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Ising model with spins $S = 1/2$ and 1 on directed and undirected Erdős–Rényi random graphs

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ABSTRACT

Using Monte Carlo simulations, we study the Ising model with spin $S = 1/2$ and 1 on *directed* and *undirected* Erdős–Rényi (ER) random graphs, with z neighbors for each spin. In the case with spin $S = 1/2$, the *undirected* and *directed* ER graphs present a spontaneous magnetization in the universality class of mean field theory, where in both *directed* and *undirected* ER graphs the model presents a spontaneous magnetization at $p = z/N$ ($z = 2, 3, \dots, N$), but no spontaneous magnetization at $p = 1/N$ which is the percolation threshold. For both *directed* and *undirected* ER graphs with spin $S = 1$, we find a first-order phase transition for $z = 4$ and 9 neighbors.

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1. Introduction

An Erdős–Rényi (ER) random graph is a set of N vertices (sites) connected by B links (bonds) [1,2]. The probability p that a given pair of sites is connected by a bond is $p = 2B/N(N - 1)$. The connectivity of a site is defined as the total number of bonds connected to it, that is $k_i = \sum_j l_{ij}$, where $l_{ij} = 1$ if there is a link between the sites i and j and $l_{ij} = 0$ otherwise. Random graphs are completely characterized by the mean number of bonds per site, or the average connectivity $z = p(N - 1)$. In the limit $N \rightarrow \infty$, the distribution of connectivities is given by the Poisson distribution. Sumour and Shabat [3,4] investigated Ising models with spin $S = 1/2$ on *directed* Barabási–Albert (BA) networks [5] using the usual Glauber dynamics. No spontaneous magnetization was found, in contrast to the case of *undirected* BA networks [6–8] where a spontaneous magnetization was found below a critical temperature which increases logarithmically with the system size. For $S = 1/2$ systems on *undirected* Small-World networks (SW) [9] with scale-free hierarchical-lattice, conventional and algebraic (Berezinskii–Kosterlitz–Thouless) ordering, with finite transition temperatures, have been found. Lima and Stauffer [10] simulated *directed* square, cubic and hypercubic lattices ranging from two to five dimensions with heat bath dynamics in order to separate the network effects from directedness. They also compared different spin-flip algorithms, including cluster flips, for Ising–BA networks. They found a freezing-in of the magnetization similar to the one in Refs. [3,4], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetization (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences spin S_i , then spin S_i in turn does not influence S_j , and there may be no well-defined total energy. Thus, they showed that for the same scale-free networks, different algorithms give different results. Lima et al. [11] studied the Ising model for spin $S = 1, 3/2$ and 2 on *directed* BA network. The Ising model with spin 1, $3/2$ and 2 seemed not to show a spontaneous magnetization and their decay time for flipping of the magnetization followed an Arrhenius law for heat bath algorithms that agrees with the results of the Ising model for spin

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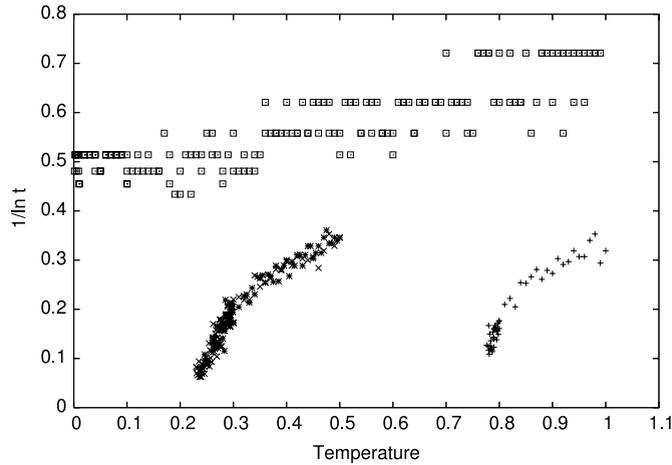


Fig. 1. Reciprocal logarithm of the relaxation times versus temperature for different probabilities $1/N$ (sq.), $2/N$ (\times), and $3/N$ ($+$); directed ER with $S = 1/2$.

