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On the arithmetic nature of hypertranscendental functions at complex points

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ABSTRACT

Most well-known transcendental functions usually take transcendental values at algebraic points belonging to their domains, the algebraic exceptions forming the so-called exceptional set. For instance, the exceptional set of the function $e^{z-\sqrt{2}}$ is the set $\{\sqrt{2}\}$, as follows from the Hermite–Lindemann theorem. In this paper, we shall use interpolation formulae to prove that any subset of $\overline{\mathbb{Q}}$ is the exceptional set of uncountably many hypertranscendental entire functions with order of growth as small as we wish. Moreover these functions are algebraically independent over $\mathbb{C}$.

1. Introduction: a brief survey on transcendental numbers

We say that a complex number α is *algebraic* if there exists a nonzero polynomial $P \in \mathbb{Q}[x]$ with $P(\alpha) = 0$. If no such polynomial exists, α is *transcendental*. The set of algebraic numbers forms a field denoted by $\overline{\mathbb{Q}}$.

Euler was probably the first person to define transcendental numbers in the modern sense (see [4]). But transcendental number theory began in 1844 with Liouville's proof [10] that if an algebraic number α has degree $n > 1$, then there exists a constant $C > 0$ such that $|\alpha - p/q| > Cq^{-n}$, for all $p/q \in \mathbb{Q} \setminus \{0\}$. Using this result, Liouville gave the first explicit examples of transcendental numbers, e.g., the "Liouville number" $\sum_{n\geq 0} 10^{-n!}$. There are several classical theorems on transcendental numbers. Let us state three of them to make this text self-contained.

In 1872 [6] proved that e is transcendental, and in 1884 [9] extended Hermite's method to prove that π is also transcendental. In fact, Lindemann proved a more general result.

Theorem 1 (*Hermite–Lindemann*)**.** *The number* e^{α} *is transcendental for any nonzero algebraic number* α.

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As a consequence, the numbers $\mathrm{e}^{\sqrt{2}}$ and e^{i} are transcendental ($\mathrm{i} = \sqrt{-1}$), as are $\log 2$ and π, since $\mathrm{e}^{\log 2} = 2$ and $\mathrm{e}^{\pi \mathrm{i}} = -1$ are algebraic.

At the 1900 International Congress of Mathematicians in Paris, as the seventh in his famous list of 23 problems, Hilbert gave a big push to transcendental number theory with his question of the arithmetic nature of the power α^β of two algebraic numbers α and β. In 1934, Gelfond and Schneider, independently, completely solved the problem (see [1, p. 9]).

Theorem 2 (*Gelfond–Schneider*)**.** *Assume that α and β are algebraic numbers, with $\alpha \neq 0$ or 1, and β irrational. Then α^β is transcendental.*

In particular, $2^{\sqrt{2}}$, $(-1)^{\sqrt{2}}$, and $\mathrm{e}^{\pi} = i^{-2i}$ are all transcendental (we refer the reader to [18,13,7] for recent results on the arithmetic nature of x^y, with both x and y transcendental). Since the sum of transcendental numbers can be algebraic (e.g., $e + (-e)$), one may ask about the nature of the sum of transcendental numbers as in the Hermite–Lindemann theorem. For instance, is $e + \mathrm{e}^{\sqrt{2}}$ transcendental? This natural question leads to a beautiful generalization of the Hermite–Lindemann theorem due to Lindemann and Weierstrass.

Theorem 3 (*Lindemann–Weierstrass*)**.** *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then $\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

An *algebraic function* is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial with complex coefficients. For instance, functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function which is not algebraic is, by definition, a *transcendental function*, for example the trigonometric functions, the exponential function, and their inverses. An interesting task is to study the arithmetic nature of a function at algebraic points. For instance, it is a simple matter to show that an *entire* function, namely a function which is analytic in $\mathbb{C}$, is a transcendental function if and only if it is not a polynomial. Thus, one may be interested in thinking only of the case of transcendental functions.

At the end of the XIXth century, after the proof by Hermite and Lindemann of the transcendence of e^α for all nonzero algebraic α, a question arose:

($*$) Does a transcendental analytic function usually take transcendental values at algebraic points?

In the example of the exponential function e^z, the word "usually" means avoiding the exception $z = 0$. After the Hermite–Lindemann theorem, it was expected that by evaluating a transcendental function f at an algebraic point of its domain, we would find a transcendental number, but exceptions can arise. All these exceptions (i.e., algebraic numbers at which the function assumes algebraic values) form the so-called *exceptional set*, denoted by S_f. This set plays an important role in transcendental number theory (see, e.g., [23] and references therein).

