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A generalized auxiliary equation method and its application to (2 + 1)-dimensional Korteweg–de Vries equations

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Abstract

A generalized auxiliary equation method is proposed for constructing more general exact solutions of nonlinear partial differential equations. With the aid of symbolic computation, we choose the (2 + 1)-dimensional Korteweg–de Vries equations to illustrate the validity and advantages of this method. As a result, many new and more general exact non-travelling wave and coefficient function solutions are obtained, which include soliton-like solutions, triangular-like solutions, single and combined non-degenerate Jacobi elliptic wave function-like solutions and Weierstrass elliptic doubly-like periodic solutions.

Keywords: Generalized auxiliary equation method; Soliton-like solutions; Triangular-like solutions; Jacobi elliptic wave function-like solutions; Weierstrass elliptic doubly-like periodic solutions

1. Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs), which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. In the past several decades, many significant methods for obtaining exact solutions of NLPDEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], tanh function method [5–7], sine–cosine method [8,9], homogenous balance method [10], homotopy perturbation method [11,12], variational method [13,14], asymptotic methods [15], non-perturbative methods [16], exp-function method [17], Adomian Pade approximation [18], Jacobi elliptic function expansion method [19], F-expansion method [20,21], Weierstrass semi-rational expansion method [22], unified rational expansion method [23], algebraic method [24–27], auxiliary equation method [28–31], and so on. Recently, Sirendaoreji [32] and Huang [33], respectively, proposed a new auxiliary equation method by introducing a new first-order nonlinear ordinary differential equation with six-degree nonlinear term and its solutions to construct exact travelling wave solutions of NLPDEs.

The present paper is motivated by the desire to generalize the work made in [24–33] to construct more general exact solutions, which contain not only the results obtained by using the methods [24–33], but also a series of new and more

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general exact solutions, in which the restriction on $\xi(x_1, x_2, \ldots, t)$ as merely a linear function of $x_1, x_2, \ldots, t$ and the restriction on the coefficients being constants are removed. For illustration, we apply this method to the $(2+1)$-dimensional Korteweg–de Vries (KdV) equations, and successfully obtain many new and more general exact solutions with two arbitrary functions.

The rest of this paper is organized as follows: in Section 2, we give the description of the generalized auxiliary equation method; in Section 3, we apply this method to the $(2+1)$-dimensional KdV equations; in Section 4, some conclusions are given.

2. Generalized auxiliary equation method

For a given NLPDE with independent variables $x = (t, x_1, x_2, \ldots, x_m)$ and dependent variable u:

$$F(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_m}, u_{x_1 t}, u_{x_2 t} \ldots, u_{x_m t}, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots, u_{x_m x_m}, \ldots) = 0, \tag{1}$$

We seek its solutions in the more general form:

$$u = a_0 + \sum_{i=1}^{n} \{a_i \phi^{-i}(\xi) + b_i \phi^i(\xi) + c_i \phi^{i-1}(\xi)\phi'(\xi) + d_i \phi^{-i}(\xi)\phi'(\xi)\}, \tag{2}$$

with $\phi(\xi)$ satisfying the new auxiliary equation:

$$\phi'^2(\xi) = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi) + h_5\phi^5(\xi) + h_6\phi^6(\xi), \tag{3}$$

where $a_0 = a_0(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$, $c_i = c_i(x)$, $d_i = d_i(x)$ $(i = 1, 2, \ldots, n)$ and $\xi = \xi(x)$ are functions to be determined, h_j $(j = 0, 1, 2 \ldots, 6)$ are real constants. To determine u explicitly, we take the following four steps:

Step 1. Determine the integer n. Substituting (2) along with (3) into Eq. (1) and balancing the highest order partial derivative term with the nonlinear terms in Eq. (1), we can obtain the value of n. For example, in the case of the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4}$$

we have $n = 4$.

Step 2. Derive a system of equations. Substituting (2) given the value of n obtained in Step 1 along with (3) into Eq. (1), collecting coefficients of $\phi^{\mu}(\xi)\phi'^{\rho}(\xi)(\rho = 0, 1; \mu = 0, \pm 1, \pm 2, \ldots)$, and then setting each coefficient to zero, we can derive a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i and ξ.

Step 3. Solve the system of equations. Solving the system of over-determined partial differential equations obtained in Step 2 with the aid of *Mathematica* and using the Wu elimination method [34], we can obtain the explicit expressions for a_0, a_i, b_i, c_i, d_i and ξ.

