Ain Shams Engineering Journal (2013) 4, 507–513

Ain Shams University

Ain Shams Engineering Journal

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Frenet frames and invariants of timelike ruled surfaces

Mehmet Önder [a,*], H. Hüseyin Uğurlu [b]

[a] *Celal Bayar University, Faculty of Science and Arts, Department of Mathematics, Muradiye Manisa, Turkey*
[b] *Gazi University, Gazi Faculty of Education, Department of Secondary Education Science and Mathematics Teaching, Mathematics Teaching Program, Ankara, Turkey*

Received 14 August 2012; revised 17 September 2012; accepted 12 October 2012
Available online 15 November 2012

KEYWORDS

Minkowski 3-space;
Timelike ruled surface;
Curvatures

Abstract In this study we give the Frenet frames and Frenet invariants of timelike ruled surfaces. We show that a timelike ruled surface and its directing cone have the same base of Frenet frame. We define instantaneous rotation vectors of the Frenet frames of timelike ruled surfaces. Also we prove the Chasles Theorem for timelike ruled surfaces.

1. Introduction

Ruled surface is a special surface generated by a continuously moving of a straight line. Since ruled surfaces have the most important positions and applications in the study of design problems in spatial mechanisms and physics, kinematics and computer aided design (CAD), these surfaces are one of the most important topics of differential geometry. Because of this position of ruled surfaces, geometers have studied on these surfaces in Euclidean space and they have investigated many properties of the ruled surfaces [1–4]. Furthermore, the differential geometry of the ruled surfaces in the Minkowski space has been studied by several authors [5–10].

The Frenet frames and invariants of skew ruled surface and of its directing cone have been given by Karger and Novak in the Euclidean 3-space [2]. These frames and invariants have an important role in kinematics and mechanics. Especially, the kinematic differential geometry of a rigid body is based on the Frenet frames and Frenet invariants of ruled surfaces. By paying attention to this fact, Wang, Liu and Xiao have given some instantaneous properties of a point trajectory and of a line trajectory in spatial kinematics and Euler-Savary analogue equations of a point trajectory and of a line trajectory [11].

Moreover, the Minkowski space is more interesting than the Euclidean space. In this space, curves and surfaces have different casual Lorentzian characters such as timelike, spacelike or null (lightlike). Then, ruled surfaces in the Minkowski space can be classify according to the Lorentzian character of their ruling and surface normal. The classification of ruled surfaces in Minkowski 3-space has been given by Kim and Yoon [9]. They have given all the types of ruled surfaces in Minkowski 3-space. Furthermore, Küçük has obtained some results on developable timelike ruled surfaces in the same space [10].

In this study, we give the Frenet frames, invariants and instantaneous rotation vectors of the Frenet frames of timelike ruled surfaces in the Minkowski 3-space. We show that a

* Corresponding author. Tel.: +90 02362412151.
E-mail addresses: mehmet.onder@cbu.edu.tr (M. Önder), hugurlu@gazi.edu.tr (H.H. Uğurlu).

Peer review under responsibility of Ain Shams University.

Production and hosting by Elsevier

2090-4479 © 2012 Ain Shams University. Production and hosting by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.asej.2012.10.003

timelike ruled surface and its directing cone have the same base. Also we give and prove the Chasles Theorem for timelike ruled surfaces.

2. Preliminaries

The Minkowski 3-space IR_1^3 is the real vector space IR^3 provided with standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of IR_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in IR_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike or null (lightlike), respectively [12]. We say that a timelike vector is future pointing or past pointing if the first compound of vector is positive or negative, respectively. The norm of the vector $\vec{v} = (v_1, v_2, v_3)$ is given by

$$\|\vec{v}\| = \sqrt{|\langle \vec{v}, \vec{v} \rangle|}.$$

For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in IR_1^3, Lorentzian vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2).$$

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in IR_1^3 are given by

$$S_1^2 = \left\{ \vec{x} = (x_1, x_2, x_3) \in E_1^3 : \langle \vec{x}, \vec{x} \rangle = r^2 \right\}$$

and

$$H_0^2 = \left\{ \vec{x} = (x_1, x_2, x_3) \in E_1^3 : \langle \vec{x}, \vec{x} \rangle = -r^2 \right\}$$

respectively [13].

Definition 2.1 [17].

(*i*) *Hyperbolic angle:* Let $\vec{x}$ and $\vec{y}$ be future pointing (or past pointing) timelike vectors in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $\langle \vec{x}, \vec{y} \rangle = -\|\vec{x}\|\|\vec{y}\| \cosh\theta$. This number is called the *hyperbolic angle* between the vectors $\vec{x}$ and $\vec{y}$ [14].