In 1886, Weierstrass found a positive answer for the question ($*$), when he gave an example of a transcendental entire function which takes rational values at all rational points. Later, [19] proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(\Sigma) \subseteq T$. Another construction due to [20] produces an entire function f whose derivatives $f^{(t)}$, for $t = 0, 1, 2, \ldots$, all map $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}$ and so $S_{f^{(t)}} = \overline{\mathbb{Q}}$. Two years later, G. Faber refined this result by showing the existence of a transcendental entire function such that $f^{(t)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for all $t \geq 0$. A more elegant discussion on this subject can be found in [11,23].

In this paper, we were able to generalize these two results, Stäckel's and Faber's. Before we state our main theorem, we need a couple of definitions. A set of functions $f_1, \ldots, f_m$ is said to be *algebraically independent over* a field K if there is no nonzero polynomial P, with coefficients in K, such that $P(f_1(z), \ldots, f_m(z))$ is the zero function. Otherwise, they are called *algebraically dependent over K*. In 1949, Morduhai and Boltovskoi introduced the term *hypotranscendental function f* by saying that there exists $n \geq 0$ such that the functions $z, f(z), \ldots, f^{(n)}(z)$ are algebraically dependent over $\mathbb{Q}$. Otherwise, the function is called *hypertranscendental*, or *transcendentally transcendental*; see [15].

Definition 1 (*Order*)**.** Let f be an entire function and $R > 0$; *the order of growth of f* is defined to be

$$\lim_{R\to\infty} \sup \frac{\log\log |f|_R}{\log R}, \quad \text{where } |f|_R = \sup_{|z|=R} |f(z)|.$$

By definition, it follows that a function f that satisfies $|f|_R \leq e^{R^\rho}$ for some $\rho > 0$ and for all R sufficiently large has order $\leq\rho$. Surprisingly, there exists a straightforward relation between the order of a function and its integer values; Chudnovsky [3] proved that if f has order ρ, then the set

$$\{z \in \mathbb{C} : f^{(t)}(z) \in \mathbb{Z} \text{ for all } t \geq 0\}$$

has cardinality at most ρ. For more see [3, Chapter 9].

Let us state our main result:

Theorem 4. *Let A be a countable subset of $\mathbb{C}$ and let ρ be a positive real number. For any integer $s \geq 0$ and any $\alpha \in A$, let $E_{\alpha,s}$ be a dense subset in $\mathbb{C}$. Then there exists a set $\mathcal{F}$ of entire functions with the following properties:*

(a) *For any $f \in \mathcal{F}$, any $\alpha \in A$ and any integer $s \geq 0$, $f^{(s)}(\alpha) \in E_{\alpha,s}$.*
(b) *Any function $f \in \mathcal{F}$ has order at most ρ;*

If $\mathcal{F}^{(s)}$ denotes the set of sth derivatives of functions in $\mathcal{F}$, that is, $\mathcal{F}^{(s)} = \{f^{(s)} : f \in \mathcal{F}\}$, then:

(c) *For any integer $m \geq 1$, any distinct functions $f_1, \ldots, f_m \in \bigcup_{s\geq 0} \mathcal{F}^{(s)}$ and any nonzero polynomial $P \in \mathbb{C}[X_0, X_1, \ldots, X_m]$, the entire function $P(z, f_1(z), \ldots, f_m(z))$ is not the zero function.*
(d) *The set $\mathcal{F}$ has the power of the continuum.*

Note that the property (c) ensures that the functions in $\mathcal{F}$ are hypertranscendental.

One basic problem in the theory of transcendental numbers is to determine S_f, or at least to find properties of this set. It is almost unnecessary to stress that this is not an easy problem. The question of the possible exceptional sets was partially solved in 1965, when Mahler [12] proved that if A is *closed relative to* $\overline{\mathbb{Q}}$, that is if $\alpha \in A$ then all its algebraic conjugates lie also in A, then it is the exceptional set of some transcendental function. Since the exceptional sets of a function and its derivative can be different, in this work we consider a more general definition (including multiplicity): Let f be an entire function. We define *the exceptional set with multiplicity of f* to be the set of pairs $(\alpha, t) \in \overline{\mathbb{Q}} \times (\mathbb{N} \cup \{0\})$ such that $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. We denote it by M_f.

In this paper we solve completely the problem of the possible exceptional sets with multiplicity of a hypertranscendental function.

Theorem 5. *If $A \times N \subseteq \overline{\mathbb{Q}} \times \mathbb{N}_0$, then there is an uncountable set $\mathcal{F}_{A,N}$ of hypertranscendental entire functions such that*

$$M_f = A \times N, \tag{1.1}$$

for all $f \in \mathcal{F}_{A,N}$. Moreover the set

$$\{f^{(t)}(\alpha) : (\alpha, t) \notin A \times N \text{ and } f \in \mathcal{F}_{A,N}\} \tag{1.2}$$

is algebraically independent over $\mathbb{C}$.

