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Unprovability of the logical characterization of bisimulation

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ABSTRACT

We quickly review *labelled Markov processes* (*LMP*) and provide a counterexample showing that in general measurable spaces, event bisimilarity and state bisimilarity differ in LMP. This shows that the Hennessy–Milner logic proposed by Desharnais does not characterize state bisimulation in non-analytic measurable spaces. Furthermore we show that, under current foundations of Mathematics, such logical characterization is unprovable for spaces that are projections of a coanalytic set. Underlying this construction there is a proof that stationary Markov processes over general measurable spaces do not have semi-pullbacks.

1. Introduction

One of the more interesting facts about the state of the art on Markov decision processes over a continuous state-space is that there exists a number of competing notions of *bisimulation*. The essential difference with the discrete case is the appearance of nonmeasurable sets, which e.g., inhibit the possibility of extending straightforwardly Larsen and Skou [15] notion of probabilistic bisimulation.

To work in a concrete setting, we will use the framework of Labelled Markov Processes (LMP). LMP have a labelled set of *actions* that encode interaction with the environment; thus LMP are a reactive model in which there are different transition subprobabilities for each action. The thesis [3] contains a thorough study and introduction to LMP.

The categorical approach to bisimulation is already present in Joyal et al. [12] and was studied for LMP in [4]. There, the notion of *zig-zag* morphism was defined and the relation of *bisimilarity* was given by a span of zig-zags. Zig-zags are exactly the coalgebra morphisms for Giry's functor Π [9]. The major obstacle for this definition of bisimulation was that transitivity of bisimilarity was proved by using structure results only available when the state-space is analytic. This was done in [4] by using a technical result by Edalat [8] that constructed a span of zig-zags given a cospan. To achieve this goal, Edalat established explicitly the existence of regular conditional probabilities for the universal completion of a Polish space. An alternative point of view, restricted to the category of Polish spaces, can be found in Doberkat [5].

A Hennessy–Milner logic was also developed and in [4] it was proved that the relation of bisimilarity was characterized by this logic in the case of an analytic state-space. Clearly, a notion of logical equivalence must be transitive, so the problem of transitivity is more general than that of the logical characterization of bisimilarity.

It was realized that a new notion of bisimulation was needed, and in [2] Danos, Desharnais, Laviolette, and Panangaden defined *event bisimulation* in terms of the measurable structure of LMP. They proved that logical equivalence and event bisimilarity coincide and that both can be phrased as a cospan of zig-zags. These results are in a way consequences of the fact that cospans are far more easy to work with in a coalgebraic setting.

In this paper, we construct a counterexample showing that in general measurable spaces, event bisimilarity and state bisimilarity differ in LMP. This shows that the Hennessy–Milner logic used in [4,3,2] does not characterize bisimulation in

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non-analytic measurable spaces. The construction includes also a counterexample to the existence of *semi-pullbacks* in the category of stationary Markov processes over general measurable spaces.

The construction of our counterexample needs the existence of a nonmeasurable set. It is known that it is consistent with current foundations of Mathematics that there exists a nonmeasurable subset of the Euclidean plane which is the continuous image of the complement of an analytic set. Hence, there are spaces in level 2 of the *projective hierarchy* of sets (level 0 occupied by Borel sets, level 1 by analytic sets and their complements) for which the logical characterization bisimulation is unprovable.

The paper is organized as follows. In Section 2 we review some background to our study, including some concepts related to measurable spaces and prior results on labelled Markov processes. Section 3 develops the consequences of Łoś and Marczewski's theorem on extension of measure, in particular the non-existence of semi-pullbacks in the category of stationary Markov processes and zig-zag morphisms. Our main counterexample, an LMP for which event and state bisimilarity differ from each other, is constructed in Section 4. A careful analysis of the set theoretical requirements of this construction is pursued in Section 5, where we show our unprovability result.

2. Background

2.1. Measurable spaces

A σ-algebra over a set S is a family of subsets of S closed under countable union and complementation. Given an arbitrary family $\mathcal{U}$ of subsets of S, we use $\sigma(\mathcal{U})$ to denote the least σ-algebra over S containing $\mathcal{U}$.

Let $\langle S, \mathcal{S}\rangle$ be a measurable space, i.e., a set S with a σ-algebra $\mathcal{S}$ over S. We say that $\langle S, \mathcal{S}\rangle$ (or $\mathcal{S}$) is *countably generated* if there is some countable family $\mathcal{U} \subseteq \mathcal{S}$ such that $\mathcal{S} = \sigma(\mathcal{U})$. Assume now that $V \subset S$. We will use $\mathcal{S}_V$ to denote $\sigma(\{V\} \cup \mathcal{S})$, *the extension of $\mathcal{S}$ by the set V*. It is immediate that

$$\mathcal{S}_V = \left\{(B_1 \cap V) \cup \left(B_2 \cap V^{\mathsf{c}}\right)\colon\ B_1, B_2 \in \mathcal{S}\right\}.$$

The *sum* of two measurable spaces $\langle S_1, \mathcal{S}_1\rangle$ and $\langle S_2, \mathcal{S}_2\rangle$ is $\langle S_1 \oplus S_2, \mathcal{S}_1 \oplus \mathcal{S}_2\rangle$, with the following abuse of notation: $S_1 \oplus S_2$ is the disjoint union (direct sum *qua* sets) and $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{Q_1 \oplus Q_2\colon\ Q_i \in \mathcal{S}_i\}$. We obtain

$$\langle S, \mathcal{S}_V\rangle \cong \langle V, \mathcal{S}|V\rangle \oplus \left\langle V^{\mathsf{c}}, \mathcal{S}|V^{\mathsf{c}}\right\rangle. \tag{1}$$

It is obvious that if $\mathcal{S}$ is countably generated so is $\mathcal{S}_V$.

If Y is a topological space, $\mathbf{B}(Y)$ will denote the σ-algebra generated by open sets in Y, hence $\langle Y, \mathbf{B}(Y)\rangle$ is a measurable space, the *Borel space of Y*.

