Decoupling of generalized mode I and II stress intensity factors in the complete contact problem of elastically dissimilar materials

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Abstract

Stress singularity of a complete contact problem is studied herein using an asymptotic analysis. It is considered that a half plane is indented by a semi-infinite sharp wedge for the contact geometry. It is also assumed that the contacting bodies have different elastic properties. It was found that the order of stress singularity is less than 0.5 and varies depending on the wedge angle and material mismatch. The variation in the stress singularity is illustrated in the Dundurs parallelogram. It provides an overall view of the material combination for reducing damage at the contact edge. An analysis is then carried out to evaluate the generalized stress intensity factors (GSIFs) of mode I and II (KI and KII) that calibrate the stress state in the vicinity of the contact edge incorporating the actual conditions. For decoupling the modes, it is developed to select an angle at which a shearing mode disappears for KI and an opening mode disappears for KII. It is found that the decoupling angle for mode I decreases and that for mode II increases as the second Dundurs parameter, β increases.

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1. Introduction

Complete contact problem does not arise often in engineering structures since it is generally advised to remove sharp edges that are in contact with the mating body. However, it can be found even in the bodies with rounded edges due to a wearing during service. It also occurs in a split part inside a surrounding component when the edges

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of the split part expand out and push the surrounding one. The importance of a complete contact problem is that the stress becomes singular at the sharp edge so that material damage is likely to occur there, compared with an incomplete contact problem where the stresses are bounded at the contact edges.

Typical mathematics to deal with the stress singularity appears in a crack problem of fracture mechanics. This implies that there is a generic analogy between the crack and complete contact problems. An asymptotic method has been an efficient tool to analyze the state of stress in the vicinity of the location of the stress singularity, e.g. the contact edge or crack tip. In the case of the complete and bonded contact, the stress equation obtained using the asymptotic analysis consists of two terms of different modes (I and II). Each term has a generalized stress intensity factor of each mode which should be evaluated to obtain the actual solutions accommodating the loads and dimensions.

This paper shows a decoupling method of mode I and II which is necessary to evaluate the generalized stress intensity factor of each mode. To this end, an eigenvalue problem is constituted for a complete contact of dissimilar materials. Fracture mechanics concept of the opening and shearing modes are adopted to separate the modes. Specific angles are looked for, which make the relevant modes disappear. The variation of the angles with respect to the material dissimilarity is discussed.

2. Eigenvalue problem

2.1. Problem description

Fig. 1 shows a typical view of the complete contact geometry. An elastic half plane is in contact with another elastic semi-infinite wedge having an internal angle, $\phi$. Here, the half plane and wedge are referred to as ‘body 1’ and ‘body 2’, respectively. It is supposed that the elastic properties of each body are different for generality. The contact interface is either adhered, slipping or combination of both. In this work, it is assumed that whole region of the interface is bonded a priori. Then, the following boundary conditions are derived.

$$\begin{align*}
\sigma_{\theta\theta}^1(r,-\pi) &= 0, \quad \sigma_{r\theta}^1(r,-\pi) = 0, \quad \sigma_{\theta\theta}^2(r,\phi) = 0, \quad \sigma_{r\theta}^2(r,\phi) = 0 \\
\sigma_{\theta\theta}^1(r,0) - \sigma_{\theta\theta}^2(r,0) &= 0, \quad \sigma_{r\theta}^1(r,0) - \sigma_{r\theta}^2(r,0) = 0, \quad u_r^1(r,0) - u_r^2(r,0) = 0, \quad u_\theta^1(r,0) - u_\theta^2(r,0) = 0
\end{align*}$$

(1)

$$\begin{align*}
\sigma_{\theta\theta}^1(r,0) - \sigma_{\theta\theta}^2(r,0) &= 0, \quad \sigma_{r\theta}^1(r,0) - \sigma_{r\theta}^2(r,0) = 0, \quad u_r^1(r,0) - u_r^2(r,0) = 0, \quad u_\theta^1(r,0) - u_\theta^2(r,0) = 0
\end{align*}$$

(2)

where the superscripts 1, 2 designate the body number.

The complete and bonded contact geometry of Fig. 1 can be treated as a sharp notch problem with a notch angle of $(\pi - \phi)$. When the contacting bodies are elastically dissimilar, it becomes a sharp notch problem of bi-materials. Therefore, the stress field in the vicinity of the contact edge, where high stress intensification occurs, can be analyzed using an asymptotic method, originally developed by Williams (1952). The key procedure is to use the Airy stress function for each body and the derived stresses and displacements, which is composed of even functions of symmetric property and odd functions of antisymmetric property as follows.
where $\Phi_i$ is the Airy stress functions of body $i$ ($i$ is the body number). Polar coordinates with the origin at the contact edge is defined as $(r, \theta)$, and $a_i \sim d_i$ are the constants to be determined from the specific conditions of the problem. Note that subscripts 1 and 2 designate the body number here.