2. Preliminary results

Some notation: throughout the paper we write $L(P)$ for the sum of absolute values of coefficients of a polynomial P, well-known as the *length of* P, $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$ and $[a, b] = \{a, a+1, \ldots, b\}$, where $a < b$ are integers.

Before upsetting the reader with plenty of technical lemmas, we start with a brief overview of our strategy for proving Theorem 1. We hope that this makes the following lemmas a little more natural. In Theorem 1, we wish to find functions with certain prescribed properties. Well, such a function will be taken as $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$, where the polynomials P_n will be appropriately chosen. First of all, we need ensure that f is an entire function, and has a prescribed growth order; for that the a_k's will be chosen as centers of balls with radii depending on P_k and of the required order. Secondly, the sequence $(P_n)_{n\geq 0}$ will be made explicit and it must be a key property—namely, for a certain sequence $(s_n)_{n\geq 0}$ (to be made explicit and depending on an enumeration of $\overline{\mathbb{Q}} = \{\alpha_1, \ldots\}$) the set of the indices for which $P_k^{(s_n)}(\alpha_n) \neq 0$ is bounded. This ensures that $f^{(s)}(\alpha)$ is actually a finite sum. After that, since $f^{(s)}(\alpha)$ is a finite sum, we can proceed by induction for finding infinite possibilities for each a_k, which can be chosen in a infinite set, namely the intersection of a ball with a dense set. Finally, the possibility of choosing a_k in an infinite set together with the property of the $P_k^{(s)}$'s guarantees the uncountability of these possible functions.

Now, let us get to the work.

Lemma 1. *Let $P(z) \in \mathbb{C}[z]$ be a polynomial and $d \geq \deg(P)$ (in the case of $P \equiv 0$, d can be taken as any non-negative real number); then*

$$|P(z)| \leq L(P)\max\{1, |z|\}^d, \quad \textit{for all } z \in \mathbb{C}. \tag{2.1}$$

Proof. Write

$$P(z) = a_0 + a_1 z + \cdots + a_{\deg(P)} z^{\deg(P)}.$$

The triangular inequality yields

$$|P(z)| \leq |a_0| + |a_1||z| + \cdots + |a_{\deg(P)}||z^{\deg(P)}|.$$

Since $|z|^k \leq \max\{1, |z|\}^{\deg(P)}$, for all $k \in [0, \deg(P)]$ and $z \in \mathbb{C}$, we get

$$\begin{aligned} |P(z)| &\leq (|a_0| + \cdots + |a_{\deg(P)}|)\max\{1, |z|\}^{\deg(P)} \\ &\leq L(P)\max\{1, |z|\}^d. \quad \square \end{aligned}$$

Lemma 2 (*Analyticity*)**.** *Let $(P_n(z))_{n\geq 0} \in \mathbb{C}[z]$ be a sequence of nonzero polynomials, and let ρ be a positive real number. Set $m_0 = 1$ and by recurrence $m_k = \max\{\lceil \frac{\deg(P_k)}{\rho} \rceil, m_{k-1} + 1\}$ for $k \geq 1$. If the sequence $(a_n)_{n\geq 0} \in \mathbb{C}$ satisfies*

$$|a_k| \leq \frac{1}{L(P_k) m_k!} \tag{2.2}$$

for all $k \geq 0$, then the series $\sum a_n P_n(z)$ converges absolutely and uniformly on any compact sets; in particular, this gives an entire function; moreover its sum $f(z)$ has order at most ρ.

Proof. We define $(Q_n(z))_{n\geq 0} \in \mathbb{C}[z]$ and $(b_n)_{n\geq 0} \in \mathbb{C}$ as follows:

$$Q_n(z) = \begin{cases} 0, & \text{if } n \neq m_k \\ P_k(z), & \text{if } n = m_k \end{cases}$$

and $b_{m_k} = a_k$ for $k \geq 0$. Since $1 = m_0 < m_1 < m_2 < \cdots$, we have that the Q_n's and b_n's are well defined and moreover $\sum_{n=0}^{\infty} a_n P_n(z) = \sum_{n=0}^{\infty} b_n Q_n(z)$. Below, one can see the gaps of zeros in the sequence $(Q_n)_{n\geq 0}$:

$$0, P_0(z), \underbrace{0, \ldots, 0}_{m_1 - m_0 - 1}, P_1(z), \underbrace{0, \ldots, 0}_{m_2 - m_1 - 1}, P_2(z), 0, \ldots.$$