$S = 1/2$ [3,4] on directed BA network. Sánchez et al. [12] on *directed* SW obtained a second-order phase transition for values of rewiring probability $p = 0.1$ and a first-order phase transition for $p = 0.9$ with $p_c \approx 0.65$ for the change of phases. The magnetic properties of Ising models defined on the triangular Apollonian network was investigated by Andrade and Herrmann [13] and no evidence of phase transition was found. In this work, we have studied the Ising model with spins $S = 1/2$ and 1 on directed and undirected ER graphs. *Undirected* ER graphs with spin $S = 1/2$ present a spontaneous magnetization in the universality class of mean field theory and for $S = 1$, we find evidences of first-order phase transition for $z \geq 2$. *Directed* ER graphs for spin $S = 1/2$ and $S = 1$ present a spontaneous magnetization for $z \geq 2$. Here z is the number of neighbors for each spin.

2. Model and simulation: Ising model on ER graphs

We consider the spin $S = 1/2$ and 1 Ising models defined by a set of spin variables S_i located on every site i , first of *directed* ER graphs, with N spins taking the values ± 1 and 0 for $S = 1$, and ± 1 for $S = 1/2$, respectively.

The probability for spin S_i to change its state in this *directed* network is

$$p_i = 1/[1 + \exp(-2E_i/k_B T)], \quad E_i = -J \sum_k S_i S_k \tag{1}$$

and enters the heat bath algorithm; k runs over all nearest neighbors of S_i . In this network, each new site added to the network selects with connectivity z already existing sites as neighbors influencing it; the newly added spin does not influence these neighbors.

To study the spin $1/2$ and 1 Ising models we start with all spins up, a number of spins equal to 2,000,000 and 4,000,000, and Monte Carlo step (MCS) time up to 200,000 and 2,000,000, respectively. In our simulations, one MCS is accomplished after all spins are updated, here, with heat bath Monte Carlo algorithm. Then we vary the temperature and study nine samples. The temperature is measured in units of the critical temperature of the square-lattice Ising model. We determine the time t after which the magnetization has flipped its sign for the first time, and then take the median value of our nine samples. So we get different values t for different temperatures. To study the critical behavior of this Ising model (with spins $1/2$ and 1), we define the variable $m = \sum_{i=1}^N S_i/N$ as normalized magnetization. The Ising model on directed BA networks has no phase transition and agrees with the modified Arrhenius law for relaxation time, $1/\ln t \propto T + \dots$, Lima et al. [11].

3. Results and discussion

3.1. Spin 1/2 Ising model

We take different probabilities for different number of nodes $N = 2,000,000$ with different temperatures in Fig. 1. There we check the first time after which the magnetization changes sign, take the median from nine samples, and plot the reciprocal of the time for three probabilities $p = z/N$ ($z = 1, 2$, and 3) in Fig. 1. The figure shows nicely the difference between probability $1/N$ (= percolation threshold) and larger probabilities. This figure shows that there is a spontaneous magnetization at $p = 2/N$ for the left curve and at $p = 3/N$ for the right curve, but no spontaneous magnetization at $p = 1/N$ which is the percolation threshold. This is substantiated by the fact that in case $p = 1/N$, there is not a phase transition at temperature greater than zero, but for the other cases we see that the system tends to a finite temperature