Step 4. Obtain exact solutions. By using the results obtained in the above steps, we can derive a series of fundamental solutions of Eq. (1) depending on the solution $\phi(\xi)$ of Eq. (3). By considering the different values of h_j $(j = 0, 1, 2 \ldots, 6)$, Eq. (3) has many kinds of solutions which can be found in [24–33]. Here we list only the solutions with $h_6 \neq 0$ as follows.

Case I

Suppose that $h_1 = h_3 = h_5 = 0$, $h_0 = 8h_2^2/27h_4$ and $h_6 = h_4^2/4h_2$.

(i) If $h_2 < 0$ and $h_4 > 0$, then Eq. (3) has the following solutions (here and thereafter $\varepsilon = \pm 1$):

$$\phi(\xi) = \left\{-\frac{8h_2 \tanh^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))}{3h_4[3 + \tanh^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))]}\right\}^{1/2}, \tag{5}$$

$$\phi(\xi) = \left\{-\frac{8h_2 \coth^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))}{3h_4[3 + \coth^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))]}\right\}^{1/2}. \tag{6}$$

(ii) If $h_2 > 0$ and $h_4 < 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{8h_2 \tan^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))}{3h_4[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))]} \right\}^{1/2}, \tag{7}$$

$$\phi(\xi) = \left\{ \frac{8h_2 \cot^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))}{3h_4[3 - \cot^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))]} \right\}^{1/2}. \tag{8}$$

Case II

Suppose that $h_0 = h_1 = h_3 = h_5 = 0$.

(i) If $h_2 > 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2 h_4 \operatorname{sech}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4^2 - h_2 h_6[1 + \varepsilon \tanh(\sqrt{h_2}(\xi + \xi_0))]^2} \right\}^{1/2}, \tag{9}$$

$$\phi(\xi) = \left\{ \frac{h_2 h_4 \operatorname{csch}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4^2 - h_2 h_6[1 + \varepsilon \coth(\sqrt{h_2}(\xi + \xi_0))]^2} \right\}^{1/2}. \tag{10}$$

(ii) If $h_2 > 0$ and $h_6 > 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2 \operatorname{sech}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{h_2 h_6} \tanh(\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{11}$$

$$\phi(\xi) = \left\{ \frac{h_2 \operatorname{csch}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{h_2 h_6} \coth(\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{12}$$

(iii) If $h_2 < 0$ and $h_6 > 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2 \sec^2(\sqrt{-h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{-h_2 h_6} \tan(\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{13}$$

$$\phi(\xi) = \left\{ -\frac{h_2 \csc^2(\sqrt{-h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{-h_2 h_6} \cot(\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{14}$$

Case III

Suppose that $h_0 = h_1 = h_3 = h_5 = 0$ and $h_4^2 - 4h_2 h_6 > 0$.

(i) If $h_2 > 0$, then Eq. (3) has the following solution:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}(2\sqrt{h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2 h_6} - h_4 \operatorname{sech}(2\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{15}$$

(ii) If $h_2 < 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{2h_2 \sec(2\sqrt{-h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2 h_6} - h_4 \sec(2\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{16}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \csc(2\sqrt{-h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2 h_6} - h_4 \csc(2\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{17}$$

(iii) If $h_2 > 0$, $h_4 < 0$ and $h_6 < 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\operatorname{sech}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0))} \right\}^{1/2}, \tag{18}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{csch}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} + \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right)\operatorname{csch}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0))} \right\}^{1/2}. \tag{19}$$

(iv) If $h_2 < 0$, $h_4 > 0$ and $h_6 < 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{-2h_2 \sec^2(\varepsilon\sqrt{-h_2}(\xi+\xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right)\sec^2(\varepsilon\sqrt{-h_2}(\xi+\xi_0))} \right\}^{1/2}, \tag{20}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \csc^2(\varepsilon\sqrt{-h_2}(\xi+\xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\csc^2(\varepsilon\sqrt{-h_2}(\xi+\xi_0))} \right\}^{1/2}. \tag{21}$$

Case IV

Suppose that $h_0 = h_1 = h_3 = h_5 = 0$ and $h_4^2 - 4h_2h_6 < 0$.

(i) If $h_2 > 0$, then Eq. (3) has the following solution:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{csch}(2\sqrt{h_2}(\xi+\xi_0))}{\varepsilon\sqrt{4h_2h_6 - h_4^2} - h_4 \operatorname{csch}(2\sqrt{h_2}(\xi+\xi_0))} \right\}^{1/2}. \tag{22}$$

Case V

Suppose that $h_0 = h_1 = h_3 = h_5 = 0$ and $h_4^2 - 4h_2h_6 = 0$.