The central example (see Theorem 3 below) is the Borel space of the open unit interval $\mathbb{I} := (0, 1)$. The σ-algebra $\mathbf{B}(\mathbb{I})$ is countably generated: it is generated by the family $\mathcal{B} := \{B_a\colon\ a \in \omega\}$ of all open subintervals of $\mathbb{I}$ with rational endpoints. This family has a property which is inherited by the whole σ-algebra: we say that a family of sets $\mathcal{S} \subseteq \mathsf{Pow}(S)$ *separates points* if for every pair of distinct points x, y in S, there is $A \in \mathcal{S}$ with $x \in A$ and $y \notin A$. Hence $\mathbf{B}(\mathbb{I})$ separates points. We have the following propositions; the first of them is immediate and we will use it without reference.

Proposition 1. *For $\mathcal{U} \subseteq \mathsf{Pow}(S)$, $\mathcal{U}$ separates points if and only if $\sigma(\mathcal{U})$ does.*

Proposition 2. *(12.1 from [14].) The following are equivalent*:

1. *$\langle S, \mathcal{S}\rangle$ is isomorphic to some $\langle Y, \mathbf{B}(Y)\rangle$, where Y is separable metrizable.*
2. *$\langle S, \mathcal{S}\rangle$ is countably generated and separates points.*

A topological space is *Polish* if it is separable and completely metrizable. Examples of Polish spaces are the Euclidean spaces $\mathbb{R}^n$ and all countable discrete spaces. Polish spaces are closed under countable product, and hence the *Baire space* $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ is Polish, assuming $\mathbb{N}$ discrete. We have the following fundamental result (see [14, 15.6]):

Theorem 3 *(The Isomorphism Theorem). Let Y be an uncountable Polish space. Then the Borel space of Y is isomorphic to $\langle \mathbb{I}, \mathbf{B}(\mathbb{I})\rangle$.*

Finally, an *analytic* (or $\boldsymbol{\Sigma}^1_1$) space is the continuous image of a Polish space.

2.2. Labelled Markov processes

The following definitions are extracted from Danos et al. [2].

Let $\langle S, \mathcal{S}\rangle$ be a measurable space. Recall that a Markov kernel on a measurable space $\langle S, \mathcal{S}\rangle$ is a function $\tau : S \times \mathcal{S} \to [0, 1]$ such that for each fixed $s \in S$, the set function $\tau(s, \cdot)$ is a (sub)probability measure, and for each fixed $X \in \mathcal{S}$, $\tau(\cdot, X)$ is a $(\mathcal{S}, \mathbf{B}([0, 1]))$-measurable function.

Now let L be any set.

Definition 4. A *labelled Markov process* (*LMP*) is a structure $\mathbf{S} = \langle S, \mathcal{S}, \{\tau_a\colon\ a \in L\}\rangle$ where $\langle S, \mathcal{S}\rangle$ is a measurable space and for $a \in L$, $\tau_a : S \times \mathcal{S} \to [0, 1]$ is a Markov kernel. We will call L the set of *labels* and $\langle S, \mathcal{S}\rangle$ the *base space* of $\mathbf{S}$.

Labelled Markov processes form a category whose arrows are given by *zig-zag morphisms*.

Definition 5. Let $\mathbf{S} = \langle S, \mathcal{S}, \{\tau_a\colon\ a \in L\}\rangle$ and $\mathbf{S}' = \langle S', \mathcal{S}', \{\tau'_a\colon\ a \in L\}\rangle$ be LMP. A *zig-zag morphism* $f : \mathbf{S} \to \mathbf{S}'$ is a surjective measurable map $f : \langle S, \mathcal{S}\rangle \to \langle S', \mathcal{S}'\rangle$ such that for all $a \in L$ we have

$$\forall s \in S,\ \forall Q \in \mathcal{S}'\colon\quad \tau_a\big(s, f^{-1}(Q)\big) = \tau'_a\big(f(s), Q\big).$$

The reader may find variants of this definition along the development of the theory. In [3] LMP are augmented with an initial state and zig-zags are not required to be surjective but to preserve initial states. Later in [4] the authors adopt the present definition. However, these are minor differences. More fundamentally, both [3,4] require the base space of an LMP to be analytic. We refer the reader to Desharnais [3] for motivation and for the fundamental results in the theory of LMP.

Some notation concerning binary relations will be needed to state the formal definitions. Let R be a binary relation over S. A set Q is *R-closed* if $Q \ni x\,R\,y$ implies $y \in Q$. $\mathcal{S}(R)$ is the σ-algebra of R-closed sets in $\mathcal{S}$. Lastly, let $\mathcal{U}$ be a subset of $\mathsf{Pow}(S)$. The relation $\mathcal{R}(\mathcal{U})$ is given by

$$(s, t) \in \mathcal{R}(\mathcal{U}) \quad\Leftrightarrow\quad \forall Q \in \mathcal{U}\colon\ \ s \in Q \quad\Leftrightarrow\quad t \in Q.$$

Fix an LMP $\mathbf{S} = \langle S, \mathcal{S}, \{\tau_a\colon\ a \in L\}\rangle$.

Definition 6.

1. A relation $R \subseteq S \times S$ is a *state bisimulation* on $\mathbf{S}$ if it is symmetric and for all $a \in L$, $s\,R\,t$ implies $\forall Q \in \mathcal{S}(R)\colon\ \tau_a(s, Q) = \tau_a(t, Q)$.
2. An *event bisimulation* on $\mathbf{S}$ is a sub-σ-algebra $\mathcal{U}$ of $\mathcal{S}$ such that $\langle S, \mathcal{U}, \{\tau_a\colon\ a \in L\}\rangle$ is an LMP (i.e., τ_a is $\mathcal{U}$-measurable for each $a \in L$). We also say that a relation R is an event bisimulation if there is an event bisimulation $\mathcal{U}$ such that $R = \mathcal{R}(\mathcal{U})$.

If there is a state (event) bisimulation R such $s\,R\,t$, we will say that *s is state- (event-)bisimilar to t*.