(*ii*) *Central angle:* Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cosh\theta$. This number is called the *central angle* between the vectors $\vec{x}$ and $\vec{y}$.

(*iii*) *Spacelike angle:* Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cos\theta$. This number is called the *spacelike angle* between the vectors $\vec{x}$ and $\vec{y}$.

(*iv*) *Lorentzian timelike angle:* Let $\vec{x}$ be a spacelike vector and $\vec{y}$ be a timelike vector in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \sinh\theta$. This number is called the *Lorentzian timelike angle* between the vectors $\vec{x}$ and $\vec{y}$.

Definition 2.2 [15]. A surface in the Minkowski 3-space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on spacelike (timelike) surface is a timelike (spacelike) vector.

3. Timelike ruled surfaces in Minkowski 3-space

Let I be an open interval in the real line IR. Let $\vec{k} = \vec{k}(u)$ be a curve in IR_1^3 defined on I and $\vec{q} = \vec{q}(u)$ be a unit direction vector of an oriented line in IR_1^3. Then we have following parametrization for a timelike ruled surface N:

$$\vec{r}(u, v) = \vec{k}(u) + v\vec{q}(u). \tag{1}$$

The parametric u-curve of this surface is a straight line of the surface which is called ruling. For $v = 0$, the parametric v-curve of this surface is $\vec{k} = \vec{k}(u)$ which is called base curve or generating curve of the surface. In particular, if the direction of $\vec{q}$ is constant, then the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

The distribution parameter (or drall) of the timelike ruled surface in (1) is given as

$$d = \frac{|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}|}{\langle \dot{\vec{q}}, \dot{\vec{q}} \rangle}, \tag{2}$$

where $\dot{\vec{k}} = \frac{d\vec{k}}{du}$, $\dot{\vec{q}} = \frac{d\vec{q}}{du}$. If $|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}| = 0$, then normal vectors are collinear at all points of same ruling and at non-singular points of the surface N, the tangent planes are identical. We then say that tangent plane contacts the surface along a ruling. Such a ruling is called a torsal ruling. If $|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}| \neq 0$, then the tangent planes of the surface M are distinct at all points of same ruling which is called non-torsal [16].

Definition 3.1. A timelike ruled surface whose all rulings are torsal is called a developable timelike ruled surface. The remaining timelike ruled surfaces are called skew timelike ruled surfaces.

Theorem 3.1 (*[5,10]*). *A timelike ruled surface is developable if and only if the distribution parameter of the surface is equal to zero, i.e., d = 0.*

For the unit normal vector $\vec{m}$ of a timelike ruled surface we have

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{(\dot{\vec{k}} + v\dot{\vec{q}}) \times \vec{q}}{\sqrt{\left| \left\langle \dot{\vec{k}}, \vec{q} \right\rangle^2 - \langle \vec{q}, \vec{q} \rangle \left\langle \dot{\vec{k}} + v\dot{\vec{q}}, \dot{\vec{k}} + v\dot{\vec{q}} \right\rangle \right|}}. \tag{3}$$

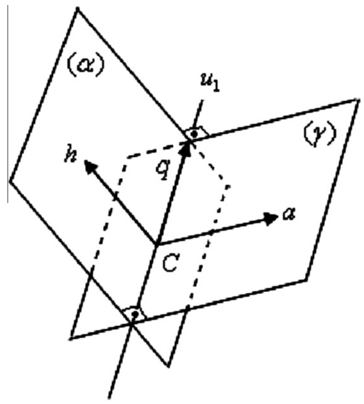

Figure 1 Asymptotic plane and central plane

From (3) at the points of a non-torsal ruling $u = u_1$ we have

$$\vec{a} = \lim_{v\to\infty} \vec{m}(u_1, v) = \frac{\dot{\vec{q}} \times \vec{q}}{\|\dot{\vec{q}}\|}. \tag{4}$$

The plane of skew timelike ruled surface N which passes through its ruling u_1 and is perpendicular to the vector $\vec{a}$ is called *asymptotic plane* α. The tangent plane γ passing through the ruling u_1 which is perpendicular to the asymptotic plane α is called *central plane*. The point at which the unit normal $\vec{m}$ is perpendicular to $\vec{a}$ is called the striction point (or central point) C on the ruling u_1 (Fig. 1). The set of central points of all rulings is called *striction curve* of the surface. The straight lines which pass through point C and are perpendicular to the planes α and γ are called *central tangent* and *central normal*, respectively.