In Eq. (3), $\lambda$ is a characteristic constant that determines the behaviour of the Airy stress function with respect to the distance from the contact edge. It is readily recognized that the use of $(\lambda + 1)$ as a power of the distance from the edge, $r$, is to intentionally make the stress components have a power of $(\lambda - 1)$, when applying the partial derivatives of the 2nd order to the Airy stress function for stress evaluation. In other words, the stress singularity at the contact edge can be described as $r^{\lambda-1}$ if $\lambda < 1$.

The stress and displacement components of body 1 and 2 are to be derived using the well-known formula as follows.

$$
\sigma_{ir} = \frac{1}{r} \frac{\partial \Phi_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \theta^2}, \quad \sigma_{i\theta} = \frac{1}{r} \frac{\partial \Phi_i}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \Phi_i}{\partial r \partial \theta},
$$

$$
\frac{\partial u_i}{\partial r} = \frac{1}{2\mu_i} \left[ \frac{1}{r} \frac{\partial \Phi_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \theta^2} \left( \frac{3-\kappa_i}{4} \right) \right], \quad \frac{\partial u_i}{\partial \theta} = \frac{1}{r} \frac{\partial \Phi_i}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \Phi_i}{\partial r \partial \theta},
$$

where $\mu_i$ and $\kappa_i$ are the shear modulus and the Kolosov constant of body $i$, respectively. In addition, $\kappa_i = (3 - 4\nu_i)$ for the plane strain, and $(3 - \nu_i)/(1 + \nu_i)$ for the plane stress conditions where $\nu_i$ is the Poisson ratio of body $i$.

On the other hand, the material dissimilarity of the two contacting bodies can be simply described using the Dundurs parameters (1969), $\alpha$ and $\beta$ which are defined as follows.

$$
\alpha = \left[ \frac{\mu_2(\kappa_1 + 1) - \mu_1(\kappa_2 + 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)} \right], \quad \beta = \left[ \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)} \right].
$$

It is readily noted that $-1 < \alpha < 1$, $-0.5 < \beta < 0.5$. Every combination of material dissimilarity can be displayed inside a parallelogram (Dundurs' parallelogram) in the $\alpha$-$\beta$ plane, whose vertices are $(\alpha, \beta) = (1,0), (1,0.5), (-1,0)$ and $(-1,-0.5)$.

2.2. Eigensolution

The substitution of Eq. (3) into Eqs. (4) and (5), and then applying Eqs. (1) and (2) yield a simultaneous equation of eight homogeneous equations with eight unknown constants, $a_1 \sim d_2$, together with a characteristic constant, $\lambda$. To avoid a trivial solution of the simultaneous equation, the determinant of the coefficient matrix of the eight homogeneous equations should be zero. This enables the $\lambda$ values to be obtained, which become the eigenvalues of the problem. In turn, the substitution of $\lambda$ back into the eight homogeneous equations provides the values of $a_1 \sim d_2$ and correspondingly, the eigenvectors such as the stress and displacement components. During this step, it may be noted that the number of solutions of $a_1 \sim d_2$ will be infinite when $\lambda$ is substituted. Thus, it is necessary to normalize the constants with respect to one of them (e.g., $a_2/a_1 \sim d_2/a_1$).

In the present case of the complete and bonded contact, two $\lambda$’s of different value occur (referred to as $\lambda_I$, $\lambda_{II}$) as long as $0 < \varphi < \pi$, which has been illustrated by Mugadu et al. (2002). When $\varphi = \pi$, it becomes a crack problem where $\lambda_I = \lambda_{II} = 0.5$. Bogy (1971) investigated the property of $\lambda$ in detail, especially in the point of the order of stress singularity. The stress becomes bounded when $\lambda_I$, $\lambda_{II}$ are real and greater than 1. The order is $O(|n|)$ and the stress oscillates if $\lambda_I$, $\lambda_{II}$ have complex numbers and $0 < \text{Re} [\lambda_I, \lambda_{II}] < 1$. In the case focused on for the present problem, both $\lambda_I$ and $\lambda_{II}$ are real and $0 < \lambda_I$, $\lambda_{II} < 1$ so that the stress becomes singular as $r \to 0$ with the order of $O(r^{\lambda_I-1})$ and $O(r^{\lambda_{II}-1})$. 