It is proved [3, Proposition 3.5.3] that whenever there exists a zig-zag morphism f between two LMP $\mathbf{S}$ and $\mathbf{T}$, the equivalence relation generated by the pairs $(s, f(s))$ with $s \in S$ is a state bisimulation on the sum $\mathbf{S} \oplus \mathbf{T}$.[2] On the other hand, for every state bisimulation R on an LMP $\mathbf{S}$, the identity map $Id : \langle S, \mathcal{S}, \{\tau_a\colon\ a \in L\}\rangle \to \langle S, \mathcal{S}(R), \{\tau_a\colon\ a \in L\}\rangle$ is a zig-zag (see [2, Lemma 4.2]).

A generalization of the notion of event bisimulation will be needed in the sequel:

Definition 7. A subfamily $\mathcal{U} \subseteq \mathcal{S}$ is *stable* with respect to $\mathbf{S}$ if for all $A \in \mathcal{U}$, $r \in [0, 1]$ and $a \in L$, $\{s \in S\colon\ \tau_a(s, A) > r\} \in \mathcal{U}$.

Since a function $f : S \to [0, 1]$ is measurable if and only if $f^{-1}((r, 1])$ is a measurable set for every $r \in [0, 1]$, an event bisimulation on $\mathbf{S}$ is the same thing as a stable sub-σ-algebra of $\mathcal{S}$. This notion of stability was further generalized by Doberkat [6] to the concept of *congruence* for stochastic systems.

It is shown that there exists a greatest state bisimulation $\sim$ (namely, the relation of *state bisimilarity*), and in [2] it is proved that event bisimulation is characterized by the logic $\mathcal{L}$ given by the following productions:

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a\rangle_q \psi$$

where $a \in L$ and $q \in \mathbb{Q} \cap [0, 1]$. Formulas in $\mathcal{L}$ are interpreted as sets of states in which they become true as follows

$$[\![\top]\!] := S, \qquad [\![\varphi_1 \wedge \varphi_2]\!] := [\![\varphi_1]\!] \cap [\![\varphi_2]\!], \qquad \big[\!\big[\langle a\rangle_q \psi\big]\!\big] := \big\{s \in S\colon\ \tau_a\big(s, [\![\psi]\!]\big) \geqslant q\big\}.$$

Let $[\![\mathcal{L}]\!] := \{[\![\varphi]\!]\colon\ \varphi \in \mathcal{L}\}$. Two states $s, t \in S$ are *logically equivalent* if $s\,\mathcal{R}([\![\mathcal{L}]\!])\,t$, i.e., if they satisfy exactly the same formulas. Given a class $\mathcal{M}$ of LMP, the problem of the *logical characterization of bisimulation* for $\mathcal{M}$ is to prove the following statement:

For all $\mathbf{S} \in \mathcal{M}$ and all $s, t \in S$, $s\,\mathcal{R}([\![\mathcal{L}]\!])\,t$ if and only if there exists a bisimulation R such that $s\,R\,t$. (We say that $\mathcal{L}$ *completely characterizes* bisimulation.)[3]

[2] The base space of the sum $\mathbf{S} \oplus \mathbf{T}$ is $\langle S, \mathcal{S}\rangle \oplus \langle T, \mathcal{T}\rangle$ and the transition function $\tau_a^{\mathbf{S}\oplus\mathbf{T}}(r, A)$ equals $\tau_a^{\mathbf{S}}(r, A \cap S)$ if $r \in S$, and $\tau_a^{\mathbf{T}}(r, A \cap T)$ if $r \in T$.

[3] Perhaps a better phrasing would be "characterization of *bisimilarity*", but we keep this one in accordance with previous works.

This depends on how do we qualify the word "bisimulation". In the case of event bisimulation, we have the following results:

Theorem 8. *(See [2, Proposition 5.5].) $\sigma(\llbracket\mathcal{L}\rrbracket)$ is the smallest stable σ-algebra.*

Theorem 9. *(See [2, Corollary 5.6].) $\sigma(\llbracket\mathcal{L}\rrbracket)$ is the least event bisimulation, and hence the logic $\mathcal{L}$ completely characterizes event bisimulation.*

In view of this result we conclude that the problem of the logical characterization of state bisimulation is equivalent to decide if event and state bisimilarity coincide. This was also proved in [2] for the class of LMP having an analytic base space. To obtain this result, one needs the logic $\mathcal{L}$ to be countable, hence also limiting the set of labels L to be at most countable. It is noteworthy that the counterexample of Section 4 conforms this restriction.

3. Extensions of measures

The reader can consult Royden [17] and Rudin [18] as general references for Measure Theory.

The key idea in the construction of our counterexample is the possibility of extending the domain of definition of a (probability) measure in a very flexible way. We will use a result due to Łoś and Marczewski [16] concerning *canonical extensions* of measures.[4] If $\mathcal{S} \subseteq \mathcal{U}$ and μ, ν are measures defined on $\langle S, \mathcal{S}\rangle$, $\langle S, \mathcal{U}\rangle$ (respectively), we say that ν *extends* μ *to* $\langle S, \mathcal{U}\rangle$ when $\nu|\mathcal{S} = \mu$. We recall that the *inner* and *outer* measures defined from μ, denoted μ_i and μ_e respectively, are the countably subadditive functions given by

$$\mu_i(A) := \sup\bigl\{\mu(M)\colon\ M \subseteq A,\ M \in \mathcal{S}\bigr\}, \qquad \mu_e(A) := \inf\bigl\{\mu(M)\colon\ M \supseteq A,\ M \in \mathcal{S}\bigr\},$$

for every $A \subseteq S$.

It is well known that the domain of definition of a measure μ can be enlarged to include all subsets A for which $\mu_i(A) = \mu_e(A)$; such sets are called μ*-measurable* and they form a σ-algebra. In the case of Lebesgue measure, we will use the name "Lebesgue measurable sets". By using the Axiom of Choice it can be proved the existence of sets in Euclidean space that are not Lebesgue measurable. For such sets the following results are most significant.