Using the perpendicularity of the vectors $\vec{q}, \dot{\vec{q}}$ and relation (4), representation of unit vector $\vec{h}$ of the central normal is given by

$$\vec{h} = \frac{\dot{\vec{q}}}{\|\dot{\vec{q}}\|}. \tag{5}$$

Substituting the parameter v of central point C into equation (3) we get $\vec{h} \times \vec{m} = 0$ and thus

$$\dot{\vec{q}} \times \left[\left(\dot{\vec{k}} + v\dot{\vec{q}}\right) \times \vec{q}\right] = \left\langle \dot{\vec{q}}, \dot{\vec{k}} \right\rangle \left(\dot{\vec{k}} + v\dot{\vec{q}}\right) + v\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle \vec{q} = 0. \tag{6}$$

From (6) we obtain

$$v = -\frac{\langle \dot{\vec{q}}, \dot{\vec{k}} \rangle}{\langle \dot{\vec{q}}, \dot{\vec{q}} \rangle}. \tag{7}$$

Thus, the parametrization of the striction curve $\vec{c} = \vec{c}(u)$ on a timelike ruled surface is given by

$$\vec{c}(u) = \vec{k}(u) + v\vec{q}(u) = \vec{k} - \frac{\langle \dot{\vec{q}}, \dot{\vec{k}} \rangle}{\langle \dot{\vec{q}}, \dot{\vec{q}} \rangle}\vec{q}. \tag{8}$$

So that, the base curve of the timelike ruled surface is its striction curve if and only if $\langle \dot{\vec{q}}, \dot{\vec{k}} \rangle = 0$.

The orthonormal system $\{C; \vec{q}, \vec{h}, \vec{a}\}$ is called Frenet frame of the ruled surface N. Here C is the central point of ruling of timelike ruled surface N and $\vec{q}, \vec{h}, \vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively.

Let now consider ruled surface N with non-null Frenet vectors and their non-null derivatives. According to the Lorentzian characters of ruling and central normal, we can give the following classifications of the timelike ruled surface N as follows:

(i) If the central normal vector $\vec{h}$ is spacelike and $\vec{q}$ is timelike, then the ruled surface N is said to be of type N_-.
(ii) If the central normal vector $\vec{h}$ and the ruling $\vec{q}$ are both spacelike, then the ruled surface N is said to be of type N_+ [9,16].

In these classifications we use subscript "+" and "−" to show the Lorentzian casual character of ruling. By using these classifications, parametrization of timelike ruled surface N can be given as follows:

$$\vec{r}(u, v) = \vec{k}(u) + v\vec{q}(u),$$

where $\langle \vec{q}, \vec{q} \rangle = \varepsilon(= \pm 1), \langle \vec{h}, \vec{h} \rangle = 1, \langle \vec{a}, \vec{a} \rangle = -\varepsilon$. Then, the tangent plane γ is a timelike plane and the asymptotic plane α is spacelike (resp. timelike) if the surface is of the type N_+ (resp. N_-), i.e. it has the same Lorentzian character with the vector $\vec{q}$.

Let us pay attention to geometrical interpretation of distribution parameter. Let the generating curve of a timelike ruled surface be its line of striction and let unnormed normal vector of the surface at the point $(u,0)$ which is striction point be $\vec{m}_0$. Then by (3) we have

$$\vec{m}_0 = \dot{\vec{k}} \times \vec{q}. \tag{9}$$

Since $\vec{h} \times \vec{m}_0 = 0$, we have

$$\dot{\vec{k}} \times \vec{q} = \beta\dot{\vec{q}}, \tag{10}$$

where $\vec{\beta} = \vec{\beta}(u)$ is a scalar function. This implies that

$$\langle \dot{\vec{k}} \times \vec{q}, \dot{\vec{q}} \rangle = \vec{\beta}\langle \dot{\vec{q}}, \dot{\vec{q}} \rangle. \tag{11}$$

Hence (2) yields $\beta = d$ and finally, $\dot{\vec{k}} \times \vec{q} = d\dot{\vec{q}}$. For $v \to \infty$ the normal vector is $\vec{m}_\infty = \dot{\vec{q}} \times \vec{q}$ and from (10) it is clear that $\vec{m}_0 \perp \vec{m}_\infty$. By (3), unnormed normal vector of timelike ruled surface is

$$\vec{m} = (\dot{\vec{k}} \times \vec{q}) + v(\dot{\vec{q}} \times \vec{q}) = \vec{m}_0 + v\vec{m}_\infty, \tag{12}$$

and depends along the ruling u on the parameter v only. Let now θ be the angle between $\vec{m}$ and $\vec{m}_0$. Then we have followings:

(i) If the timelike ruled surface N is of the type N_+, then

$$\begin{cases} \langle \vec{m}, \vec{m}_0 \rangle = \|\vec{m}\|\|\vec{m}_0\| \cosh\theta, \\ \langle \vec{m}, \vec{m}_\infty \rangle = \|\vec{m}\|\|\vec{m}_\infty\| \sinh\theta, \end{cases} \tag{13}$$

where θ is central angle. Then from (13) we get

$$\tanh\theta = -\frac{v}{d}. \tag{14}$$

(ii) If the timelike ruled surface N is of the type N_-, then we have

$$\begin{cases} \langle \vec{m}, \vec{m}_0 \rangle = \|\vec{m}\|\|\vec{m}_0\| \cos\theta, \\ \langle \vec{m}, \vec{m}_\infty \rangle = \|\vec{m}\|\|\vec{m}_\infty\| \sin\theta, \end{cases} \tag{15}$$

where θ is spacelike angle. Then from (15) we get

$$\tan\theta = \frac{v}{d}. \tag{16}$$

So that, we give the following theorem which is known as Chasles Theorem for timelike ruled surfaces.

Theorem 3.2. *Let the base curve of a timelike ruled surface be its striction curve. For the angle θ between tangent plane of timelike ruled surface at the point (u,v) of a non-torsal ruling u and central plane we have*

- **(i)** $\tanh\theta = -\frac{v}{d}$, if the timelike ruled surface N is of the type N_+,
- **(ii)** $\tan\theta = \frac{v}{d}$, if the timelike ruled surface N is of the type N_-.

Here d is the distribution parameter of ruling u and central point has the coordinates $(u,0)$.

4. Frenet equations and Frenet invariants of timelike ruled surfaces

Let $\{C;\vec{q},\vec{h},\vec{a}\}$ be the Frenet frame of timelike ruled surface N. For this frame we have

$$\vec{q}\times\vec{h} = \varepsilon\vec{a}, \vec{h}\times\vec{a} = -\varepsilon\vec{q}, \vec{a}\times\vec{q} = -\vec{h}. \tag{17}$$

The set of all bound vectors $\vec{q}(u)$ at the point O constitutes *directing timelike cone* of the timelike ruled surface N. If $\varepsilon = -1$ (resp. $\varepsilon = 1$), then end points of the vectors $\vec{q}(u)$ drive a spherical spacelike (resp. timelike) curve k_1 on hyperbolic unit sphere H_0^2 (resp. on Lorentzian unit sphere S_1^2), called the *hyperbolic (resp. Lorentzian) spherical image* of the ruled surface N, whose arc is denoted by s_1.

Let now define the Frenet frame of the directing timelike cone by the orthonormal frame $\{O;\vec{q},\vec{n},\vec{z}\}$ where

$$\vec{n} = \frac{d\vec{q}}{ds_1} = \vec{q}\,'. \tag{18}$$

Since we have

$$\vec{q}\,' = \frac{\dot{\vec{q}}}{\|\dot{\vec{q}}\|} = \vec{h}, \tag{19}$$

by the aid of Eq. (5), we see that tangent planes of directing cone are parallel to the asymptotic planes of timelike ruled surface. Finally, we have

$$\vec{z} = \vec{q}\times\vec{h} = \varepsilon\vec{a}. \tag{20}$$

From (18)–(20) we have the following theorem:

Theorem 4.1. *Frenet frame of the directing timelike cone has the same base of Frenet frame of the timelike ruled surface N.*

Let now compute the derivatives of the vectors $\vec{h}$ and $\vec{a}$ with respect to the arc s_1 of generating curve k_1. We have $\langle\vec{h},\vec{h}\rangle = 1$, thus $\langle\vec{h},\vec{h}'\rangle = 0$ where $\vec{h}' = \frac{d\vec{h}}{ds_1}$. Consequently we may write

$$\vec{h}' = b_1\vec{q} + b_2\vec{a}, \tag{21}$$

where b_1, b_2 are functions of s_1. From $\langle\vec{h},\vec{q}\rangle = 0$, it follows that

$$\langle\vec{h}',\vec{q}\rangle + \langle\vec{h},\vec{q}\,'\rangle = \varepsilon b_1 + 1 = 0, \tag{22}$$

and if we put $b_2 = \kappa$ we get

$$\vec{h}' = -\varepsilon\vec{q} + \kappa\vec{a}, \tag{23}$$

where κ is called the conical curvature of the directing cone. From $\langle\vec{h},\vec{a}\rangle = 0$ we have

$$\langle\vec{h}',\vec{a}\rangle + \langle\vec{h},\vec{a}\,'\rangle = -\varepsilon\kappa + \langle\vec{h},\vec{a}\,'\rangle = 0, \tag{24}$$

and thus we get

$$\langle\vec{h},\vec{a}\,'\rangle = \varepsilon\kappa. \tag{25}$$

Further, $\langle\vec{a},\vec{a}\rangle = -\varepsilon$ and $\langle\vec{a},\vec{q}\rangle = 0$ imply the relations $\langle\vec{a},\vec{a}\,'\rangle = 0$ and

$$\langle\vec{a}\,',\vec{q}\rangle + \langle\vec{a},\vec{q}\,'\rangle = \langle\vec{a}\,',\vec{q}\rangle = 0. \tag{26}$$

