



The area of cyclic polygons: Recent progress on Robbins' conjectures

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Abstract

In his works [Discrete Comput. Geom. 12 (1994) 223–236; Amer. Math. Monthly 102 (1995) 523–530] David Robbins proposed several interrelated conjectures on the area of the polygons inscribed in a circle as an algebraic function of its sides. Most recently, these conjectures have been established in the course of several independent investigations. In this note we give an informal outline of these developments.

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1. Introduction

Let a_1, a_2, \dots, a_n be the side lengths of a convex polygon inscribed in a circle. What is the area $S = S(a_1, a_2, \dots, a_n)$ of the polygon as a function of the sides? This question goes back to Heron of Alexandria (the case $n = 3$) and Brahmagupta (the case $n = 4$). It seems, David Robbins was the first to address this question in full generality and suggest the way of phrasing the answer [8,9]. We start with the general remarks on the problem (largely due to Robbins) and then outline recent developments in an informal essay style.

Following Robbins, we call polygons inscribed in a circle the *cyclic polygons*. We denote the vertices by A_1, A_2, \dots, A_n and the center by O .

First, observe that $S(a_1, a_2, \dots, a_n)$ is well defined, that is there exists at most one cyclic polygon with the given (ordered list of) side lengths. Indeed, start with a large enough

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circle and place $n + 1$ vertices A_0, A_1, \dots, A_n at distance $|A_i - A_{i+1}| = a_i$. Continuously decreasing the radius we obtain a unique convex polygon with $A_0 = A_n$, as desired.

Second, observe that S is a symmetric function in a_i . Indeed, this follows from the fact that we can interchange triangles $[OA_{i-1}A_i]$ and $[OA_iA_{i+1}]$. The details are straightforward.

Our final observation is the fact that S is an algebraic function of the side lengths a_i . First, notice that it is polynomial in the coordinates of vertices $A_i = (x_i, y_i)$:

$$S = \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \dots + \frac{1}{2} \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}. \tag{o}$$

Here each summand is equal to the (signed) area of the triangle $[OA_iA_{i+1}]$, and it is easy to see that they add up to the area of the polygon ($\mathbf{0}$ denotes the origin).

Now, move the polygon in such way that $A_1 = (0, 0)$ and $A_2 = (a_1, 0)$. There are $2(n - 2)$ free variables for the remaining vertex coordinates and 2 variables for the coordinates of the center O . Together these give $2n - 2$ variables. Similarly, the remaining side lengths give $n - 1$ equations, the equality of the distance to the origin give another $n - 1$ equations, which total $2n - 2$ equations. One can show that these equations are algebraically independent so all free coordinates are in fact the algebraic functions of the side lengths a_i . Thus, from the formula (o), so is the area S .

Note that depending on the orientation of the polygon, the (signed) area S given by (o) is either positive or negative. Also, in the quadratic equations for the distances

$$(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 = a_i^2$$

only the squared edge lengths appear. Thus, for each n there exist a minimal polynomial equation $P_n(S^2, a_1^2, \dots, a_n^2) = 0$ which has the squared area as its root. Changing the first variable, we obtain $\alpha_n(16S^2, a_1^2, \dots, a_n^2) = 0$, which Robbins called the *generalized Heron polynomials*.

As we mentioned above, polynomials α_3, α_4 were well known. In his work [8,9], Robbins calculated α_5, α_6 and made a number of conjectures on the general form of polynomials α_n . By now, his conjectures have largely been established in a series of recent development. Before we move on to outline their solutions, let us mention that Varfolomeev, unaware of Robbins' work, recently rediscovered some of his results and independently made a number of advances on the subject [12,13].

As the reader will see, we do not include any technical details, nor do we present a formal survey. Instead, give the reader a quick introduction to the subject and its basic ideas, aiming to ease the entrance barrier and to simplify navigation through recent developments.

2. The first coefficient

Based on small examples, David Robbins conjectured that the polynomials α_n are *monic* in variable $z = 16S^2$ ("monic" means that the highest coefficient is equal to one). This seemingly random observation is in fact very interesting and is strongly related to Sabitov's

theory and the proof of the bellows conjecture. This connection was independently discovered by Connelly [3] and Varfolomeev [12], who gave two different proofs. Let us first elaborate on Sabitov's work.

The story goes back to Connelly's celebrated discovery of flexible (nonconvex) polyhedra and his *bellows conjecture*¹ as to whether the volume remains invariant under flexing (continuous face-preserving deformations). We refer to [2] for background and references. Later both Connelly and Sabitov conjectured that in fact the volume is integral over the ring generated by squares of polyhedra's edge lengths. This immediately implies the conjecture since nonzero polynomials have only finitely many roots, and thus allow only a finite set of possible volume values.