Also, if Q_n is nonzero, then $n = m_k$ for some $k \geq 0$. Hence

$$\deg(Q_n) = \deg(Q_{m_k}) = \deg(P_k) \leq m_k \rho = n\rho.$$

Thus we get by Lemma 1

$$|Q_n(z)| \leq L(Q_n)\max\{1, |z|\}^{n\rho},$$

for all $n \geq 0$. Let $K \subseteq \mathbb{C}$ be a compact set; then $|z| \leq R$, for some $R > 0$ and any $z \in K$. Therefore, in K, we have

$$\begin{aligned}\left|b_{m_k} Q_{m_k}(z)\right| &\leq |b_{m_k}||L(Q_{m_k})| \max\{1, |z|\}^{m_k \rho} \\ &= |a_k||L(P_k)| \max\{1, |z|\}^{m_k \rho} \\ &\leq \frac{\max\{1, |z|\}^{m_k \rho}}{m_k!}.\end{aligned}$$

We conclude that $|b_{m_k} Q_{m_k}(z)| \leq M_k$, for all $z \in K$, where $M_k = \frac{\max\{1,R\}^{m_k \rho}}{m_k!}$. On the other hand,

$$\sum_{k=0}^{\infty} M_k \leq \sum_{n=0}^{\infty} \frac{\max\{1, R\}^{n\rho}}{n!} = e^{\max\{1,R\}^{\rho}}. \tag{2.3}$$

Therefore $f(z) = \sum_{k=0}^{\infty} b_{m_k} Q_{m_k}(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ is an entire function (by the Weierstrass M-test). From the inequality in (2.3), we deduce that f has order at most ρ. □

Now, let us enumerate the set A in Theorem 4 as $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. All integer numbers $n \geq 1$ can be written uniquely in the form $n = \frac{m_n(m_n+1)}{2} + j_n$; for $m_n \geq 0$ and $1 \leq j_n \leq m_n + 1$, define $\gamma_n = \alpha_{m_n+2-j_n}$. Now, let us construct a sequence of polynomials as follows:

$$P_0(z) = 1 \quad \text{and} \quad P_n(z) = (z - \gamma_1) \cdots (z - \gamma_n) \quad \text{for } n \geq 1.$$

Here we list the first few polynomials:

$$\begin{aligned}&P_0(z) = 1\\ &P_1(z) = (z - \alpha_1)\\ &P_2(z) = (z - \alpha_1)(z - \alpha_2)\\ &P_3(z) = (z - \alpha_1)^2(z - \alpha_2)\\ &P_4(z) = (z - \alpha_1)^2(z - \alpha_2)(z - \alpha_3)\\ &P_5(z) = (z - \alpha_1)^2(z - \alpha_2)^2(z - \alpha_3)\\ &P_6(z) = (z - \alpha_1)^3(z - \alpha_2)^2(z - \alpha_3)\\ &P_7(z) = (z - \alpha_1)^3(z - \alpha_2)^2(z - \alpha_3)(z - \alpha_4)\\ &\vdots\end{aligned}$$

The pattern can be seen by following the arrows and picking up the corresponding term at each node in Fig. 1.

With the same notation, we set $s_n = j_n - 1$.

Lemma 3 (*Truncation*)**.** *For $n \geq 1$, we have $P_{n-1}^{(s_n)}(\gamma_n) \neq 0$ and $P_l^{(s_n)}(\gamma_n) = 0$ when $l \geq n$.*

Proof. Let us partition the set of these polynomials into infinitely many disjoint sets, in the following way:

$$D_0 = \{P_0\} \quad \text{and} \quad D_m = \{P_{d_m}, P_{d_m+1}, \ldots, P_{d_m+(m-1)}\}$$

where $d_m = m + \frac{(m-1)(m-2)}{2}$, for $m > 0$. Explicitly, the m polynomials in D_m are defined as

$$P_{d_m}(z) = (z - \alpha_1)^{m-1}(z - \alpha_2)^{m-2} \cdots (z - \alpha_{m-2})^2(z - \alpha_{m-1})(z - \alpha_m)$$

and for $j \in [1, m-1]$,

$$P_{d_m+j}(z) = P_{d_m}(z) \prod_{t=1}^{j}(z - \alpha_t).$$

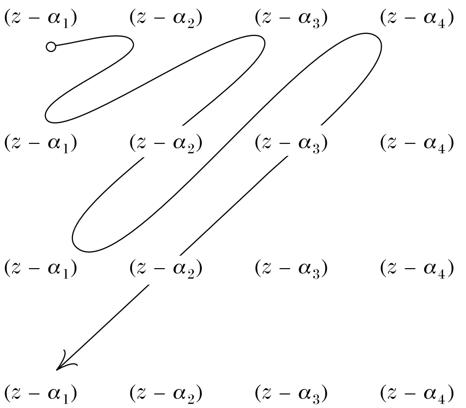

Fig. 1. Building the P_n's.