Theorem 10. *Let μ be a finite measure defined in $\langle S, \mathcal{S}\rangle$, and let $V \subseteq S$. Then $\underline{\mu}$ and $\bar{\mu}$ defined as*

$$\underline{\mu}(E) = \mu_i(E \cap V) + \mu_e\bigl(E \cap V^c\bigr),$$
$$\bar{\mu}(E) = \mu_e(E \cap V) + \mu_i\bigl(E \cap V^c\bigr)$$

for every $E \in \mathcal{S}_V$ are measures that extend μ to $\langle S, \mathcal{S}_V\rangle$ and satisfy

$$\underline{\mu}(V) = \mu_i(V), \qquad \bar{\mu}(V) = \mu_e(V).$$

Proof. By Theorems 4, 2, and 1 in [16]. The proof follows elementarily from these facts:

1. For all A, B such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$ we have $\mu_i(A) + \mu_e(B) = \mu(A + B)$ (see, for instance, [10, 14.H]).
2. For $E_j \in \mathcal{S}_V$ $(j \in \omega)$ pairwise disjoint there are $M_j, N_j \in \mathcal{S}$ such that $E_j = (M_j \cap V) \cup (N_j \cap V^c)$ and M_j (N_j) pairwise disjoint.
3. If $A_j \subseteq M_j$ with $M_j \in \mathcal{S}$ pairwise disjoint, then $\mu_i(\bigcup_j A_j) = \sum_j \mu_i(A_j)$ and $\mu_e(\bigcup_j A_j) = \sum_j \mu_e(A_j)$. □

Corollary 11. *Let μ be a finite measure defined in $\langle S, \mathcal{S}\rangle$ and let $V \subseteq S$ be non-μ-measurable. Then there are extensions μ_1 and μ_2 to $\mathcal{S}_V$ of μ such that $\mu_1(V) \neq \mu_2(V)$.*

Proof. Immediate by definition of (non-)μ-measurable set. □

At this point it is possible to give a hint for the failure of the logical characterization of bisimulation. The logic can be seen as an encoding for the family $\llbracket\mathcal{L}\rrbracket$ of measurable sets, which can be enlarged to the σ-algebra $\sigma(\llbracket\mathcal{L}\rrbracket)$. This σ-algebra cannot "weigh" a set V which is not measurable "respect to $\sigma(\llbracket\mathcal{L}\rrbracket)$"; more precisely, one can have two measures that are equal on $\sigma(\llbracket\mathcal{L}\rrbracket)$ ("logically equal") but they differ on $\sigma(\llbracket\mathcal{L}\rrbracket)_V$.

With this tool at hand we are now ready to witness a failure for the existence of semi-pullbacks [8] in the category of (labelled) Markov processes over general measurable spaces and zig-zag morphisms. A category *has semi-pullbacks* if for

[4] Łoś and Marczewski use the term "measure" to mean a *finitely* additive set function while reserving "σ-measure" for a standard (σ-additive) measure. In any case, they prove the result for both finitely and countably additive set functions.

$$\begin{tikzcd}
 & \langle S, \Sigma, \rho\rangle \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\langle \bar{\mathbb{I}}, \mathcal{S}_V, \zeta\rangle \arrow[dr, "Id"'] & & \langle \bar{\mathbb{I}}, \mathcal{S}_V, \theta\rangle \arrow[dl, "Id"] \\
 & \langle \bar{\mathbb{I}}, \mathcal{S}, \tau\rangle &
\end{tikzcd}$$

Fig. 1. A semi-pullback.

every diagram consisting of objects $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{T}$ and arrows $f_i : \mathbf{S}_i \to \mathbf{T}$ ($i = 1, 2$; a *cospan*) there exist an object $\mathbf{S}$ and arrows $\pi_i : \mathbf{S} \to \mathbf{S}_i$ (a *span*) such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$. Recall also that a *stationary Markov process* [8] is an LMP with a single Markov kernel (i.e., the label set is a singleton). We have:

Theorem 12. *The category of stationary Markov processes and zig-zag morphisms does not have semi-pullbacks.*

Proof. Let m be Lebesgue measure on the closed interval $\bar{\mathbb{I}} := [0, 1]$, let $\mathcal{S} := \mathbf{B}(\bar{\mathbb{I}})$, and let V be a subset of $\bar{\mathbb{I}}$ that is not Lebesgue measurable. Take two extensions m_0 and m_1 of m to $\mathcal{S}_V$ such that $m_0(V) \neq m_1(V)$ as in Corollary 11. Let χ_M be the indicator function of the set $M \subset \bar{\mathbb{I}}$. This function is $\mathcal{S}_V$-measurable if and only if $M \in \mathcal{S}_V$. Now define

$$\zeta(r, A) := \chi_{(0,1]}(r) \cdot \delta_0(A) + \chi_{\{0\}}(r) \cdot m_0(A),$$

$$\theta(r, A) := \chi_{(0,1]}(r) \cdot \delta_0(A) + \chi_{\{0\}}(r) \cdot m_1(A),$$

$$\tau(r, B) := \chi_{(0,1]}(r) \cdot \delta_0(B) + \chi_{\{0\}}(r) \cdot m(B),$$

for every $0 \leqslant r \leqslant 1$, $A \in \mathcal{S}_V$ and $B \in \mathcal{S}$. We will prove that ζ is a Markov kernel over $\langle \bar{\mathbb{I}}, \mathcal{S}_V\rangle$; the proofs for θ and that τ is a Markov kernel over $\langle \bar{\mathbb{I}}, \mathcal{S}\rangle$ are exactly analogous. To accomplish this, we have to check that $\zeta(r, \cdot)$ is a (sub)probability measure for each $r \in \bar{\mathbb{I}}$, and $\zeta(\cdot, A)$ is measurable for each $A \in \mathcal{S}_V$.

The first part is immediate since $\zeta(r, \cdot)$ is a convex linear combination of two probability measures on $\langle \bar{\mathbb{I}}, \mathcal{S}_V\rangle$, namely Dirac's δ_0 concentrated at 0 and m_0. For the second part, just observe that $\zeta(\cdot, A)$ is well defined for $A \in \mathcal{S}_V$ and it is a linear combination of $\mathcal{S}_V$-measurable real functions, hence $\mathcal{S}_V$-measurable.