This implies that the vector $\vec{a}\,'$ is collinear with spacelike vector $\vec{h}$, i.e. $\vec{a}\,' = b_3\vec{h}$, here b_3 is a function of s_1. By the equality (25) we get

$$\langle\vec{h},\vec{a}\,'\rangle = b_3 = \varepsilon\kappa, \tag{27}$$

and thus

$$\vec{a}\,' = \varepsilon\kappa\vec{h}. \tag{28}$$

For the spherical curve k_3 with arc s_3 circumscribed by bound vector $\vec{a}$ at the point O we have

$$\frac{ds_3}{ds_1} = \|\vec{a}\,'\| = \kappa. \tag{29}$$

Thus, with (19), (23) and (28) we have the following theorem.

Theorem 4.2. *For the derivatives of the Frenet vectors of timelike ruled surface N and of its directing timelike cone with respect to the arc s_1, we have following Frenet formulae*

$$\begin{bmatrix} d\vec{q}/ds_1 \\ d\vec{h}/ds_1 \\ d\vec{a}/ds_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon & 0 & \kappa \\ 0 & \varepsilon\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}. \tag{30}$$

The Frenet formulae can be interpreted kinematically as follows: If a moving line $\vec{q}$ makes a motion along a curve in such a way that s_1 is the time parameter, then the moving frame $\{\vec{q},\vec{h},\vec{a}\}$ moves in accordance with (30). This motion contains, apart from an instantaneous translation, and instantaneous rotation with angular velocity vector given by the Darboux vector or instantaneous rotation vector $\vec{w}_1 = \varepsilon\kappa\vec{q} - \vec{a}$. Thus, for the derivatives in (30) we can write

$$\vec{q}\,' = \vec{w}_1\times\vec{q}, \quad \vec{h}' = \vec{w}_1\times\vec{h}, \quad \vec{a}\,' = \vec{w}_1\times\vec{a}.$$

Let now s be the arc of striction curve of timelike ruled surface N. Furthermore, we call the first curvature $\frac{ds_1}{ds} = \kappa_1$ and the second curvature $\frac{ds_3}{ds} = \kappa_2$ of timelike ruled surface N or rather of its directing cone then we have

$$\kappa_2 = \kappa\kappa_1. \tag{31}$$

Timelike ruled surfaces for which $\kappa_1\kappa_2 \neq 0$ and $\kappa = (\kappa_2/\kappa_1) =$ *constant* have a timelike cone of revolution as their directing cone. If $\kappa_1 \neq 0$, $\kappa_2 = 0$, then we obtain a directing timelike plane instead of a timelike directing cone and these timelike ruled surfaces, satisfying $\kappa_1 \neq 0$, $\kappa_2 = 0$, are called timelike conoids.

Multiplying (30) by the first curvature $\frac{ds_1}{ds} = \kappa_1$ we have the following theorem.

Theorem 4.3. *For the derivatives of vectors of Frenet frame $\{O;\vec{q},\vec{h},\vec{a}\}$ of timelike ruled surface N and of its directing timelike cone with respect to the arc of striction curve of surface we have*

$$\begin{bmatrix} d\vec{q}/ds \\ d\vec{h}/ds \\ d\vec{a}/ds \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\varepsilon\kappa_1 & 0 & \kappa_2 \\ 0 & \varepsilon\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{32}$$

where $\kappa_1 = \frac{ds_1}{ds}$, $\kappa_2 = \frac{ds_3}{ds}$ and s_1, s_3 are the arcs of the spherical curves k_1, k_3 circumscribed by the bound vectors $\vec{q}$ and $\vec{a}$, respectively.

For the derivatives of vectors of Frenet frame $\{C; \vec{q}, \vec{h}, \vec{a}\}$ with respect to the arc of striction curve of the surface, the instantaneous rotation vector can be given by $\vec{w}_2 = \varepsilon\kappa_2\vec{q} - \kappa_1\vec{a}$. Thus, for the derivatives in (32) we can write

$$\frac{d\vec{q}}{ds} = \vec{w}_2 \times \vec{q}, \quad \frac{d\vec{h}}{ds} = \vec{w}_2 \times \vec{h}, \quad \frac{d\vec{a}}{ds} = \vec{w}_2 \times \vec{a}.$$