Also, we may deduce that $\gamma_{d_m+k} = \alpha_{m-k}$ and $s_{d_m+k} = k$. Now, by construction of the polynomials, it is enough to prove the lemma for $k = n$. Let us distinguish two cases. The first one is that where P_{n-1} and P_n are in D_m, for some $m \geq 2$. Thus $P_{n-1} = P_{d_m+k}$ and $P_n = P_{d_m+k+1}$, for some $k \in [0, m-2]$. Therefore we must prove that $P_{d_m+k}^{(k+1)}(\alpha_{m-k-1}) \neq 0$ and $P_{d_m+k+1}^{(k+1)}(\alpha_{m-k-1}) = 0$. The result follows because α_{m-k-1} is a zero of P_{n-1} with multiplicity $k+1$, which means $P_{n-1}^{(s_n)}(\gamma_n) \neq 0$ and, on the other hand, α_{m-k-1} is a zero of $P_n(z)$ with multiplicity $k+2$, which implies $P_n^{(s_n)}(\gamma_n) = 0$.

The second case is that where $P_{n-1} \in D_{m-1}$ and $P_n \in D_m$, for some $m \geq 1$. In this case $P_n(z) = P_{n-1}(z)(z - \alpha_m)$, where

$$P_{n-1}(z) = (z - \alpha_1)^{m-1} \cdots (z - \alpha_{m-2})^2 (z - \alpha_{m-1}).$$

It is easy see that $P_{n-1}^{(s_n)}(\gamma_n) = P_{n-1}(\alpha_m) \neq 0$ and $P_n^{(s_n)}(\gamma_n) = P_n(\alpha_m) = 0$. □

Lemma 4 (*Identity*). *If $\sum_{k=0}^{\infty} a_k P_k(z) = \sum_{k=0}^{\infty} b_k P_k(z)$ for all $z \in \mathbb{C}$, then $a_k = b_k$ for each $k \geq 0$.*

Proof. It suffices to prove that if $f(z) := \sum_{k=0}^{\infty} a_k P_k(z) = 0$ for all $z \in \mathbb{C}$, then $(a_k)_{k\geq 0}$ is identically 0. Notice that $a_0 = f(\alpha_1) = 0$. Assuming that $a_0, a_1, \ldots, a_{n-1}$ are all 0, by Lemma 3, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} a_k P_k^{(s_{n+1})}(\alpha_{\gamma_{n+1}}) \\ &= \sum_{k=0}^{n-1} a_k P_k^{(s_{n+1})}(\gamma_{n+1}) + a_n P_n^{(s_{n+1})}(\alpha_{j_{n+1}}) + \sum_{k=n+1}^{\infty} a_k P_k^{(s_{n+1})}(\alpha_{j_{n+1}}) \\ &= a_n P_n^{(s_{n+1})}(\gamma_{n+1}). \end{aligned}$$

Since $P_n^{(s_{n+1})}(\gamma_{n+1}) \neq 0$, we have $a_n = 0$. Hence the proof will be completed by induction. □

Now we are able to prove our first theorem.

3. Proof of Theorem 1

We are going to construct the desired entire function by fixing the coefficients in the series $\sum_{k=0}^{\infty} a_k P_k(z)$ recursively, where the sequence $(P_k)_{k\geq 0}$ has been defined in Section 2.

First, with the same notation as in Lemma 2, the condition $|a_k| \leq t_k := \frac{1}{L(P_k)m_k!}$ will ensure $\sum_{k=0}^{\infty} a_k P_k(z)$ to be entire with order at most ρ.

Next, we will fix the coefficients a_k recursively. For $n \geq 1$, we define $E_n = E_{\gamma_n,s_n}$ and let the numbers $\beta_n := f^{(s_n)}(\gamma_n) = \sum_{k=0}^{\infty} a_k P_k^{(s_n)}(\gamma_n)$. We are going to choose the value of a_k such that $\beta_n \in E_{\gamma_n,s_n} = E_n$ for all $n \geq 1$.

By Lemma 2, we know that $P_l^{(s_n)}(\gamma_n) = 0$ when $l \geq n$, so β_n is actually the finite sum $\sum_{k=0}^{n-1} a_k P_k^{(s_n)}(\gamma_n)$. Notice that $\beta_1 = a_0 P_0^{(0)}(\alpha_1) = a_0$ and E_1 is dense, so we can choose a value for a_0 in an infinite set I_0 such that $0 < |a_0| \leq t_0$ and $\beta_1 \in E_1$. Now suppose that the values of $\{a_0, a_1, \ldots, a_{n-1}\}$ are well fixed respectively in infinite sets I_k such that $0 < |a_k| \leq t_k$ and $\beta_k \in E_k$ for $0 \leq k \leq n-1$. By Lemma 3, we know that $P_n^{(s_{n+1})}(\gamma_{n+1}) \neq 0$; set

$$I_n := \left(\frac{E_n - A_n}{P_n^{(s_{n+1})}(\gamma_{n+1})} \right) \cap B(0; t_n) \setminus \{0\},$$

where $A_n := \sum_{k=0}^{n-1} a_k P_k^{(s_{n+1})}(\gamma_{n+1})$. So we can pick a proper value of a_n in the infinite set I_n; thus $0 < |a_n| \leq t_n$ and $\beta_n = \sum_{k=0}^{n-1} a_k P_k^{(s_{n+1})}(\gamma_{n+1}) + a_n P_n^{(s_{n+1})}(\gamma_{n+1}) \in E_n$.