(i) If $h_2 > 0$, then Eq. (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}[1 + \varepsilon \tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))] \right\}^{1/2}, \tag{23}$$

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}[1 + \varepsilon \coth(\varepsilon\sqrt{h_2}(\xi+\xi_0))] \right\}^{1/2}. \tag{24}$$

3. Application of the method

Let us consider the $(2+1)$-dimensional KdV equations:

$$u_t + u_{xxx} - 3v_x u - 3v u_x = 0, \tag{25}$$

$$u_x - v_y = 0. \tag{26}$$

Boiti et al. [35] first derived this system by using the idea of the weak Lax pair. Lou et al. [36] pointed out that this system can also be obtained from the inner parameter-dependent symmetry constraint of the Kadomtsev–Petviashvili (KP) equation, and that it is an asymmetric part of the Nizhnik–Novikov–Vesselov (NNV) equation. Ring types of solutions, periodic solutions and localized coherent solutions of Eqs. (25) and (26) can be found in [37–41].

According to Step 1, we get $n = 4$ for u and v. We assume that Eqs. (25) and (26) have the following formal solutions:

$$\begin{aligned} u = {} & a_0 + a_1\phi^{-1}(\xi) + a_2\phi^{-2}(\xi) + a_3\phi^{-3}(\xi) + a_4\phi^{-4}(\xi) + b_1\phi(\xi) + b_2\phi^2(\xi) + b_3\phi^3(\xi) \\ & + b_4\phi^4(\xi) + c_1\phi'(\xi) + c_2\phi(\xi)\phi'(\xi) + c_3\phi^2(\xi)\phi'(\xi) + c_4\phi^3(\xi)\phi'(\xi) + d_1\phi^{-1}(\xi)\phi'(\xi) \\ & + d_2\phi^{-2}(\xi)\phi'(\xi) + d_3\phi^{-3}(\xi)\phi'(\xi) + d_4\phi^{-4}(\xi)\phi'(\xi), \end{aligned} \tag{27}$$

$$\begin{aligned} v = {} & A_0 + A_1\phi^{-1}(\xi) + A_2\phi^{-2}(\xi) + A_3\phi^{-3}(\xi) + A_4\phi^{-4}(\xi) + B_1\phi(\xi) + B_2\phi^2(\xi) + B_3\phi^3(\xi) \\ & + B_4\phi^4(\xi) + C_1\phi'(\xi) + C_2\phi(\xi)\phi'(\xi) + C_3\phi^2(\xi)\phi'(\xi) + C_4\phi^3(\xi)\phi'(\xi) + D_1\phi^{-1}(\xi)\phi'(\xi) \\ & + D_2\phi^{-2}(\xi)\phi'(\xi) + D_3\phi^{-3}(\xi)\phi'(\xi) + D_4\phi^{-4}(\xi)\phi'(\xi), \end{aligned} \tag{28}$$

where $a_0 = a_0(y,t)$, $a_i = a_i(y,t)$, $b_i = b_i(y,t)$, $c_i = c_i(y,t)$, $d_i = d_i(y,t)$, $A_0 = A_0(y,t)$, $A_i = A_i(y,t)$, $B_i = B_i(y,t)$, $C_i = C_i(y,t)$, $D_i = D_i(y,t)$ $(i = 1,2,3,4)$, $\eta = \eta(y,t)$, $\xi = kx + \eta$, k is a constant.

With the aid of *Mathematica*, substituting (27) and (28) along with Eq. (3) into Eqs. (25) and (26), then setting each coefficient of $\phi^{\mu}(\xi)\phi'^{\rho}(\xi)$ $(\rho = 0, 1;\ \mu = 0, \pm 1, \pm 2, \ldots)$ to zero, we get a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i, A_0, A_i, B_i, C_i, D_i and η. Solving the system of over-determined partial differential equations by the use of *Mathematica*, we obtain the following results:

Case 1

$$a_0 = -\frac{(3C - k^2h_2 \pm 6k^2\sqrt{h_0h_4})f_1(y)}{3k}, \quad a_1 = \frac{kh_1f_1(y)}{2}, \quad a_2 = kh_0f_1(y), \tag{29}$$

$$a_3 = 0, \quad a_4 = 0, \quad b_1 = \frac{kh_3f_1(y)}{2}, \quad b_2 = kh_4f_1(y), \quad b_3 = 0, \quad b_4 = 0, \tag{30}$$

$$c_1 = \pm k\sqrt{h_4}f_1(y), \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = \pm k\sqrt{h_0}f_1(y), \tag{31}$$

$$d_3 = 0, \quad d_4 = 0, \quad A_0 = \frac{3kC + f_2'(t)}{3k}, \quad A_1 = \frac{k^2h_1}{2}, \quad A_2 = k^2h_0, \quad A_3 = 0, \tag{32}$$

$$A_4 = 0, \quad B_1 = \frac{k^2h_3}{2}, \quad B_2 = k^2h_4, \quad B_3 = 0, \quad B_4 = 0, \quad C_1 = \pm k^2\sqrt{h_4}, \tag{33}$$

$$C_2 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad D_1 = 0, \quad D_2 = \pm k^2\sqrt{h_0}, \quad D_3 = 0, \quad D_4 = 0, \tag{34}$$

$$\eta = \int f_1(y)\mathrm{d}y + f_2(t), \quad h_5 = 0, \quad h_6 = 0, \quad h_1\sqrt{h_4} \pm h_3\sqrt{h_0} = 0, \tag{35}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k is a non-zero constant, C is an arbitrary constant. The sign "$\pm$" in C_1 and D_2 means that all possible combinations of "$+$" and "$-$" can be taken. If the same sign is used in C_1 and D_2, then "$-$" must be used in a_0 and "$+$" must be used in (35). If different signs are used in C_1 and D_2, then "$+$" must be used in a_0 and "$-$" must be used in (35). Furthermore, the same sign must be used in c_1 and C_1. Also the same sign must be use in d_2 and D_2. Hereafter, the sign "$\pm$" always stands for this meaning in the similar circumstances.

Case 2

$$a_0 = -\frac{(3C - k^2h_2)f_1(y)}{3k}, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad b_1 = \frac{kh_3f_1(y)}{2}, \tag{36}$$

$$b_2 = kh_4f_1(y), \quad b_3 = 0, \quad b_4 = 0, \quad c_1 = \pm k\sqrt{h_4}f_1(y), \quad c_2 = 0, \quad c_3 = 0, \tag{37}$$

$$c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad A_0 = \frac{3kC + f_2'(t)}{3k}, \quad A_1 = 0, \quad A_2 = 0, \tag{38}$$

$$A_3 = 0, \quad A_4 = 0, \quad B_1 = \frac{k^2h_3}{2}, \quad B_2 = k^2h_4, \quad B_3 = 0, \quad B_4 = 0, \quad C_1 = \pm k^2\sqrt{h_4}, \tag{39}$$

$$C_2 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad D_1 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0, \tag{40}$$

$$\eta = \int f_1(y)\mathrm{d}y + f_2(t), \quad h_5 = 0, \quad h_6 = 0, \tag{41}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k is a non-zero constant, and C is an arbitrary constant.

Case 3

$$a_0 = -\frac{(3C - 4k^2h_2)f_1(y)}{3k}, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad b_1 = 0, \tag{42}$$

$$b_2 = 2kh_4f_1(y), \quad b_3 = 0, \quad b_4 = 4kh_6f_1(y), \quad c_1 = 0, \quad c_2 = \pm 4k\sqrt{h_6}f_1(y), \tag{43}$$

$$c_3 = 0, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad A_0 = \frac{3kC + f_2'(t)}{3k}, \quad A_1 = 0, \tag{44}$$

$$A_2 = 0, \quad A_3 = 0, \quad A_4 = 0, \quad B_1 = 0, \quad B_2 = 2k^2h_4, \quad B_3 = 0, \quad B_4 = 4k^2h_6, \tag{45}$$

$$C_1 = 0, \quad C_2 = \pm 4k^2\sqrt{h_6}, \quad C_3 = 0, \quad C_4 = 0, \quad D_1 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0, \tag{46}$$

$$\eta = \int f_1(y)\mathrm{d}y + f_2(t), \quad h_0 = h_0, \quad h_1 = 0, \quad h_3 = 0, \quad h_5 = 0, \tag{47}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k is a non-zero constant, and C is an arbitrary constant.