Now let $\mathbf{S}_1 := \langle \bar{\mathbb{I}}, \mathcal{S}_V, \zeta\rangle$, $\mathbf{S}_2 := \langle \bar{\mathbb{I}}, \mathcal{S}_V, \theta\rangle$ and $\mathbf{T} := \langle \bar{\mathbb{I}}, \mathcal{S}, \tau\rangle$. The identity maps $Id : \mathbf{S}_i \to \mathbf{T}$ are obviously zig-zag, since m_0 and m_1 agree with m over $\mathcal{S}$. We will see that there are no $\mathbf{S} := \langle S, \Sigma, \rho\rangle$ and zig-zag morphisms f and g that make the diagram in Fig. 1 commutative.

If such f and g exist, they must be equal as functions from S to $\bar{\mathbb{I}}$ because of the commutativity of the diagram. Now let $u \in S$ such that $f(u) = 0$ (recall that a zig-zag is surjective). Hence $g(u) = 0$. By the definition of zig-zag, we have

$$\rho\big(u, f^{-1}(V)\big) = \zeta(0, V) = m_0(V) \neq m_1(V) = \theta(0, V) = \rho\big(u, g^{-1}(V)\big).$$

From this we reach a contradiction, since we have $f^{-1}(V) = g^{-1}(V)$. $\square$

Given the relation between semi-pullbacks and regular conditional probabilities (cf. [8]), this failure of existence of semi-pullbacks can be traced to the fact that if $m_i(V) = 0$ and $m_e(V) = 1$, then there is no regular conditional probability for $\frac{1}{2}(\overline{m} + \underline{m})$ on $\mathbf{B}(\bar{\mathbb{I}})_V$ given $\mathbf{B}(\bar{\mathbb{I}})$ (see [1, p. 81] and [7, p. 624]).

From Theorem 12 we infer that the method of proof (i.e., the construction of a semi-pullback) used in [4] to show the logical characterization of bisimulation cannot be applied in non-analytic spaces. It could be argued that the existence of semi-pullbacks is not equivalent to the transitivity of bisimulation defined as a span of zig-zags. In spite of this, in the next section we will see that this sort of extension of measures ensures that the transitivity of bisimilarity cannot be proved in general.

4. The counterexample

Following the same line of thought of the proof of Theorem 12, let m be Lebesgue measure on $\mathbb{I}$, let $\mathcal{S} := \mathbf{B}(\mathbb{I})$, and let V be a subset of $\mathbb{I}$ that is not Lebesgue measurable. Take two extensions m_0 and m_1 of m to $\mathcal{S}_V$ such that $m_0(V) \neq m_1(V)$. Let $s, t, x \notin \mathbb{I}$ be mutually distinct; we may view m_0 and m_1 as measures defined on the sum $\mathbb{I} \oplus \{s, t, x\}$, supported on $\mathbb{I}$. Recall that $\mathcal{S}$ is generated by the countable family $\mathcal{B} := \{B_a\colon\ a \in \omega\}$.

Let $L_3 := \omega \cup \{\infty\}$. Now define an LMP $\mathbf{S_3} = \langle S_3, \mathcal{S}_3, \{\tau_a\colon\ a \in L_3\}\rangle$ such that

$$\langle S_3, \mathcal{S}_3\rangle := \big\langle \mathbb{I} \oplus \{s,t,x\}, \mathcal{S}_V \oplus \mathsf{Pow}\big(\{s,t,x\}\big)\big\rangle,$$
$$\tau_a(r, A) := \chi_{B_a}(r) \cdot \delta_x(A),$$
$$\tau_\infty(r, A) := \chi_{\{s\}}(r) \cdot m_0(A) + \chi_{\{t\}}(r) \cdot m_1(A)$$

when $a \in \omega$ and $A \in \mathcal{S}_3$.

Lemma 13. $\mathbf{S_3}$ *is an LMP.*

Proof. We have to check that for all $l \in L_3$, $\tau_l(r, \cdot)$ is a subprobability measure for each $r \in S_3$, and $\tau_l(\cdot, A)$ is measurable for each $A \in \mathcal{S}_3$.

The first part follows from the fact that for all r, $0 \leqslant \chi_{B_a}(r) \leqslant 1$ and $0 \leqslant \chi_{\{s\}}(r) + \chi_{\{t\}}(r) \leqslant 1$.

For the second part, we infer measurability by the same reasoning in the proof of Theorem 12: $\tau_l(\cdot, A)$ is always a linear combination of measurable functions. □

Lemma 14. *s and t are event-bisimilar.*

Proof. We will check that s and t will not be separated by a certain stable σ-algebra $\mathcal{U}$. Hence they cannot be separated by the *smallest* such σ-algebra, which is (as a relation) the *greatest* event bisimulation.

Let $\mathcal{U} := \sigma(\mathcal{B} \cup \{\{s,t\}, \{x\}\})$. As it easily seen from the proof of Lemma 13, $\tau_a(\cdot, A)$ is $\mathcal{U}$-measurable for all $a \in \omega$ and $A \in \mathcal{S}_3$ (a fortiori, for $A \in \mathcal{U}$). Since m_0 and m_1 are equal on $\sigma(\mathbf{B}(\mathbb{I}) \cup \mathsf{Pow}(\{s,t,x\}))$, for every $A \in \mathcal{U}$, $\tau_\infty(s, A) = \tau_\infty(t, A)$, and hence for any $B \subseteq [0,1]$, s belongs to $\tau_\infty(\cdot, A)^{-1}(B)$ if and only if t does. □

Theorem 15. *Event and state bisimilarity differ in* $\mathbf{S_3}$.

Proof. To prove them different it is enough to show that s and t are not state-bisimilar (and hence event bisimilarity is not included in state bisimilarity). The strategy is simple: we show that state bisimilarity on $\mathbf{S_3}$ is the identity relation, and hence cannot contain the pair (s,t).

It is easy to show that the singleton formed with the (only) null state x must be a $\sim$-class. For any other $r \in S_3$, there exists $l \in L_3$ such that $\tau_l(r, S_3) = 1$ but $\tau_l(x, S_3) = 0$ (and S_3 is obviously $\sim$-closed). Now take $y \neq z$ in $\mathbb{I}$; we will show that y and z cannot be related by $\sim$. Since $\mathcal{B}$ generates $\mathbf{B}(\mathbb{I})$, there exists $a \in \omega \subset L_3$ such that B_a separates y from z. Without loss of generality, assume $\{y,z\} \cap B_a = \{y\}$. Then $\tau_a(y, \{x\}) = 1$ but $\tau_a(z, \{x\}) = 0$. We conclude that $\sim$ restricted to $\mathbb{I} \cup \{x\}$ is the identity and (in particular) $V \subset \mathbb{I}$ is $\sim$-closed.