So now by induction, all the a_k are well chosen such that for all $k \geq 0$ we have $0 < |a_k| \leq t_k$ and $\beta_{k+1} \in E_{k+1}$. Thus f is an entire function satisfying the conditions (a) and (b).

Let $\mathcal{F}_A$ be the set of all entire functions satisfying the conditions (a) and (b). Set $I = I_0 \times I_1 \times \cdots$ and consider the function $\phi : I \to \mathcal{F}_A$ given by $\phi(a_0, a_1, \ldots) = \sum_{n=0}^{\infty} a_n P_n(z)$. That ϕ is well defined follows from proof above; also Lemma 4 implies that ϕ is one-to-one. Hence $\mathcal{F}_A$ is uncountable, since I is.

There exists an uncountable set $\mathcal{B} := \{\xi\} \cup \{T_{\lambda,s}\}_{\lambda \in \Lambda, s \geq 0}$ algebraically independent over $\overline{\mathbb{Q}}$ (for instance the transcendental basis of the field extension $\mathbb{C}/\mathbb{Q}$). Consider $A' = \{\xi\} \cup A$. Fix $\lambda \in \Lambda$; set $E_{\xi,s}^{\lambda} = \{\alpha T_{\lambda,s} : \alpha \in \overline{\mathbb{Q}} \setminus \{0\}\}$ and $E_{\alpha_n,s}^{\lambda} = E_{\alpha,s}$ for all $\alpha \in A$ and $s \geq 0$. By the all previous discussion, there exists a set $\mathcal{F}_\lambda$ of entire functions satisfying the conditions (b) and (d), as well as the condition (a) for the new set A' (which is still countable). Next, for each $\lambda \in \Lambda$ take a unique function $f_\lambda \in \mathcal{F}_\lambda$. Set $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$; we shall prove that this is our desired set. In fact, by construction, this set satisfies the conditions (a), (b) and (d). To prove (c), take distinct functions $f_1, \ldots, f_m \in \bigcup_{s \geq 0} \mathcal{F}^{(s)}$. Therefore $f_j(z) = f_{\lambda_j}^{(s_j)}(z)$ for $j = 1, \ldots, m$ and for some pairwise distinct pairs $(\lambda_1, s_1), \ldots, (\lambda_m, s_m) \in \Lambda \times \mathbb{N}_0$. It follows that $f_j(\xi) = \gamma_j T_{\lambda_j,s_j}$ for $j = 1, \ldots, m$ and some γ's $\in \overline{\mathbb{Q}} \setminus \{0\}$. This yields that the numbers $\xi, f_1(\xi), \ldots, f_m(\xi)$ are algebraically independent and then that (c) holds. □

Before going further, it is worth noting some interesting consequences of Theorem 4 which give generalizations for classical results on this subject. The suitable choices of A, $E_{\alpha,s}$ are noted in parentheses.

Corollary 1 (*Generalization of the First Stäckel Theorem*). *For each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$ there is a hypertranscendental entire function f such that $f^{(s)}(\Sigma) \subseteq T$ for $s \geq 0$. ($A = \Sigma$, $E_{\alpha,s} = T$.)*

Corollary 2 (*Generalization of the Second Stäckel Theorem*). *Let $A \subseteq \mathbb{C}$ be countable and dense in $\mathbb{C}$; then there is a hypertranscendental entire function f such that $f^{(s)}(A) \subseteq A$, for $s \geq 0$. ($E_{\alpha,s} = A$.)*

Corollary 3 (*Generalization of Faber's Theorem*). *There exists a hypertranscendental entire function such that $f^{(s)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for $s \geq 0$. ($A = \overline{\mathbb{Q}}$, $E_{\alpha,s} = \mathbb{Q}(i)$.)*

4. Applications to exceptional sets: proof of Theorem 2

4.1. An overview on exceptional sets

Weierstrass (see [11]) initiated the notion of investigating the set of algebraic numbers where a given transcendental entire function f takes algebraic values. For an entire function f, we define the *exceptional set* of f as follows:

$$S_f = \{\alpha \in \overline{\mathbb{Q}} : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

The study of exceptional sets started in 1886 with a letter from Weierstrass to Strauss. This study was later developed by Strauss, Stäckel, and Faber. Further results are due to van der Poorten, Gramain, Surroca and others (see [5,21]).