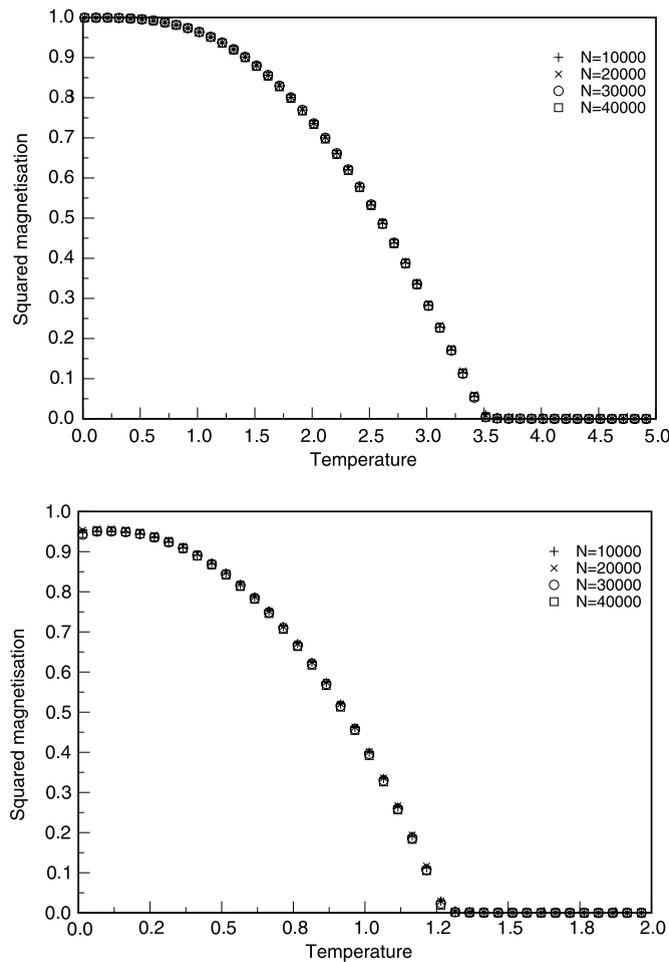


Fig. 2. Squared normalized magnetization versus temperatures, $S = 1/2$, $p = 4/N$, for different sizes N of the *undirected* ER graph (top) and *directed* ER graph (bottom).

as the system size grows, where we have approximate values of $T \approx 0.22$ and 0.78 for $p = 2/N$ (\times) and $p = 3/N$ ($+$), respectively. In Fig. 2, we show the dependence of the magnetization M on the temperature, obtained for directed and undirected ER graphs with $S = 1/2$; we use only one probability equal $p = 4/N$, because it gives a clear answer compatible with the mean-field universality class, as expected because of the infinite range of the symmetric interaction. For *undirected* ER graphs, if A is a neighbor of B then, in contrast to the directed case, also B is a neighbor of A . From our simulation we see that the undirected version has a spontaneous magnetization, to which the system relaxes similarly to the standard Ising square lattice. Then we plot the square of normalized magnetization versus temperature in Fig. 2. For T below T_c we have a spontaneous magnetization and above T_c we do not have one as we see in Fig. 2 (part (a)). In equilibrium there is a Curie temperature. The squared magnetization vanishes at this $T_c \approx 3.5J/K_B$ linearly in temperature. This behavior corresponds, not unexpectedly, to a mean field critical exponent. Unexpectedly, this same behavior occurs also for *directed* ER graphs (part (b)) that do not present an infinite range of the symmetric interaction as occurs with *undirected* ER graphs. The squared magnetization vanishes at this $T_c \approx 1.2J/K_B$. These results show that the behaviors of $S = 1/2$ Ising model spin on ER graphs are similar, whether these networks are *directed* or *undirected*.