Case 4

$$a_0 = -\frac{(3C - 4k^2h_2)f_1(y)}{3k}, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad b_1 = 0, \tag{48}$$

$$b_2 = 4kh_4f_1(y), \quad b_3 = 0, \quad b_4 = 8kh_6f_1(y), \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \tag{49}$$

$$c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad A_0 = \frac{3kC + f_2'(t)}{3k}, \quad A_1 = 0, \tag{50}$$

$$A_2 = 0, \quad A_3 = 0, \quad A_4 = 0, \quad B_1 = 0, \quad B_2 = 4k^2h_4, \quad B_3 = 0, \quad B_4 = 8k^2h_6, \tag{51}$$

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad D_1 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0, \tag{52}$$

$$\eta = \int f_1(y)\mathrm{d}y + f_2(t), \quad h_4^2 = 4h_2h_6, \quad h_0 = 0, \quad h_1 = 0, \quad h_3 = 0, \quad h_5 = 0, \tag{53}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k is a non-zero constant, and C is an arbitrary constant.

From (27) and (28), Cases 1–2 and Cases I–V in [26], we can obtain many kinds of solutions of Eqs. (25) and (26) depending on the special choices for $h_i\,(i = 0, 1, 2, \ldots, 6)$.

3.1

If $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2rq + p^2$, $h_3 = 2pq$, $h_4 = q^2$, $h_5 = h_6 = 0$, then $\phi(\xi)$ is one of the 24 ϕ_l^{I} $(l = 1, 2, \ldots, 24)$.

For example, if we select $l = 10$, from Case 1 we obtain soliton-like solutions of Eqs. (25) and (26):

$$\begin{aligned}
u \;=\; & -\frac{(3C - k^2(2rq + p^2) \pm 6k^2|rq|)f_1(y)}{3k} + krpf_1(y)\phi^{-1}(\xi) + kr^2f_1(y)\phi^{-2}(\xi) + kpqf_1(y)\phi(\xi) \\
& + kq^2f_1(y)\phi^2(\xi) \pm k|q|f_1(y)\phi'(\xi) \pm k|r|f_1(y)\phi^{-2}(\xi)\phi'(\xi), \\
v \;=\; & \frac{(3kC + f_2'(t))}{3k} + k^2rp\phi^{-1}(\xi) + k^2r^2\phi^{-2}(\xi) + k^2pq\phi(\xi) + k^2q^2\phi^2(\xi) \\
& \pm k^2|q|\phi'(\xi) \pm k^2|r|\phi^{-2}(\xi)\phi'(\xi),
\end{aligned}$$

$$\phi(\xi) = \frac{2r\cosh(\sqrt{p^2 - 4qr}\xi)}{\sqrt{p^2 - 4qr}\sinh(\sqrt{p^2 - 4qr}\xi) - p\cosh(\sqrt{p^2 - 4qr}\xi) \pm \mathrm{i}\sqrt{p^2 - 4qr}},$$

$$\phi'(\xi) = \frac{2r(p^2 - 4qr)(-1 \pm \mathrm{i}\sinh(\sqrt{p^2 - 4qr}\xi))}{[\sqrt{p^2 - 4qr}\sinh(\sqrt{p^2 - 4qr}\xi) - p\cosh(\sqrt{p^2 - 4qr}\xi) \pm \mathrm{i}\sqrt{p^2 - 4qr}]^2},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$. If "+" is used in a_0, then $qr > 0$. If "−" is used in a_0, then $qr < 0$.

3.2

If $h_0 = r^2$, $h_1 = 2rp$, $h_2 = h_5 = h_6 = 0$, $h_3 = 2pq$, $h_4 = q^2$ *and* $p^2 = -2rq$, *then* $\phi(\xi)$ *is one of the* 12 ϕ_l^{II} $(l = 1, 2, \ldots, 12)$.

For example, if we select $l = 12$, from Case 1, we obtain soliton-like solutions of Eqs. (25) and (26):

$$\begin{aligned} u &= -\frac{(C + 2k^2qr)f_1(y)}{k} + krpf_1(y)\phi^{-1}(\xi) + kr^2f_1(y)\phi^{-2}(\xi) + kpqf_1(y)\phi(\xi) + kq^2f_1(y)\phi^2(\xi) \\ &\quad \pm k|q|f_1(y)\phi'(\xi) \pm k|r|f_1(y)\phi^{-2}(\xi)\phi'(\xi), \\ v &= \frac{(3kC + f_2'(t))}{3k} + k^2rp\phi^{-1}(\xi) + k^2r^2\phi^{-2}(\xi) + k^2pq\phi(\xi) + k^2q^2\phi^2(\xi) \\ &\quad \pm k^2|q|\phi'(\xi) \pm k^2|r|\phi^{-2}(\xi)\phi'(\xi), \end{aligned}$$