The bellows conjecture was established by Sabitov, who gave several consequently improving expositions of his proof in a series of papers (see, e.g., [10,11]). Let ℓ_i denote the edge lengths of the polytope with edge graph G , and let V denote its volume. Sabitov showed that there exists a nontrivial polynomial equation $P_G(V^2, \ell_1^2, \ell_2^2, \dots) = 0$. The difficult part in the proof is not computing this polynomial but checking that the leading coefficient is not zero. In fact, after a change of variables $\tilde{P}_G(144V^2, \ell_1^2, \ell_2^2, \dots) = 0$ all coefficients become integral, and the polynomial \tilde{P}_G is monic in $(144V^2)$.

Unfortunately, Sabitov's proof is based on elimination theory and is more technical than enlightening. Sabitov's approach was later modified in [4] where the *theory of places* is used to prove the bellows conjecture.

Note that when the polytope is a simplex the resulting polynomial equation can be viewed as (a different) generalization of the Heron formula [10]. The striking similarity of two problems led Varfolomeev to rediscovery of some of Robbins' ideas and results. He used Sabitov's methods to show that polynomials α_n are monic [12].

Connelly came to his proof [3] independently, after [6] advertised Robbins' efforts. He observed the similarity as well, and used the theory of places, to obtain a beautiful and concise proof of this Robbins conjecture.

In the spirit of the bellows conjecture, both authors address the question as to when cyclic polytopes are flexible. The immediate implication of the above result is the fact that the (symplectic) area is unchanged under flexing. In fact, as was shown by Connelly much earlier [1], the area of flexible cyclic polygons is always zero. This was also rediscovered by Varfolomeev [12] (see also [3]).

3. The degree

The most aesthetically pleasing conjecture of Robbins is his proposed formula for the degree of generalized Heron polynomials (in the variable $z = 16S^2$):

$$\deg \alpha_{2k+1} = \Delta_k, \quad \deg \alpha_{2k+2} = 2\Delta_k, \quad \text{where } \Delta_k = \frac{2k+1}{2} \binom{2k}{k} - 2^{2k-1}. \quad (*)$$

¹ Bob Connelly declines to take credit for the bellows conjecture and wrote to me that it was communicated to him by various people, all of whom refer yet again to other people. Therefore, the conjecture is a folklore in the area, while Connelly deserves a great deal of credit for its advancement.

Let us explain the origin of this formula for Δ_n . Observe that formula (o) works not only for (the usual) cyclic polygons, but also for those with self-intersections, still inscribed in a circle. Therefore, the minimal polynomial $\alpha_n(z)$ has at least as many real roots as the number of different areas of these self-intersecting polygons. Then, Robbins showed that for nearly equal side lengths a_1, a_2, \dots, a_n all self-intersecting cyclic polygons have different areas. A simple combinatorial argument gives the r.h.s. in (*) for the number of different self-intersecting cyclic polygons, and implies the desired lower bound on the degree of α_n .

Robbins' conjecture for the degree, the formula (*), was established in [5], and later by a simpler but related argument in [7]. Both proofs first obtain the corresponding formula for the degree of polynomial equations on the radius of the circle, and then move to the degree of α_n . The study in [5] goes much deeper, as the authors establish formal connections with Sabitov's theory, which we outline below.

It was observed by Sabitov that not only the volume, but other "polynomial invariants" of polytopes are roots of polynomial relations with coefficients being polynomial in the squared side lengths. It is a natural general question to compute the degree of these minimal polynomials. We should emphasize that here we discuss only convex polytopes, so certain technical difficulties of Sabitov's approach do not appear in this case. Now, for convex polytopes not only the volume, but *all* diagonals are the roots of polynomial equations and thus one can think of them as of an extension of the Cauchy rigidity theorem. Following this logic, in [5] we refined a known algebraic proof of the Cauchy rigidity theorem and added an argument from algebraic geometry. We obtained a general upper bound on the degree of minimal polynomial relations for all polynomial invariants of convex polytopes, including the volume, (squared) diagonal lengths, etc. In this special case the upper bound we obtain for P_G is in terms of *complex* realizations of the graph G (realizations in \mathbb{C}^3) and in the worst case gives 2^m , where m is the number of edges in G (= the number of edges in the polytope).

Now, after we learned from [6] about Robbins' conjectures, we discovered a formal connection between our work (Sabitov's theory) and that of Robbins. Consider a bipyramid with (large enough) equal length edges leaving north or south poles, and the edge lengths a_1, a_2, \dots, a_n in the middle. Clearly, the middle edges form the desired cyclic polygon, and in fact different (real) realizations of this bipyramid produce different (self-intersecting) cyclic polygons. Also, the *main* (north to south pole) *diagonal* is related by Pythagoras theorem to the radius. It may seem like our main upper bound is directly applicable in this case to obtain the degree of the minimal polynomial relation for the radius. The problem is that the number of complex realizations is a difficult quantity to compute in most cases. In fact, our logic moves backwards and is more convoluted.