3.2. Spin 1 Ising model

Fig. 3 is analogous to Fig. 1 except that now $S = 1$ instead of $1/2$ for $N = 4,000,000$ up sites. In Fig. 4, we show magnetization versus temperature on *directed* ER networks (part (a)) and also on *undirected* ER networks (part (b)) for different probabilities $p = z/N$ with $z = 4$ (left) and 9 (right) for system size $N = 16,000$ sites. The shapes of these figures show qualitatively that they present evidence of first-order phase transition and also show that the behaviors of magnetization versus temperature are identical for the same probabilities regardless of whether the networks are *directed* or *undirected*. In order to verify the order of the transition, we apply finite-size scaling (FSS) for $N = 250, 500, 1000, 2000, 4000, 8000$, and $16,000$ sites. Initially we search for the minima of the energetic fourth-order cumulant:

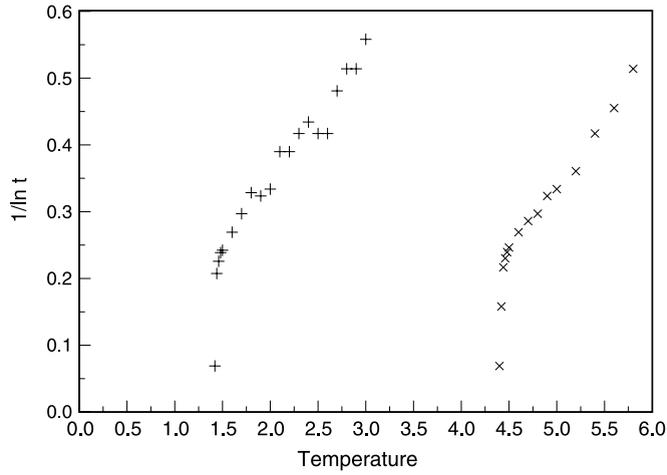


Fig. 3. The same behavior Fig. 1, but now with $S = 1$ for different probabilities $p = z/N$ with $z = 2(+)$ and $9(x)$, $N = 4,000,000$.

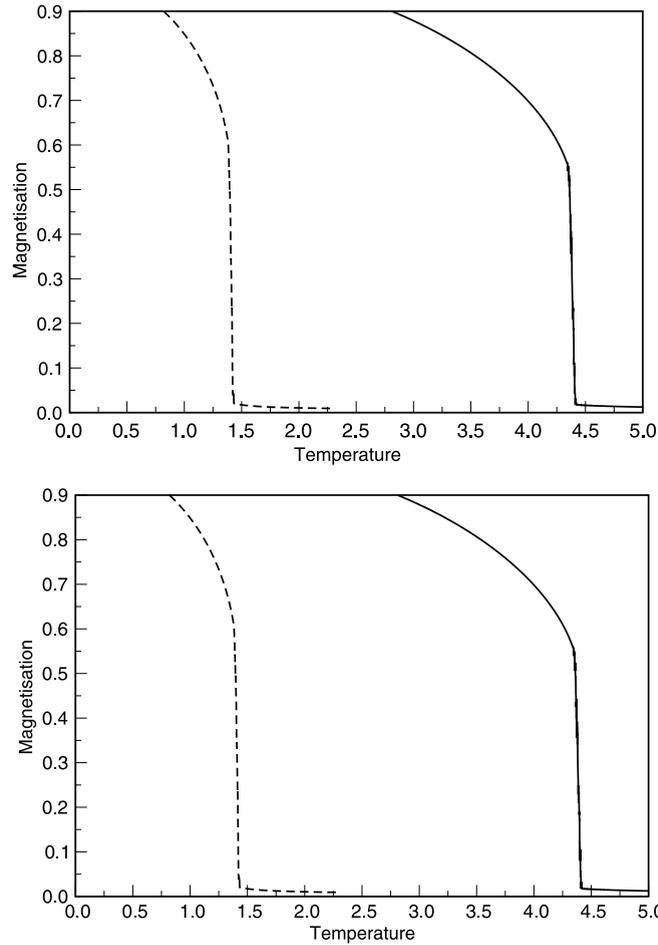


Fig. 4. Magnetization versus temperature for spin $S = 1$ on directed (top) and undirected (bottom) ER graphs.