Now, we will show that the tangent of striction curve of timelike ruled surface at central point C is perpendicular to the vector $\vec{h}$. By differentiating the equation of the striction curve given in (8), we have

$$\frac{d\vec{c}}{du} = \dot{\vec{c}} = \dot{\vec{k}} + v\dot{\vec{q}} + \dot{v}\vec{q},$$

and further by using (5) and (7) we get

$$\langle \vec{h}, \dot{\vec{c}} \rangle = \langle \vec{h}, \dot{\vec{k}} \rangle + \langle v\dot{\vec{q}}, \vec{h} \rangle = \frac{\langle \dot{\vec{k}}, \dot{\vec{q}} \rangle}{\|\dot{\vec{q}}\|} - \frac{\langle \dot{\vec{k}}, \dot{\vec{q}} \rangle \langle \dot{\vec{q}}, \dot{\vec{q}} \rangle}{\|\dot{\vec{q}}\| \langle \dot{\vec{q}}, \dot{\vec{q}} \rangle} = 0.$$

Now, we can deal from two conditions whether the striction curve $\vec{c}$ is timelike or spacelike.

Case 1: Line of striction $\vec{c}$ is timelike

Let $\vec{c}$ be a timelike curve and $\varphi \in IR$ be an angle between striction curve and ruling, i.e. $\varphi \measuredangle \vec{q}\vec{t}$, where $\vec{t}$ is the tangent vector of line of striction. Then we can write

$$\vec{t} = \frac{d\vec{c}}{ds} = \mu(\varphi)\vec{q} + \eta(\varphi)\vec{a}, \tag{33}$$

where

$$\mu(\varphi) = \begin{cases} \sinh\varphi, & \textit{if } N \textit{ is of the type } N_+, \\ \cosh\varphi, & \textit{if } N \textit{ is of the type } N_-, \end{cases} \quad \text{and}$$

$$\eta(\varphi) = \begin{cases} \cosh\varphi, & \textit{if } N \textit{ is of the type } N_+, \\ \sinh\varphi, & \textit{if } N \textit{ is of the type } N_-. \end{cases}$$

Thus, while the equation of the timelike ruled surface is

$$\vec{r}(s, v) = \vec{c}(s) + v\vec{q}(s), \tag{34}$$

the equation of the striction curve is

$$c(s) = \int [\mu(\varphi)\vec{q} + \eta(\varphi)\vec{a}] ds. \tag{35}$$

For the parameter of distribution, using (2) and (32) we have

$$d = \frac{\left|\vec{q}, \frac{d\vec{q}}{ds}, \frac{d\vec{c}}{ds}\right|}{\left\langle \frac{d\vec{q}}{ds}, \frac{d\vec{q}}{ds} \right\rangle} = \frac{|\vec{q}, \vec{h}, \vec{t}|}{\kappa_1}. \tag{36}$$

From (33) and (36) it follows that

$$\eta(\varphi) = \langle \vec{a}, \vec{t} \rangle = \varepsilon \langle \vec{q} \times \vec{h}, \vec{t} \rangle = \varepsilon |\vec{q}, \vec{h}, \vec{t}| = \varepsilon\kappa_1 d. \tag{37}$$

Let s_1 be the arc of spherical curve k_1 of the direction cone of timelike ruled surface. By (33) and (37), Frenet formulae of timelike ruled surface are

$$\begin{cases} \frac{d\vec{c}}{ds_1} = \frac{\mu(\varphi)}{\kappa_1}\vec{q} + \frac{\eta(\varphi)}{\kappa_1}\vec{a} = f\vec{q} + \varepsilon d\vec{a}, \\ \frac{d\vec{q}}{ds_1} = \vec{h}, \\ \frac{d\vec{h}}{ds_1} = -\varepsilon\vec{q} + \kappa\vec{a}, \\ \frac{d\vec{a}}{ds_1} = \varepsilon\kappa\vec{h}. \end{cases} \tag{38}$$

Case 2: Line of striction $\vec{c}$ is spacelike

Let now $\vec{c}$ be a spacelike curve and $\theta \in IR$ be an angle between striction curve and ruling, i.e. $\varphi \measuredangle \vec{q}\vec{t}$, where $\vec{t}$ is the tangent vector of striction curve. Then we can write

$$\vec{t} = \frac{d\vec{c}}{ds} = \mu(\theta)\vec{q} + \eta(\theta)\vec{a}, \tag{39}$$

where

$$\mu(\theta) = \begin{cases} \cosh\theta, & \textit{if } N \textit{ is of the type } N_+ \\ \sinh\theta, & \textit{if } N \textit{ is of the type } N_- \end{cases} \quad \text{and}$$

$$\eta(\theta) = \begin{cases} \sinh\theta, & \textit{if } N \textit{ is of the type } N_+ \\ \cosh\theta, & \textit{if } N \textit{ is of the type } N_-. \end{cases}$$