\[
\Phi_i = r^{\lambda_i} [a_i \cos(\lambda_i + 1) \theta + b_i \sin(\lambda_i + 1) \theta + c_i \cos(\lambda_i - 1) \theta + d_i \sin(\lambda_i - 1) \theta], \quad (i=1,2). 
\]
The eigenvalue, $\lambda$, varies depending on the material dissimilarity, $\alpha$ and $\beta$ as well as the contact angle, $\varphi$. Therefore, a lot of $\lambda$ values can be generated as $\alpha$, $\beta$ and $\varphi$ vary. We confine ourselves here to the case of $\varphi = \pi/2$ to investigate primarily the influence of material dissimilarity on the $\lambda$ values. The result is provided in the Dundurs parallelogram as shown in Fig. 2. We define $\lambda_I < \lambda_{II}$ so that $\lambda_I$ is related with the stronger stress singularity than $\lambda_{II}$. It is shown that $0.5 < \lambda_I, \lambda_{II} < 1$ in Fig. 2. Since $\lambda_I = \lambda_{II} = 0.5$ in the crack problem as aforementioned, the stress singularity at the contact edge is weaker than that at the crack tip. It implies that the material damage induced by a contact stress can be less in the complete contact problem compared with the crack problem if the applied loading condition is the same.

![Fig. 2. Eigenvalues, $\lambda_I, \lambda_{II}$ of a complete contact problem of bonded and elastically dissimilar bodies in the case of the contact angle, $\varphi = \pi/2$ (see Fig. 1).](image)

Consequently, the stress equation in the vicinity of the contact edge can be expressed in an asymptotic form as follows.

$$\sigma_{ij}(r, \theta) = K_I r^{\lambda_I - 1} f_{ij}^I(\theta) + K_{II} r^{\lambda_{II} - 1} f_{ij}^{II}(\theta).$$  \(7\)

where $K_I, K_{II}$ are the scaling factors of the stresses evaluated using $\lambda_I$ and $\lambda_{II}$, respectively. $f_{ij}^k(\theta)$, $(ij = r, \theta; k = I, II)$ are the functions consisting of angle around the contact edge ($\theta$, see Fig. 1), which appear during the evaluation of the eigenvectors with $\lambda_k (k = I, II)$. Note that an actual stress should originally include the nonsingular bounded terms in addition to Eq. (7). However, it can be neglected since its contribution becomes very small compared with the singular terms of Eq. (7) as the contact edge approaches ($r \to 0$).

$K_I, K_{II}$ are termed as generalized stress intensity factors of mode I and mode II, respectively. It is similar to those in fracture mechanics. Although there is no implication of crack driving force of opening and shearing modes in the contact problem, the opening property of $K_I$ in fracture mechanics can be adopted as the pulling-apart property at a point in the contacting bodies, and similarly, the shearing property of $K_{II}$ as the slipping property at the point. It is compulsory to calculate $K_I, K_{II}$ to express the stress field of an actual problem that includes the loading and size conditions because $r^{-k} f_{ij}^k(\theta)$, $(ij = r, \theta; k = I, II)$ describes the characteristics of the semi-infinite problem only (i.e., order of singularity and stress variation inside the contacting bodies). To this end, decoupling of mode I and II should precede, which will be explained in the next section.

3. Mode decoupling

3.1. Behaviour of angle function

Due to the analogy of $K_I, K_{II}$ in the crack and complete contact problems, it is adopted that $K_I$ is directly associated with $\sigma_{r0}$ and $K_{II}$ with $\sigma_{\theta r}$. Each stress component is written from Eq. (7) as follows.
\[ \sigma_{\theta \theta}(r, \theta) = K_I r^{-\lambda_I - 1} f_{\theta \theta}^I(\theta) + K_{II} r^{-\lambda_{II} - 1} f_{\theta \theta}^{II}(\theta), \] 
\[ \sigma_{r \theta}(r, \theta) = K_I r^{-\lambda_I - 1} f_{r \theta}^I(\theta) + K_{II} r^{-\lambda_{II} - 1} f_{r \theta}^{II}(\theta). \] (8) (9)

Now, it is necessary to remove the \( K_{II} \)-term in Eq. (8) to make \( K_I \) be related with \( \sigma_{\theta \theta} \) only, and similarly, to remove the \( K_I \)-term in Eq. (9) to make \( K_{II} \) be related with \( \sigma_{r \theta} \) only. So we can obtain the following equations with specific angles, \( \theta_I \) and \( \theta_{II} \). Fig. 3 illustrates the method of selecting \( \theta_I \) and \( \theta_{II} \).

\[ f_{\theta \theta}^{II}(\theta_I) = 0, \quad f_{r \theta}^{II}(\theta_{II}) = 0. \] (10)

Fig. 3. Angle function, \( f_{\theta \theta}^I(\theta), f_{\theta \theta}^{II}(\theta) \), and mode decoupling angles in the case of \( (\alpha, \beta) = (-0.3, -0.2), \phi = \pi/2 \).