It remains to observe that $\tau_\infty(s, V) \neq \tau_\infty(t, V)$, and hence s and t are not state-bisimilar. □

Corollary 16. *The logic $\mathcal{L}$ does not characterize state bisimulation for LMP having a non-analytic base space. Moreover, the logical characterization of state bisimulation fails for the class of LMP having separable metrizable base spaces.*

Proof. The first assertion is immediate from Theorem 15. Since $\mathcal{S}_3$ is countably generated and separates points, the second assertion follows from Theorem 15 and Proposition 2. □

It is known that in a general coalgebraic setting state bisimilarity (defined as the existence of a span of zig-zags) is transitive, provided the functor preserves weak pullbacks [19]. By dropping alternatively states s and t in $\mathbf{S_3}$ one may show that this is not the case for LMP over general measurable spaces.

Corollary 17. *The relation of bisimilarity (as given by a span of zig-zags) is not transitive for general measurable spaces.*

Sketch of proof. Let $\mathbf{S_3} \setminus \{\mathbf{s}\} = \langle S_3 \setminus \{s\}, \mathcal{S}_3|(S_3 \setminus \{s\}), \{\tau_a\colon\ a \in L_3\}\rangle$ be the result of "deleting" the state s from $\mathbf{S_3}$, let $\mathbf{S_3} \setminus \{\mathbf{t}\} = \langle S_3 \setminus \{t\}, \mathcal{S}_3|(S_3 \setminus \{t\}), \{\tau_a\colon\ a \in L_3\}\rangle$, and $\mathbf{T} = \langle S_3 \setminus \{s\}, \mathcal{S} \oplus \mathsf{Pow}(\{t,x\}), \{\bar{\tau}_a\colon\ a \in L_3\}\rangle$, where $\bar{\tau}_a$ and τ_a coincide for $a \in \omega$ and for $A \in \mathcal{S} \oplus \mathsf{Pow}(\{t,x\})$,

$$\bar{\tau}_\infty(t, A) = m(A), \qquad \bar{\tau}_\infty(r, A) = 0 \quad \text{for } r \neq t$$

(note that in $\mathbf{T}$ we are restricting ourselves to measurable subsets of the form $B \oplus X$, where $B \in \mathcal{S} = \mathbf{B}(\mathbb{I})$ and $X \subseteq \{t,x\}$). The identity map Id of $S_3 \setminus \{s\}$ and the map $F : S_3 \setminus \{t\} \to S_3 \setminus \{s\}$ which sends s to t and such that $F|(\mathbb{I} \oplus \{x\})$ is the identity, are zig-zag morphisms $Id : \mathbf{S_3} \setminus \{\mathbf{s}\} \to \mathbf{T}$ and $F : \mathbf{S_3} \setminus \{\mathbf{t}\} \to \mathbf{T}$, respectively. Hence both $\mathbf{S_3} \setminus \{\mathbf{t}\}$ and $\mathbf{S_3} \setminus \{\mathbf{s}\}$ are state-bisimilar to $\mathbf{T}$, but they are not state-bisimilar to each other. □

We can recast this last corollary in our relational framework for bisimulation and show a serious categorical drawback of the concept of state bisimilarity: it is not reflected by direct sum.

Example 1. Consider the LMP $\mathbf{S_3} \setminus \{\mathbf{s}\}$ and $\mathbf{S_3} \setminus \{\mathbf{t}\}$ from the proof of Corollary 17 and let $\mathbf{T}' = \langle (S_3 \setminus \{s,t\}) \cup \{t'\}, \mathcal{S} \oplus \mathsf{Pow}(\{t',x\}), \{\bar{\tau}_a\colon\ a \in L_3\}\rangle$ be the result of renaming t to t' in $\mathbf{T}$. Then s and t are state-bisimilar in the sum $\mathbf{U} := \mathbf{S_3} \setminus \{\mathbf{t}\} \oplus \mathbf{S_3} \setminus \{\mathbf{s}\} \oplus \mathbf{T}'$ but in $\mathbf{S_3} \setminus \{\mathbf{t}\} \oplus \mathbf{S_3} \setminus \{\mathbf{s}\}$ they are not.

Indeed, it is immediate that s, t are not state-bisimilar in $\mathbf{S_3} \setminus \{\mathbf{t}\} \oplus \mathbf{S_3} \setminus \{\mathbf{s}\}$ by using the argument of Theorem 15. But in the sum $\mathbf{S_3} \setminus \{\mathbf{t}\} \oplus \mathbf{S_3} \setminus \{\mathbf{s}\} \oplus \mathbf{T}'$ they are. Take the equivalence relation R whose classes are $\{s,t,t'\}$ and all other triples having corresponding elements in each of $\mathbf{S_3} \setminus \{\mathbf{t}\}$, $\mathbf{S_3} \setminus \{\mathbf{s}\}$, and $\mathbf{T}'$. Then R is a state bisimulation. For this, note that if $\langle U, \mathcal{U}\rangle$ is the base space of $\mathbf{U}$, then every $\mathcal{U}$-measurable R-closed subset of U must be of the form $(B \oplus B \oplus B) \cup F$, where $B \subseteq \mathbf{B}(\mathbb{I})$ and F a finite set (in particular, V cannot be the $(\mathbf{S_3} \setminus \{\mathbf{t}\})$-part of a set in $\mathcal{U}(R)$). For these sets the transition functions behave identically.

Hence, state bisimilarity in LMP over general measurable spaces has an undesirable non-local character. One possible conclusion of this would be to abandon state bisimilarity and to use the event-based version, which is the main point of [2]. But one must not overlook that the artifact of using a non-Lebesgue measurable set is rather tricky: the Banach–Tarski Paradox, stating that a ball of radius 1 can be decomposed in finitely many pieces that can be reassembled to form two balls of radius 1, relies on the same device. We therefore should ask under what circumstances we may encounter a non-Lebesgue measurable set. We discuss this in the next section.