Some exceptional sets:

Example 1. Any finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \overline{\mathbb{Q}}$ is exceptional. In fact, if $f_1(z) = e^{(z-\alpha_1)\cdots(z-\alpha_k)}$, then the Hermite–Lindemann theorem implies $S_{f_1} = \{\alpha_1, \ldots, \alpha_k\}$.

Example 2. The empty set is also exceptional. Indeed, if $f_2(z) = e^z + e^{z+1}$, the Lindemann–Weierstrass theorem implies $S_{f_2} = \emptyset$.

Example 3. Some infinite sets are also known to be exceptional. For instance, if $f_3(z) = 2^z, f_4(z) = e^{i\pi z}$, then $S_{f_3} = S_{f_4} = \mathbb{Q}$, by the Gelfond–Schneider theorem.

We point out that is not known whether an elementary function[1] with an exceptional set is in $\mathbb{Z}$ or $\mathbb{N}$. For obtaining such examples, we appeal to Schanuel's conjecture, one of the main open problems in transcendental number theory.

Conjecture 1 (*Schanuel*). *If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$, then among the numbers $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$, at least n are algebraically independent.*

This conjecture was introduced in the 1960's by Schanuel in a course given by [8]. Several classical consequences of this conjecture, together with an elegant exposition of it, can be found in [17, Chapter 10, Section 7G]. Very recent consequences can be found in [2,14,22].

Example 4. Assume that Schanuel's conjecture is true. If $f_5(z) = \sin(\pi z)e^z, f_6(z) = 2^{3^z}$ and $f_7(z) = 2^{2^{2^{z-1}}}$, then $S_{f_5} = S_{f_6} = \mathbb{Z}$ and $S_{f_7} = \mathbb{N}$.

Summarizing, the sets $\emptyset$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$ (take $\Sigma = T = \overline{\mathbb{Q}}$ in the first Stäckel theorem) and all finite sets are exceptional. But, what are all the possible exceptional sets?

Before answering this question, observe that the exceptional sets of a function f and its derivative f' can be distinct. For instance, if $f(z) = 2^z$, then $S_f = \mathbb{Q}$. However, $f'(z) = 2^z \log 2$ and thus $S_{f'} \cap S_f = \emptyset$ (since $\log 2$ is transcendental). This motivates a more general definition where multiplicities are included. Let f be an entire function. We define *the exceptional set with multiplicity of f* to be

$$M_f = \{(\alpha, t) \in \overline{\mathbb{Q}} \times \mathbb{N}_0 : f^{(t)}(\alpha) \in \overline{\mathbb{Q}}\}.$$

Example 5. If $f(z) = e^z + \sum 10^{-n!}$, $g(z) = e^z + e^{z+1}$ and $h(z) = e^z$, then $M_f = \{0\} \times \mathbb{N}$, $M_g = \emptyset$ and $M_h = \{0\} \times \mathbb{N}_0$.

A relation between S_f and M_f is given in the next result.

Proposition 1. *If $M_f = A \times N$, then $S_{f^{(t)}} = A$ for all $t \in N$.*

Proof. If $t \in N$ and $\alpha \in \overline{\mathbb{Q}}$, then $\alpha \in S_{f^{(t)}}$, if and only if $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. Since $M_f = A \times N$ and $t \in N$, then $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$ if and only if $\alpha \in A$. □

In view of the previous proposition, we can restate our question: what are the possible subsets of $\overline{\mathbb{Q}} \times \mathbb{N}_0$ which are exceptional sets with multiplicity of a transcendental function?

[1] A function built from a finite number of exponentials, logarithms, constants, one variable, and nth roots through composition and combinations using the four elementary operations $(+, -, \times, \div)$. By allowing these functions (and constants) to be complex numbers, trigonometric functions and their inverses become included in the elementary functions.

How about the previous question where we replace transcendental functions by hypertranscendental functions? Recall that by a hypertranscendental function, we mean a function which does not satisfy any algebraic differential equations. Clearly, hypertranscendental functions are transcendental. The exponential function e^z gives an example of a transcendental function which is not hypertranscendental and the well-known zeta function ($\zeta(z)$) and Gamma function ($\Gamma(z)$) are hypertranscendental; see [16]. Moreover (see [16]), sums, products, differences, quotients and compositions of hypotranscendental functions are again hypotranscendental; e.g., the function $\sin(e^{e^{1/z}} - 2^{\pi \log z})$ is hypotranscendental.

In view of that, we note that all the previous functions f_i, with $i \in [1, 7]$, are hypotranscendental. Hence there arises a very much stronger question: what are the possible exceptional sets with multiplicity of hypertranscendental functions?