$$\phi(\xi) = \frac{4r\sinh\left(\frac{\sqrt{-6qr}}{4}\xi\right)\cosh\left(\frac{\sqrt{-6qr}}{4}\xi\right)}{\mp 2\sqrt{-2qr}\sinh(\frac{\sqrt{-6qr}}{4}\xi)\cosh\left(\frac{\sqrt{-6qr}}{4}\xi\right) + 2\sqrt{-6qr}\cosh^2\left(\frac{\sqrt{-6qr}}{4}\xi\right) - \sqrt{-6qr}},$$

$$\phi'(\xi) = \frac{3r}{\left(\sqrt{3}\cosh\left(\frac{\sqrt{-6qr}}{2}\xi\right) \mp \sinh\left(\frac{\sqrt{-6qr}}{2}\xi\right)\right)^2},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$, $qr < 0$.

3.3

If $h_0 = h_1 = h_5 = h_6 = 0$, h_2, h_3 *and* h_4 *are arbitrary constants, then* $\phi(\xi)$ *is one of the 10* ϕ_l^{III} $(l = 1, 2, \ldots, 10)$.

For example, if we select $l = 4$, then $h_2 = 4$, $h_3 = 4(d - 2b)/a$, $h_4 = (c^2 + 4b^2 - 4bd)/a^2$; from Case 1 we obtain soliton-like solutions of Eqs. (25) and (26):

$$\begin{aligned} u &= -\frac{(3C - 4k^2)f_1(y)}{3k} + \frac{2k(d - 2b)f_1(y)}{a}\phi(\xi) + \frac{k(c^2 + 4b^2 - 4bd)f_1(y)}{a^2}\phi^2(\xi) \\ &\quad \pm \frac{k\sqrt{c^2 + 4b^2 - 4bd}\,f_1(y)}{|a|}\phi'(\xi), \end{aligned}$$

$$v = \frac{(3kC + f_2'(t))}{3k} + \frac{2k^2(d - 2b)}{a}\phi(\xi) + \frac{k^2(c^2 + 4b^2 - 4bd)}{a^2}\phi^2(\xi) \pm \frac{k^2\sqrt{c^2 + 4b^2 - 4bd}}{|a|}\phi'(\xi),$$

$$\phi(\xi) = \frac{a\,\mathrm{csch}^2(\xi)}{b\,\mathrm{csch}^2(\xi) + c\coth(\xi) + d},$$

$$\phi'(\xi) = -\frac{4a(c\cosh(2\xi) + d\sinh(2\xi))}{(2b - d + d\cosh(2\xi) + c\sinh(2\xi))^2},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

3.4

If $h_1 = h_3 = h_5 = h_6 = 0$, h_0, h_2 *and* h_4 *are arbitrary constants, then* $\phi(\xi)$ *is one of the 16* ϕ_l^{IV} $(l = 1, 2, \ldots, 16)$.

For example, if we select $l = 13$, then $h_0 = 1/4$, $h_2 = (1 - 2m^2)/2$, $h_4 = 1/4$; from Case 1 we obtain combined non-degenerative Jacobi elliptic wave function-like solutions of Eqs. (25) and (26):

$$\begin{aligned} u &= -\frac{(6C - k^2(1 - 2m^2) \pm 3k^2)f_1(y)}{6k} + \frac{kf_1(y)}{4}\phi^{-2}(\xi) + \frac{kf_1(y)}{4}\phi^2(\xi) \\ &\quad \pm \frac{kf_1(y)}{2}\phi'(\xi) \pm \frac{kf_1(y)}{2}\phi^{-2}(\xi)\phi'(\xi), \end{aligned}$$

$$v = \frac{(3kC + f_2'(t))}{3k} + \frac{k^2}{4}\phi^{-2}(\xi) + \frac{k^2}{4}\phi^2(\xi) \pm \frac{k^2}{2}\phi'(\xi) \pm \frac{k^2}{2}\phi^{-2}(\xi)\phi'(\xi),$$
$$\phi(\xi) = \mathrm{ns}\,\xi \pm \mathrm{cs}\,\xi, \quad \phi'(\xi) = -(\mathrm{cs}\,\xi \pm \mathrm{ns}\,\xi)\mathrm{ds}\xi,$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