5. Further analysis of the construction

We know by the work of Desharnais et al. [4] that in the class of LMP over analytic spaces, the logic $\mathcal{L}$ indeed characterizes state bisimulation and hence (obviously[5]) our construction must give a non-analytic base space. It is then natural to ask if by imposing some regularity assumptions on the base space we can be certain to avoid the pathological examples of the previous sections.

Since our counterexamples need a non-Lebesgue measurable subset to start with, the first question is how complex should be the base measurable space $\langle S, \mathcal{S}\rangle$ as to allow non-μ-measurable subsets among the sets in $\mathcal{S}$. The measure of "complexity" we are taking into account is the place S occupies in the *projective hierarchy* of Descriptive Set Theory [14,11]. The first level of this hierarchy is inhabited by analytic sets and their complements (*coanalytic* or $\boldsymbol{\Pi}^1_1$ sets). We will only be interested in the first two levels, so we give the formal definition of the class of sets in level two and state some of their properties.

Let X be a Polish space. A subset of X is in $\boldsymbol{\Sigma}^1_2(X)$ if it is expressible as a projection of the coanalytic set:

$$\boldsymbol{\Sigma}^1_2(X) = \left\{\mathsf{proj}_X(C)\colon\ C \text{ coanalytic in } X \times Y,\ Y \text{ Polish}\right\}. \tag{2}$$

A set is in $\boldsymbol{\Pi}^1_2(X)$ if its complement is $\boldsymbol{\Sigma}^1_2(X)$; finally define $\boldsymbol{\Delta}^1_2(X) := \boldsymbol{\Sigma}^1_2(X) \cap \boldsymbol{\Pi}^1_2(X)$. We say that a measurable space $\langle S, \mathcal{S}\rangle$ is $\boldsymbol{\Sigma}^1_2$ (resp., $\boldsymbol{\Pi}^1_2$, $\boldsymbol{\Delta}^1_2$) if there exist a Polish space X and $Y \in \boldsymbol{\Sigma}^1_2(X)$ (resp., $\boldsymbol{\Pi}^1_2(X)$, $\boldsymbol{\Delta}^1_2(X)$) such that $\langle S, \mathcal{S}\rangle \cong \langle Y, \mathbf{B}(X)|Y\rangle$. All these classes of sets are closed under countable unions and intersections and stable under restriction (if $X \subseteq Y$ are both Polish and Γ is $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Pi}^1_2$, or $\boldsymbol{\Delta}^1_2$, then $\Gamma(Y)|X \subseteq \Gamma(X)$). Moreover, since the class of Polish spaces (and their Borel spaces) are closed under sum, this property is inherited by $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Pi}^1_2$ and $\boldsymbol{\Delta}^1_2$ measurable spaces.

Since every Polish space is the continuous image of the Baire space $\mathcal{N}$, it can be proved that in Eq. (2) we can replace Y by $\mathcal{N}$. Given this preponderant role of the space of functions from $\mathbb{N}$ to $\mathbb{N}$, recursion theory has an impact in the development of descriptive set theory by the introduction of the *lightface* hierarchy Σ^1_n, Π^1_n and Δ^1_n; here the notion of closed set is replaced by an effective one. We repeat the definitions in [11, 25.1].

Definition 18.

1. A set $A \subseteq \mathcal{N}$ is Σ^1_1 if there exists a recursive set $R \subseteq \bigcup_{n=0}^{\infty}(\mathbb{N}^n \times \mathbb{N}^n)$ such that for all $x = (x_0, x_1, \ldots) \in \mathcal{N}$,

$$x \in A \quad \Leftrightarrow \quad \exists y \in \mathcal{N},\ \forall n \in \mathbb{N}\colon\quad R(x|n, y|n),$$

   where $x|n := (x_0, \ldots, x_{n-1})$.
2. Let $a \in \mathcal{N}$. A set $A \subseteq \mathcal{N}$ is $\Sigma^1_1(a)$ (Σ^1_1 in a) if there exists a set R recursive in a such that for all $x \in \mathcal{N}$,

$$x \in A \quad \Leftrightarrow \quad \exists y \in \mathcal{N},\ \forall n \in \mathbb{N}\colon\quad R(x|n, y|n, a|n).$$

3. $A \subseteq \mathcal{N}$ is Π^1_n (in a) if A^c is Σ^1_n (in a).
4. $A \subseteq \mathcal{N}$ is Σ^1_{n+1} (in a) if it is the projection of a Π^1_n (in a) subset of $\mathcal{N} \times \mathcal{N}$.
5. $A \subseteq \mathcal{N}$ is Δ^1_n (in a) if A is both Σ^1_n (in a) and Π^1_n (in a).

[5] Recall Lusin proved [14, Theorem 21.10] that every analytic subset of $\mathbb{R}$ is Lebesgue measurable.

We have

$$\boldsymbol{\Sigma}^1_n(\mathcal{N}) = \bigcup_{a\in\mathcal{N}} \Sigma^1_n(a).$$

These notions can be extended to subsets of $\mathbb{R}^n$ and in particular we obtain $\Sigma^1_2(\mathbb{R}^n) \subseteq \boldsymbol{\Sigma}^1_2(\mathbb{R}^n)$ (as well as $\Pi^1_2 \subseteq \boldsymbol{\Pi}^1_2$ and $\Delta^1_2 \subseteq \boldsymbol{\Delta}^1_2$).

The reason for stopping at level 2 of the projective hierarchy is that a classical result by Gödel shows it is consistent with current foundations of mathematics (as given by Zermelo–Fraenkel set theory with Choice, ZFC) that we may find a $\boldsymbol{\Delta}^1_2$ measurable space with non-Lebesgue measurable sets in its σ-algebra. Actually, it is consistent with ZFC that there exists a $\Delta^1_2(\mathbb{R}^2)$ set that is not Lebesgue measurable.

