Using this notation and our knowledge of the $d_{i,j}$ from the resolution, we can write $\deg(J)$ in terms of u_+, u_-, v_+, v_-, p, r, s, and t.

Since α and β are homogeneous polynomials, there are relations between the eight variables u_+, u_-, v_+, v_-, p, r, s, and t which arise because the degrees of the terms in the binomials are equal. Using these relations, we can eliminate u_- and v_- and write $\deg(J)$ in terms of the other six variables, u_+, v_+, p, r, s, and t. So,

$$\deg(J) = u_+v_+ + u_+p + v_+p + p^2 - pr + u_+s + ps + v_+t + pt.$$

Now, what are the possibilities for M_1 and M_2? M_1 could be $\deg(\alpha)$, $\deg(\beta)$, $\deg(\gamma)$, or $\deg(\delta)$ and M_2 could be $\deg(\gamma)+s$, $\deg(\gamma)+t$, $\deg(\delta)+p$, or $\deg(\delta)+r$. We proceed by investigating these cases.

We begin by using the fact that the syzygies are homogeneous to describe some relations on the exponents.

$$v_+ + p + \deg(\alpha) = u_- + r + \deg(\beta) = t + \deg(\gamma),$$

$$v_- + r + \deg(\alpha) = u_+ + p + \deg(\beta) = s + \deg(\gamma),$$

$$v_- + t + \deg(\alpha) = u_- + s + \deg(\beta) = p + \deg(\delta),$$

$$v_+ + s + \deg(\alpha) = u_+ + t + \deg(\beta) = r + \deg(\delta).$$

From these equalities, we can distill the inequalities $\deg(\gamma) + \deg(\delta) \geqslant 2\deg(\alpha)$ and $\deg(\gamma) + \deg(\delta) \geqslant 2\deg(\beta)$. Hence $M_1 = \deg(\gamma)$ or $\deg(\delta)$. Since γ and δ are interchangeable, we can assume $M_1 = \deg(\delta)$.

This leaves us with four cases to check corresponding to the four possible values of M_2. In each case we consider the expression for $M_1M_2 - 2\deg(J)$. We expand the expression in terms of u_+, u_-, v_+, v_-, p, r, s and t and then eliminate two of the variables using the equations arising from the homogeneity conditions. The choice of which variables to eliminate is not obvious, but there is always a nice choice which makes it easier to show that the expression is nonnegative. Then, in each case, we can show that the expression is nonnegative by using the inequalities that arise from the choices of M_1 and M_2. Which inequalities were necessary and how to use them were not obvious at first glance so a computer program PORTA [2] was used to help reduce the inequalities. Once it was clear what we should look for, it was easy to do these by hand.

Consider the case where $M_2 = \deg(\gamma) + t$. If we eliminate u_- and v_+, the expression for $M_1M_2 - \deg(J)$ can be rewritten as

$u_+^2 + v_-^2 + u_+(p-s) + v_-(r-s) + u_+(t-r) + u_+t + v_-(t-p) + v_-t + 2t^2$.

The choices of M_1 and M_2 in this case imply that $p \geqslant s$, $r \geqslant s$, $t \geqslant r$, and $t \geqslant p$. It is clear, therefore, that the above expression is nonnegative.

The other three cases look similar although different eliminations and inequalities are used for each case.

So, for every possible choice of M_1 and M_2, the expression $M_1M_2 - 2\deg(J)$ is nonnegative. Therefore all ideals J arising from a syzygy quadrangle satisfy the bound of Conjecture 3. □

Proof of Theorem 5. If $R/I_{\mathscr{L}}$ is Cohen–Macaulay, then we know from Herzog and Srinivasan's paper [4] that it satisfies the bound. So, let us assume $R/I_{\mathscr{L}}$ is not Cohen–Macaulay. We may construct a resolution for $R/I_{\mathscr{L}}$ via its syzygy quadrangles.

Let J be an ideal whose four generators are associated to a single syzygy quadrangle of $I_{\mathscr{L}}$. Since, according to Theorem 4, the syzygies from this resolution are also syzygies of $R/\mathscr{L}$, we know that $M_1(J) \leqslant M_1(I_{\mathscr{L}})$ and $M_2(J) \leqslant M_2(I_{\mathscr{L}})$. Together these imply that

$$\frac{M_1(J)M_2(J)}{2} \leqslant \frac{M_1(I_{\mathscr{L}})M_2(I_{\mathscr{L}})}{2}.$$

On the other hand, since $J \subset I_{\mathscr{L}}$, we have that $\deg(I_{\mathscr{L}}) \leqslant \deg(J)$.

By Lemma 6, we know that the bound holds for J. That is

$$\deg(J) \leqslant \frac{M_1(J)M_2(J)}{2}.$$

Hence,

$$\deg(I_{\mathscr{L}}) \leqslant \deg(J) \leqslant \frac{M_1(J)M_2(J)}{2} \leqslant \frac{M_1(I_{\mathscr{L}})M_2(I_{\mathscr{L}})}{2}. \qquad \square$$

4. Further thoughts

We have shown that the conjecture of Herzog and Srinivasan is true for codimension two lattice ideals. For non-Cohen–Macaulay codimension 2 lattice ideals, this bound cannot be tight, that is, we cannot force the expression $M_1M_2 - 2\deg(J)$ to be zero. Suppose we try to force the expression to be zero, then the squared variables would have to be zero. In the case where $M_2 = \deg(\gamma) + t$ for instance, it would force u_+, v_-, and t to be zero. So $M_2 = \deg\gamma$ which forces $p = r = s = t = 0$. Hence $I_{\mathscr{L}}$ is Cohen–Macaulay and we have a contradiction. The other cases are similar.

Although we cannot find an ideal where equality holds, one way to try to make the above expression small, that is, to make M_1M_2 close to $2\deg(J)$, is to choose an ideal where $p=r=s=t$. Doing this also forces $u_+=u_-$ and $v_+=v_-$. Hence the expression $M_1M_2-2\deg(J)=u_+^2+v_+^2+u_+p+u_-p+2p^2$ for all choices of M_1 and M_2. If we then let $u_+=v_+=0$ and $p=1$, we get some ideal of degree 2 with $M_1=2$ and $M_2=3$. For example, in four variables the lattice generated by $(1,-1,-1,1)$ and $(1,-1,1,-1)$ gives the ideal $(ad-bc, ac-bd, a^2-b^2, c^2-d^2)$ which has this form. Thus the bound is quite close to being tight. On the other hand, if we do not require $u_+=v_+=0$, the expression $M_1M_2-2\deg(J)$ increases like $(\deg(\alpha))^2$ as u_+, u_-, or p increases.

The general form of a resolution for codimension 3 or higher lattice ideals is unknown. These higher codimension lattice ideals do not seem to lend themselves to the same sort of decomposition as in codimension 2, so extending the method we used here to prove the bound for codimension 3 does not seem promising. The case of non-Cohen–Macaulay ideals of codimension 2 other than lattice ideals is also still open.

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