$$B = 1 - \left[\frac{\langle e^4 \rangle}{3\langle e^2 \rangle^2} \right]_{av} \tag{2}$$

where $e = E/N$ is the energy per spin. It is known that this parameter takes a minimum value B_{\min} at the effective transition temperature $T_c(N)$. One can show [14] that for a second-order transition $\lim_{N \rightarrow \infty} (2/3 - B_{\min}) = 0$, even at T_c , while at

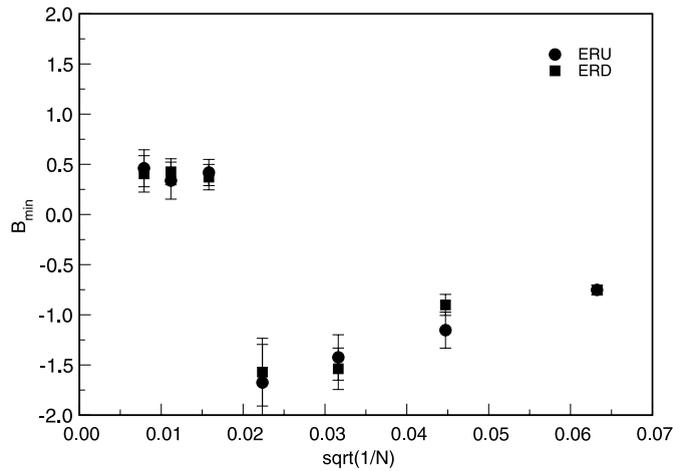


Fig. 5. Energetic Binder cumulant B_{\min} versus $1/N$ for $z = 9$.

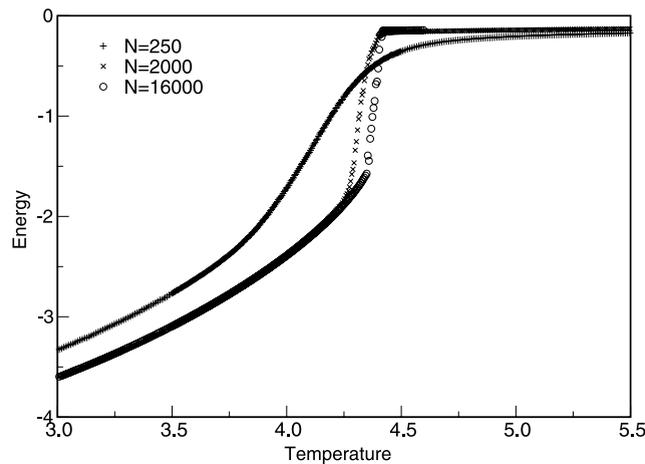


Fig. 6. Energy versus temperature on ERD graphs for $z = 9$.

a first-order transition the same limit is different from zero ($\neq 0$). In Fig. 5, we plot the Binder minimum parameter B_{\min} versus $1/N$ (Eq. (2)) for $z = 9$, and several system sizes. The Binder parameter goes to a value which is different from $2/3$. This is a sufficient condition to characterize a first-order transition. The order of transition can be confirmed by plotting the values of energy versus temperature (see Fig. 6), where we present a jump when system sizes increase. This behavior is evidence for a first-order phase transition for $z = 9$; this same behavior occurs also for $z = 4$.

4. Conclusion

In conclusion, we have presented the Ising model for spins $S = 1/2$ and 1 on *directed* ER and *undirected* ER graphs, because our main objective in this paper was to verify the existence or not of phase transitions and also the kind of phase transition.

For spin $S = 1/2$ Ising models, both *directed* or *undirected* ER graphs have a phase transition temperature below which a spontaneous magnetization exists, where ER graphs have a spontaneous magnetization in the universality class of mean field theory. For spin $S = 1$ Ising models, on *directed* and *undirected* ER graphs the results are identical, i.e., are independent of the nature of the graphs studied here and have both a good evidence of a first-order phase transition different from spin $S = 1/2$. Our results agree with the results of nonequilibrium model on directed and undirected ER graphs studied by Pereira and Brady [15] and Lima et al. [16].

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