Then making the similar calculations given in Case 1 and considering (30) Frenet formulae of timelike ruled surface are given as follows

$$\begin{cases} \frac{d\vec{c}}{ds_1} = \frac{\mu(\theta)}{\kappa_1}\vec{q} + \frac{\eta(\theta)}{\kappa_1}\vec{a} = f\vec{q} + \varepsilon d\vec{a}, \\ \frac{d\vec{q}}{ds_1} = \vec{h}, \\ \frac{d\vec{h}}{ds_1} = -\varepsilon\vec{q} + \kappa\vec{a}, \\ \frac{d\vec{a}}{ds_1} = \varepsilon\kappa\vec{h}. \end{cases} \tag{40}$$

The functions $f(s_1)$, $d(s_1)$, $\kappa(s_1)$ are the invariants of timelike ruled surface and they determine the timelike ruled surface uniquely up to its position in the space.

Example 4.1. Conoid of the 2nd kind. Let consider the timelike ruled surface N defined by

$$\vec{r}(u, v) = (v\cosh u, \cos u, v\sinh u).$$

This parametrization defines a non-cylindrical ruled surface of the type N_- which is said to be a conoid of the 2nd kind in IR^3_1 (Fig. 2) [9]. The base curve and ruling of N are

$$\vec{k}(u) = (0, \cos u, 0) \text{ and } \vec{q}(u) = (\cosh u, 0, \sinh u),$$

respectively. The distribution parameter of N is $d = \sin u \neq 0$. So that, the surface N is a skew timelike ruled surface. The striction curve of N is given by

$$\vec{c}(u) = \vec{k}(u) = (0, \cos u, 0).$$

Since $\|\dot{\vec{c}}(u)\| = \sin u \neq 1$, u is not the arc parameter of striction curve. By changing the parameter, the arc parameter of the striction curve is $u = \arccos(1 - s)$. Thus, the striction curve and Frenet vectors of N with respect to the arc parameter s are

$$\begin{aligned} &\vec{c}(s) = (0, 1 - s, 0), \\ &\vec{q}(s) = (\cosh(\arccos(1 - s)), 0, \sinh(\arccos(1 - s))), \\ &\vec{h}(s) = (-\sinh(\arccos(1 - s)), 0, -\cosh(\arccos(1 - s))), \\ &\vec{a}(s) = (0, 1, 0). \end{aligned}$$

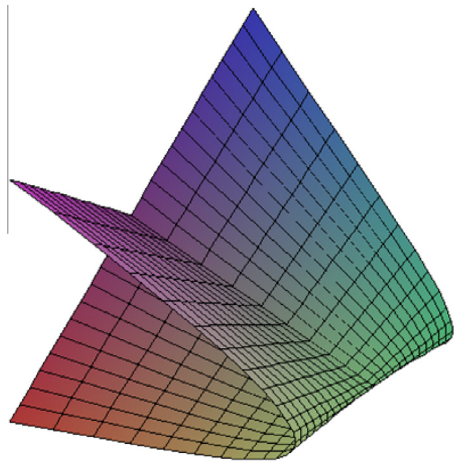

Figure 2 A conoid of the 2nd kind

Then the derivative of Frenet vectors with respect to the arc parameter s are

$$\frac{d\vec{q}}{ds} = \frac{1}{\sqrt{2s - s^2}}(\sinh(\arccos(1-s)), 0, \cosh(\arccos(1-s))),$$
$$\frac{d\vec{h}}{ds} = \frac{1}{\sqrt{2s - s^2}}(-\cosh(\arccos(1-s)), 0, -\sinh(\arccos(1-s))),$$
$$\frac{d\vec{a}}{ds} = (0,0,0).$$

Thus we have

$$\begin{bmatrix} \frac{d\vec{q}}{ds} \\ \frac{d\vec{h}}{ds} \\ \frac{d\vec{a}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2s-s^2}} & 0 \\ -\frac{1}{\sqrt{2s-s^2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}.$$

So that, for the curvatures of the timelike ruled surface we have $\kappa_1 = -\frac{1}{\sqrt{2s-s^2}}$, $\kappa_2 = 0$. It means that timelike directing cone of the surface is a directing timelike plane. Moreover, Darboux vector of the Frenet frame is $\vec{w}_2 = \frac{1}{\sqrt{2s-s^2}}\vec{a}$.