All this mystery is finished by Theorem 5: every $A \times N \subseteq \overline{\mathbb{Q}} \times \mathbb{N}_0$ is the exceptional set with multiplicity of uncountably many hypertranscendental entire functions with order of growth as small as we wish. In particular, when $N = \mathbb{N}_0$, $A \subseteq \overline{\mathbb{Q}}$, Theorem 5 and Proposition 1 yield:

Corollary 4. *If $A \subseteq \overline{\mathbb{Q}}$, then there is an uncountable set, $\mathcal{F}_A$, of hypertranscendental entire functions such that, if $f \in \mathcal{F}_A$, then*

$$S_{f^{(t)}} = A \quad \text{for } t \geq 0.$$

Moreover, the set

$$\{f^{(n)}(\alpha) : \alpha \in \overline{\mathbb{Q}} \setminus A,\ n \geq 0 \text{ and } f \in \mathcal{F}_A\} \tag{4.1}$$

is algebraically independent.

Thus, all that remains is to prove Theorem 5.

4.2. Proof of Theorem 5

Suppose that A, $\overline{\mathbb{Q}} \setminus A$, N and $\mathbb{N}_0 \setminus N$ are all infinite sets; thus we can enumerate $\overline{\mathbb{Q}} = \{\alpha_0, \alpha_1, \ldots\}$ and $\mathbb{N}_0 = \{s_0, s_1, \ldots\}$ where $A = \{\alpha_2, \alpha_4, \ldots, \alpha_{2n}, \ldots\}$ and $N = \{s_2, s_4, \ldots, s_{2n}, \ldots\}$. Consider $\{T_{\lambda,m,l} : \lambda \in \Lambda \text{ and } (m, l) \in \mathbb{N}_0 \times \mathbb{N}_0\}$, an uncountable set and algebraically independent, and set $A_{\lambda,m,l} = \{\gamma T_{\lambda,m,l} : \gamma \in \overline{\mathbb{Q}} \setminus \{0\}\}$, a dense subset of $\mathbb{C}$. For $\lambda \in \Lambda$, define

$$E^{\lambda}_{\alpha_n,s_k} = \begin{cases} \mathbb{Q}(i), & \text{se } (n, k) \in (2\mathbb{Z})^2 \\ A_{\lambda,n,k}, & \text{se } (n, k) \notin (2\mathbb{Z})^2. \end{cases}$$

Now by Theorem 4, there exists an uncountable set $\mathcal{F}_\lambda$ of hypertranscendental entire functions f with $f^{(l_{2k})}(\alpha_{2m}) \in \mathbb{Q}(i)$ and $f^{(l)}(\alpha_m) \in A_{\lambda,m,l}$, for each $(\alpha_m, l) \notin A \times N$. Therefore it is plain that $M_f = A \times N$. For all $\lambda \in \Lambda$, we take only one function $f_\lambda \in \mathcal{F}_\lambda$. Set $\mathcal{F}_{A,N} = \{f_\lambda\}_{\lambda \in \Lambda}$; so $M_{f_\lambda} = A \times N$ for all $\lambda \in \Lambda$. Also, for all pairwise distinct ternaries $(\lambda_1, \alpha_{n_1}, t_1), \ldots, (\lambda_k, \alpha_{n_k}, t_k)$ with $(\alpha, t)'s \notin A \times N$ and $\lambda's \in \Lambda$, the numbers $f^{(t_1)}_{\lambda_1}(\alpha_{n_1}), \ldots, f^{(t_k)}_{\lambda_k}(\alpha_{n_k})$ lie respectively in $A_{\lambda_1,n_1,t_1}, \ldots, A_{\lambda_k,n_k,t_k}$; hence they are algebraically independent.

For the case where A is finite, we can suppose that $A = \{\alpha_1, \ldots, \alpha_m\}$. Take $E^{\lambda}_{\alpha_k,s_{2l}} = \mathbb{Q}(i)$ for any $k \in [1, m]$ and any $l \geq 0$; define $E^{\lambda}_{\alpha_k,l} = A_{\lambda,k,l}$ for each $(\alpha_k, l) \in \overline{\mathbb{Q}} \times \mathbb{N}_0 \setminus A \times N$. Then for this case we proceed as in the proof above. The other possibilities are solved in the same way. □

Returning to the exceptional sets, we still have the following last corollary.

Corollary 5. *Let $P(z_1, \ldots, z_n)$ be a non-constant polynomial with algebraic coefficients. If $f_1, \ldots, f_n \in \bigcup_{s \geq 0} \mathcal{F}^{(s)}_A$, then*

$$S_{P(f_1,\ldots,f_n)} = A. \tag{4.2}$$

Proof. In the case $A = \overline{\mathbb{Q}}$ the result follows easily. If there is an $\alpha \in \overline{\mathbb{Q}} \setminus A$, then by (4.1) the numbers $f_1(\alpha), \ldots, f_n(\alpha)$ are algebraically independent, and therefore $P(f_1, \ldots, f_n)(\alpha) \in \overline{\mathbb{Q}}$ if and only if $\alpha \in A$. In other words $S_{P(f_1,\ldots,f_n)} = A$. □

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