If both contacting bodies are elastically similar, i.e. \( (\alpha, \beta) = (0, 0), \theta_I = \theta_{II} = (\phi - \pi)/2 \). So to speak, \( \theta_I = \theta_{II} = -\pi/4 \) for \( \phi = \pi/2 \) presently. When \( (\alpha, \beta) \) deviates from \( (0, 0) \), \( \theta_I \neq \theta_{II} \) and they will have various angles.

### 3.2. Angle of mode decoupling depending on the material dissimilarity

The variation of \( \theta_I \) and \( \theta_{II} \) depending on the material dissimilarity is investigated here and the result is depicted in the Dundurs parallelogram (Fig. 4). The values of \( (\alpha, \beta) \) when \( \theta_I = \theta_{II} = -\pi/4 \) is provided with a bold (red) line. It is certainly confirmed that the bold line passes \( (0,0) \). It is interesting to see that \( \theta_I \) and \( \theta_{II} \) reveal opposite tendencies. As \( \beta \) increases, \( \theta_I \) decreases but \( \theta_{II} \) increases. It is also found that \( \theta_I \) varies faster and wider than \( \theta_{II} \) does.

Now, \( K_I, K_{II} \) are to be evaluated using \( \theta_I \) and \( \theta_{II} \) and from the following equations.

\[ K_I = \lim_{r \to 0} \left[ \sigma_{\theta \theta}(r, \theta_I) \cdot r^{1-\lambda_I} \right], \quad K_{II} = \lim_{r \to 0} \left[ \sigma_{r \theta}(r, \theta_{II}) \cdot r^{1-\lambda_{II}} \right]. \] (11)

Note that \( f_{\theta \theta}^{II}(\theta_I) = f_{r \theta}^{II}(\theta_{II}) = 1 \), which is also found in Fig. 3 as an example case.

With the actual conditions of the loading and body size, a simple finite element method can be used to obtain \( K_I \) and \( K_{II} \). This is not explained here since the fundamental procedure has been presented in detail by Kim et al. (2013) elsewhere. The feature of the finite element result is that \( \sigma_{\theta \theta}(r, \theta_I) \cdot r^{1-\lambda_I} \) and \( \sigma_{r \theta}(r, \theta_{II}) \cdot r^{1-\lambda_{II}} \) converge to constant values as \( r \to 0 \), which become the generalized stress intensity factors, \( K_I \) and \( K_{II} \), respectively.
4. Conclusions

This work is a part of an analysis of a complete contact problem. An asymptotic method is used to obtain the stress field in the vicinity of the contact edge where a stress singularity takes place and thus, material damage is likely to occur. The order of singularity and variation of the stresses are obtained from the characteristic equation derived from the boundary conditions of the problem. It is found that two different eigenvalues appear when the contact surfaces are bonded and $0 < \varphi < \pi$ (see Fig. 1). Resultantly, the stresses are displayed as a form of 

$$\sigma_{ij}(r,\theta) = \sum_k K_k r^{-1/2} j^k_{ij}(\theta),$$

where $j^k_{ij}$ are the eigenvalues and $K_i, K_{II}$ are the generalized stress intensity factors, of mode I and II, respectively. $f^0_{ij}(\theta)$ is the angle functions that display the stress variation in the contacting bodies. To evaluate the actual stresses incorporating the actual dimension and loading conditions, $K_i, K_{II}$ should be evaluated while accommodating those conditions. Therefore, a technique to decouple the mode I and II is required. The present idea is to adopt the fracture mechanics concept of the opening and shearing properties of $K_i$ and $K_{II}$. This is carried out by selecting the angles, $\theta_I$ and $\theta_{II}$, at which $f^0_{ij}(\theta_I) = 0$ for $K_I$ and $f^0_{ij}(\theta_{II}) = 0$ for $K_{II}$. The influence of the material dissimilarity on the mode decoupling angle is investigated using a plane of the Dundurs parameters, $\alpha$ and $\beta$. The result shows that $\theta_I$ decreases but $\theta_{II}$ increases as $\beta$ increases. This technique supports the finite element analysis of obtaining $K_I$ and $K_{II}$ for an actual problem as a part of the analysis process of the complete contact problem of elastically dissimilar bodies subject to a bonded condition.

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