In the limit case when $m \to 1$, we obtain combined soliton-like solutions of Eqs. (25) and (26):

$$u = -\frac{(6C + k^2 \pm 3k^2) f_1(y)}{6k} + \frac{kf_1(y)}{4}\phi^{-2}(\xi) + \frac{kf_1(y)}{4}\phi^2(\xi) \pm \frac{kf_1(y)}{2}\phi'(\xi) \pm \frac{kf_1(y)}{2}\phi^{-2}(\xi)\phi'(\xi),$$
$$v = \frac{(3kC + f_2'(t))}{3k} + \frac{k^2}{4}\phi^{-2}(\xi) + \frac{k^2}{4}\phi^2(\xi) \pm \frac{k^2}{2}\phi'(\xi) \pm \frac{k^2}{2}\phi^{-2}(\xi)\phi'(\xi),$$
$$\phi(\xi) = \coth\xi \pm \mathrm{csch}\,\xi, \quad \phi'(\xi) = -(\mathrm{csch}\,\xi \pm \coth\xi)\mathrm{csch}\,\xi,$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

When $m \to 0$, we obtain triangular-like solutions of Eqs. (25) and (26):

$$u = -\frac{(6C - k^2 \pm 3k^2) f_1(y)}{6k} + \frac{kf_1(y)}{4}\phi^{-2}(\xi) + \frac{kf_1(y)}{4}\phi^2(\xi) \pm \frac{kf_1(y)}{2}\phi'(\xi) \pm \frac{kf_1(y)}{2}\phi^{-2}(\xi)\phi'(\xi),$$
$$v = \frac{(3kC + f_2'(t))}{3k} + \frac{k^2}{4}\phi^{-2}(\xi) + \frac{k^2}{4}\phi^2(\xi) \pm \frac{k^2}{2}\phi'(\xi) \pm \frac{k^2}{2}\phi^{-2}(\xi)\phi'(\xi),$$
$$\phi(\xi) = \csc\xi \pm \cot\xi, \quad \phi'(\xi) = -(\cot\xi \pm \csc\xi)\csc\xi,$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

3.5

If $h_2 = h_4 = h_5 = h_6 = 0$, h_0, h_1 and h_3 are arbitrary constants, then $\phi(\xi)$ is the only ϕ_1^{V}.

From (35), we get $h_0 = 0$ or $h_3 = 0$, Eqs. (25) and (26) have no solutions for Case 1. Fortunately, from Case 2, we obtain Weierstrass elliptic doubly-like periodic solutions of Eqs. (25) and (26):

$$u = -\frac{Cf_1(y)}{k} + \frac{kh_3 f_1(y)}{2}\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right), \tag{54}$$

$$v = \frac{3kC + f_2'(t)}{3k} + \frac{k^2 h_3}{2}\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) \tag{55}$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$, $h_3 > 0$, $g_2 = -4h_1/h_3$, $g_3 = -4h_0/h_3$.

From (27) and (28), Cases 3–4, and Cases I–V listed in the present paper, we can obtain many kinds of solutions of Eqs. (25) and (26) depending on the special choices for h_i $(i = 0, 1, 2, \ldots, 6)$.

3.6

If $h_1 = h_3 = h_5 = 0$, $h_0 = 8h_2^2/27h_4$ and $h_6 = h_4^2/4h_2$, then $\phi(\xi)$ is one of the (7) and (8).

For example, if we select (7), from Case 3 we obtain triangular-like solutions of Eqs. (25) and (26):

$$u = -\frac{(3C - 4k^2h_2) f_1(y)}{3k} + 2kh_4 f_1(y)\phi^2(\xi) + \frac{kh_4^2 f_1(y)}{h_2}\phi^4(\xi) \mp \frac{2kh_4\sqrt{h_2} f_1(y)}{h_2}\phi(\xi)\phi'(\xi),$$
$$v = \frac{3kC + f_2'(t)}{3k} + 2k^2h_4\phi^2(\xi) + \frac{k^2h_4^2}{h_2}\phi^4(\xi) \mp \frac{2k^2h_4\sqrt{h_2}}{h_2}\phi(\xi)\phi'(\xi),$$
$$\phi(\xi) = \left\{\frac{8h_2\tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{3h_4[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]}\right\}^{1/2},$$
$$\phi'(\xi) = \frac{4\varepsilon h_2^{3/2}\sin(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{\sqrt{3}h_4[1 + 2\cos(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]^2\phi(\xi)},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