First, we use an ad hoc combinatorial argument to compute explicitly the minimal polynomial relation for the radius and its degree. The corresponding polynomial relation turns out to have a nice closed formula amenable to direct calculation. Then we use the relationship described above to obtain the polynomial relation for the main diagonal, and thus bound the number of complex realizations, which we show is equal to the number of real realizations (self-intersecting cyclic polygons). Finally, we apply our upper bound theorem to obtain the upper bound on the degree of α_n , the minimal polynomial for the area of cyclic polygons. With Robbins' matching lower bound we obtain the result.

It is interesting to note that we never actually obtain any useful formula for α_n . In fact, the only polynomial invariant that stands out in this case is the radius—all others play a supporting role. For example, instead of the area we could be proving formula (*) for the degree of the minimal polynomial relation on the sum or squares of all diagonals in a cyclic polygon.

Let us say a few words on the proof in [7, §5]. This work started out by David Robbins and Julie Roskies just a few month before Robbins' premature death, and continued later with the help of Miller Maley. Their proof of the degree formula starts with the use of Möbius formulas for the radius of the cyclic polygon, which are essentially equivalent to our formulas. Rather than utilize the general theory we develop in [5], the authors use a simplified algebraic geometry argument adjusted in this particular case to obtain the result. Basically, their argument is the same as our argument for the special case of a bipyramid.

Finally, let us note that Varfolomeev also studies explicit formulas for the radius in cyclic polygons [12]. He also guesses the answer in terms of self-intersecting polygons, but never obtains a general formula nor even calculates their number beyond few small cases.

4. Explicit formulas

One more Robbins' conjecture concerns the form of polynomials of α_{2k-1} versus that of α_{2k} . Roughly speaking, he claimed that given the formula for α_{2k-1} one can easily obtain the formula for α_{2k} as a product of the formula for α_{2k-1} and its variation. This conjecture was established by Varfolomeev [12] by a direct argument (see also [7]). As a corollary, calculations of Robbins et al. for cyclic pentagons and heptagons immediately translate to give the formulas for cyclic hexagon and octagons [7,8].

It was Robbins' wish to obtain a concise formula for α_7 and although he did not live to finish the project, such a formula was recently obtained in [7]. Of course, Robbins already showed that some kind of formula exists, but given the large number of terms one can ask if there is a way to simplify it. In view of our earlier impression (see above) that a nice formula may exist only for the radius, we find it amazing that the authors were able to obtain a concise formula for the area.

Let us mention that already the Robbins' formula for α_5 is very interesting as it expresses the area as a discriminant of a certain "mystery cubic" [9]. It remains unclear where this cubic comes from and what is its role in the grand scheme of things. For example, Varfolomeev [12] does not notice this formula and uses rather elaborate explicit formulas for α_5 .

Now, in [7] the authors obtain a closed formula for α_7 in terms of a resultant of two concise, but not generalizable polynomials. The resulting formula is nice but again very mysterious. It remains open whether this work can be extended to obtain concise formulas for α_n , where $n \geq 9$.

5. Final remarks and open problems

From the point of view of Sabitov's theory and our paper [5], it would be natural to ask for the minimal degree polynomial relations for the volume or the diagonal lengths in various families of convex polytopes. As noted in [5], even for relatively small polytopes this problem is computationally intractable and new ideas are needed even to obtain the exact asymptotic behavior. At the moment, the precise formulas that we found for regular bipyramids are clearly beyond reach in most cases.

An interesting twist on cyclic polygons was proposed in [7] where the authors define what they call "semicyclic polygons", where one side is forced to be a diameter and its length is not specified. It seems that much of the work extends to this case with little difficulty. We propose to consider an equivalent model of the centrally symmetric cyclic polygons with given edge lengths. This version has the advantage of being possible to generalize to cyclic polygons with $\mathbb{Z}/k\mathbb{Z}$ cyclic symmetry. It would be interesting to see if the analysis extends to this case. In general, one can consider general polytopes with a given symmetry group. Developing the corresponding "equivariant Sabitov theory" seems like a fruitful direction.

When it comes to the area and generalized Heron polynomials α_n , it is probably too much to ask for a concise general formula. Still, we remain optimistic of other research venues. In his latest work [13], Varfolomeev calculated the Galois group of α_5 and showed that it is the full group of permutations S_7 (he did this also for the radius). There seem to be no immediate implications of this result except perhaps the impossibility of "construction" of the cyclic pentagon with a ruler and a compass, given the generic lengths of edges (such construction of a regular pentagon is well known). In any case, it would be nice to extend these calculations for general α_n .

Finally, further connections to rigidity theory are waiting to be explored. We refer to final remarks in [3] for directions and motivation. Also, an intriguing construction of a finitely generated infinite-dimensional Lie algebra was announced in [13] and promised to be the subject of the future investigations. We are anxious to see how this theory further develops.

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