More precisely, Gödel's axiom constructibility $V = L$ (which is relative consistent with ZFC) implies by Theorem 25.26 in Jech [11] and subsequent Corollary 25.28 that there exists a Δ^1_2 relation on $\mathbb{R}$ (i.e. a set W in $\Delta^1_2(\mathbb{R}^2)$) such that $\langle \mathbb{R}, W\rangle$ is a well-order isomorphic to $\langle \omega_1, <\rangle$, where ω_1 is the first uncountable ordinal. And it is known that such a relation W cannot be Lebesgue measurable as a subset of $\mathbb{R}^2$. From this set we will be able to reconstruct our counterexample.

Firstly, we manufacture a subset of $\mathbb{I} \times \mathbb{I}$ that is not Lebesgue measurable.

Lemma 19. *It is consistent with ZFC that there exists a* (*Lebesgue*) *nonmeasurable subset W' in $\boldsymbol{\Delta}^1_2(\mathbb{I} \times \mathbb{I})$.*

Proof. By the preceding discussion, it is consistent to assume $W \in \boldsymbol{\Delta}^1_2(\mathbb{R}^2)$ and $\langle \mathbb{R}, W\rangle \cong \langle \omega_1, <\rangle$. As a consequence, $\mathbb{R}$ and $\mathbb{I}$ are both equinumerous with ω_1. Define W' to be the restriction of W to $\mathbb{I}$, i.e., $W' = W \cap (\mathbb{I} \times \mathbb{I})$; we have $W' \in \boldsymbol{\Delta}^1_2(\mathbb{I}^2)$. Again, we obtain a well-order $\langle \mathbb{I}, W'\rangle$ of type ω_1. Finally, a standard argument shows that such a W' cannot be a Lebesgue measurable subset of $\mathbb{I}^2$ (see, for example [13, Section 17.1]). □

By using this set W' the construction of our counterexample can be carried out with inessential changes.

Theorem 20. *The logical characterization of state bisimulation cannot be proved* (*on ZFC basis*) *for the class of LMP with $\boldsymbol{\Delta}^1_2$ base spaces.*

Proof. In the construction of Section 4, we may replace $\mathbb{I}$ by $\mathbb{I}^2$ with no trouble since $\mathbf{B}(\mathbb{I}^2)$ is likewise countably generated (e.g. take $\mathcal{B}$ to be the family of open squares with rational vertices). Hence we now have $S := \mathbb{I}^2$ and $\mathcal{S} := \mathbf{B}(\mathbb{I}^2)$.

Observe that $\mathcal{S} \subset \boldsymbol{\Delta}^1_2(\mathbb{I}^2)$ and since $\boldsymbol{\Delta}^1_2(\mathbb{I}^2)$ is closed under intersection and complementation, the sets $B \cap W'$ and $B \cap W'^{\mathsf{c}}$ belong to $\boldsymbol{\Delta}^1_2(\mathbb{I}^2)$ for $B \in \mathcal{S}$. By Eq. (1),

$$\left\langle \mathbb{I}^2, \mathcal{S}_{W'}\right\rangle \cong \left\langle W', \mathcal{S}|W'\right\rangle \oplus \left\langle W'^{\mathsf{c}}, \mathcal{S}|W'^{\mathsf{c}}\right\rangle$$

is the sum of two $\boldsymbol{\Delta}^1_2$ spaces, hence a $\boldsymbol{\Delta}^1_2$ space it is.

The construction of $\langle S_3, \mathcal{S}_3\rangle$ will then result in a $\boldsymbol{\Delta}^1_2$ space since it is the sum of $\langle \mathbb{I}^2, \mathcal{S}_{W'}\rangle$ and a discrete space. □

Theorem 20 places a limit on what can be proved in ZFC alone. Since the Axiom of Constructibility cannot be proved in ZFC, we only know that it is *consistent* that an LMP such as $\mathbf{S_3}$ can be constructed over a $\boldsymbol{\Delta}^1_2$ space.

6. Conclusions & an open problem

We constructed an LMP over a non-analytic measurable space in which state bisimilarity and event bisimilarity differ from each other. Since the latter is completely characterized by the modal logic $\mathcal{L}$, we have an LMP such that state bisimulation is not characterized by $\mathcal{L}$. Among the consequences of this construction we recall the non-locality of state bisimulation: state-bisimilarity is not reflected by direct sum.

We also showed that it is consistent relative to Zermelo–Fraenkel set theory with Choice (ZFC) that the logical characterization of bisimulation cannot be proved for the class of LMP having $\boldsymbol{\Delta}^1_2$ base spaces. This was accomplished by means of a classical result of Gödel that shows the consistency of the existence of a $\boldsymbol{\Delta}^1_2(\mathbb{R}^2)$ set that is not Lebesgue measurable. $\boldsymbol{\Delta}^1_2$ sets lie in the second level of the projective hierarchy (the first one being occupied by analytic and coanalytic subsets of Polish spaces) and for uncountable spaces, this hierarchy has ω levels properly. So this cuts out the possibility of proving the logical characterization of state bisimulation for "almost every" projective space in the ZFC framework.

We face two possibilities: to abandon state bisimulation completely in favour of the event based one; or to consider extending our mathematical foundations. In the second scenario, one may investigate the consequences of the *axioms of determinacy*. These provide a smooth theory for the structure of analytic and projective subsets of Polish (resp., standard Borel) spaces and they are gaining wide acceptance. In particular, the axiom of *Analytic Determinacy* (*AD*) (see [14] for details) implies that every set in $\boldsymbol{\Sigma}^1_2(\mathbb{R}^n)$ or $\boldsymbol{\Pi}^1_2(\mathbb{R}^n)$ is Lebesgue measurable and enjoys various other regularity properties.[6]

[6] From AD we also obtain regularity properties for coanalytic sets that are not available under ZFC: for instance, the *perfect set property* [14, 32.2].

A successful application of AD in this situation will probably depend on presenting the problem of the logical characterization (or those problems that imply it, like the existence of semi-pullbacks or regular conditional distributions) as an infinite game over an analytic space.

A more fundamental question is whether we actually need a non-Lebesgue measurable set to furnish such a counterexample, but we do not have an answer yet. Coanalytic sets are Lebesgue measurable, and hence the immediate problem is to decide whether the results in [4,2] can be extended to the class of LMP with coanalytic base spaces.

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