3.7

If $h_0 = h_1 = h_3 = h_5 = 0$, h_2, h_4 *and* h_6 *are arbitrary constants, then* $\phi(\xi)$ *is one of the* (9)–(17) and (22).
For example, if we select (10), from Case 3 we obtain soliton-like solutions of Eqs. (25) and (26):

$$u = -\frac{(3C - 4k^2h_2)f_1(y)}{3k} + 2kh_4 f_1(y)\phi^2(\xi) + 4kh_6 f_1(y)\phi^4(\xi) \pm 4k\sqrt{h_6} f_1(y)\phi(\xi)\phi'(\xi)$$
$$v = \frac{3kC + f_2'(t)}{3k} + 2k^2h_4\phi^2(\xi) + 4k^2h_6\phi^4(\xi) \pm 4k^2\sqrt{h_6}\phi(\xi)\phi'(\xi)$$
$$\phi(\xi) = \left\{\frac{h_2h_4\mathrm{csch}^2(\sqrt{h_2}(\xi+\xi_0))}{h_4^2 - h_2h_6[1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0))]^2}\right\}^{1/2},$$
$$\phi'(\xi) = \frac{h_2^{3/2}h_4\mathrm{csch}^4(\sqrt{h_2}(\xi+\xi_0))[2\varepsilon h_2h_6\cosh(2\sqrt{h_2}(\xi+\xi_0)) + (2h_2h_6 - h_4^2)\sinh(2\sqrt{h_2}(\xi+\xi_0))]}{2[h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2]^2\phi(\xi)},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

3.8

If $h_0 = h_1 = h_3 = h_5 = 0$ *and* $h_4^2 - 4h_2h_6 = 0$, *then* $\phi(\xi)$ *is one of the* (23) and (24).
For example, if we select (23), from Case 3 we obtain soliton-like solutions of Eqs. (25) and (26):

$$u = -\frac{(3C - 4k^2h_2)f_1(y)}{3k} + 2kh_4 f_1(y)\phi^2(\xi) + \frac{kh_4^2 f_1(y)}{h_2}\phi^4(\xi) \pm 2k\sqrt{\frac{h_4^2}{h_2}} f_1(y)\phi'(\xi),$$
$$v = -\frac{3kC + f_2'(t)}{3k} + 2k^2h_4\phi^2(\xi) + \frac{k^2h_4^2}{h_2}\phi^4(\xi) \pm 2k^2\sqrt{\frac{h_4^2}{h_2}}\phi'(\xi),$$
$$\phi(\xi) = \left\{-\frac{h_2}{h_4}[1+\varepsilon\tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))]\right\}^{1/2}, \quad \phi'(\xi) = -\frac{h_2^{3/2}\mathrm{sech}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0))}{2h_4\phi(\xi)},$$

where $\xi = kx + \int f_1(y)\mathrm{d}y + f_2(t)$.

From (27) and (28), Cases 1–4, we can obtain other exact solutions of Eqs. (25) and (26); here we omit them for simplicity.

Remark 1. The solutions obtained from Cases 1–2 can be obtained by using the method from [27], but all the solutions obtained from Cases 3–4 cannot be obtained by using the methods in [24–33], which are new and have not been reported yet. All the results reported in this paper have been checked with *Mathematica*. By using our method, we can also obtain many new and more general exact solutions of the other NLPDEs in [24–33], including all the solutions given there as special cases of our method. It shows that our method is more powerful than the others in constructing exact solutions of NLPDEs.

4. Conclusion

In summary, we have presented a generalized auxiliary equation method to construct more general exact solutions of NLPDEs. With the help of *Mathematica*, this method provides a powerful mathematical tool to obtain more general exact solutions of a great many NLPDEs in mathematical physics, such as the (3 + 1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (2 + 1)-dimensional Nizhnik–Novikov–Vesselov (NNV) equations, Broer–Kaup–Kupershmidt (BKK) equations, breaking soliton (BS) equations, Broer–Kaup (BK) equations, dispersive long wave (DLW) equations, and so on. Compared with the existing methods [6,20,21,24–33], this method is more powerful. It can be used to construct more general exact solutions which contain not only the results obtained by using the methods [6,20,21,24–33], but also a series of new and more general exact solutions. Applying the proposed method to the (2 + 1)-dimensional KdV equations, we have obtained many new and more general exact non-travelling wave and coefficient function solutions, which include soliton-like solutions, triangular-like solutions, single and combined non-degenerate Jacobi elliptic wave function-like solutions, and Weierstrass elliptic doubly-like periodic solutions. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. These may be important to explain some physical phenomena.

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