Example 4.2 (*Conoid of the 3rd kind*). Let consider the timelike ruled surface N of type N_+ defined by

$$\vec{r}(u, v) = (\cos(u+1), v\cos u, v\sin u).$$

which is said to be a conoid of the 3rd kind in IR_1^3 (Fig. 3) [9]. The base curve and ruling of N are $\vec{k}(u) = (\cos(u+1), 0, 0)$ and $\vec{q}(u) = (0, \cos u, \sin u)$, respectively. The distribution parameter of N is $d = -\sin(u+1)$. So that, the surface N is a skew timelike ruled surface. The striction curve of N is given by

$$\vec{c}(u) = \vec{k}(u) = (\cos(u+1), 0, 0).$$

Then, arc parameter of the striction curve is $u = \arccos(-s - \cos 1) - 1$. Thus, the striction curve and the Frenet vectors of N are

$$\vec{c}(s) = (-s - \cos 1, 0, 0),$$
$$\vec{q}(s) = (0, \cos(\arccos(-s-\cos 1) - 1), \sin(\arccos(-s-\cos 1) - 1)),$$
$$\vec{h}(s) = (0, -\sin(\arccos(-s-\cos 1) - 1), \cos(\arccos(-s-\cos 1) - 1)),$$
$$\vec{a}(s) = (1,0,0).$$

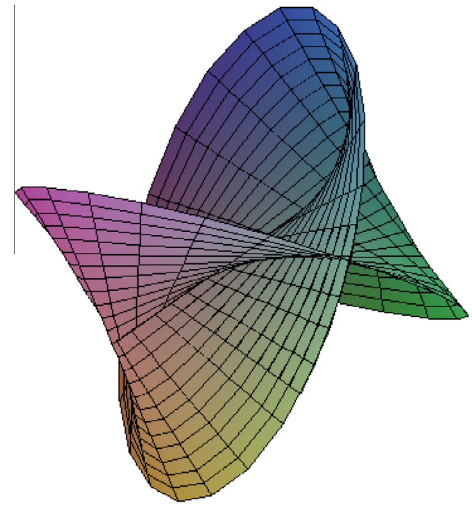

Figure 3 A conoid of the 3rd kind

The derivatives of Frenet vectors are

$$\frac{d\vec{q}}{ds} = \frac{1}{\sqrt{1-(s+\cos 1)^2}}(0, -\sin(\arccos(-s-\cos 1) - 1), \cos(\arccos(-s-\cos 1) - 1)),$$
$$\frac{d\vec{h}}{ds} = \frac{1}{\sqrt{1-(s+\cos 1)^2}}(0, -\cos(\arccos(-s-\cos 1) - 1), -\sin(\arccos(-s-\cos 1) - 1)),$$
$$\frac{d\vec{a}}{ds} = (0,0,0).$$

Thus we get

$$\begin{bmatrix} \frac{d\vec{q}}{ds} \\ \frac{d\vec{h}}{ds} \\ \frac{d\vec{a}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{1-(s+\cos 1)^2}} & 0 \\ -\frac{1}{\sqrt{1-(s+\cos 1)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}.$$

So that, for the curvatures of the timelike ruled surface and instantaneous rotation vector of the Frenet frame we get $\kappa_1 = \frac{1}{\sqrt{1-(s+\cos 1)^2}}$, $\kappa_2 = 0$, and $\vec{w}_2 = -\frac{1}{\sqrt{1-(s+\cos 1)^2}}\vec{a}$, respectively. Since $\kappa_1 \neq 0$, $\kappa_2 = 0$, the directing cone of the surface is a timelike plane.

5. Conclusions

Ruled surfaces have an important role in some areas such as design problems in spatial mechanisms and physics, kinematics and computer aided design (CAD). Especially, the frames and invariants of these surfaces have important applications in these sciences. Moreover, the study of ruled surfaces in the Minkowski 3-space is more interesting than the the Euclidean case. According to the classifications of ruled surfaces, they have different values for derivative of the vectors. So, the kinematics and geometric interpretations can be more different. In this paper, we introduce the Frenet frames and invariants of timelike ruled surfaces with timelike and spacelike rulings which can be used to give the instantaneous properties of a point trajectory and of a line trajectory in Lorentzian spatial kinematics.

References

[1] Guggenheimer H. Differential geometry, McGraw- Hill Book Comp. Inc. London, Lib. Cong. Cat. Card No. 68-12118; 1963. p. 164.

[2] Karger A, Novak J. Space kinematics and lie groups. Prague, Czechoslovakia: STNL Publishers of Technical Lit.; 1978.

[3] Peternel M, Pottmann H, Ravani B. On the computational geometry of ruled surfaces. Comp Aid Geom Des 1999;31:17–32.

[4] Ravani B, Ku TS. Bertrand offsets of ruled and developable surfaces. Comp Aid Geom Des 1991;23(2).

[5] Abdel-All NH, Abdel-Baky RA, Hamdoon FM. Ruled surfaces with timelike rulings. Appl Math Comput 2004;147:241–53.

[6] Çalışkan A, Uğurlu HH, Kılıç O. On the geometry of timelike congruences. Hadron J Suppl 2003;18:311–28.

[7] Dillen F, Sodsiri W. Ruled surfaces of Weingarten type in Minkowski 3-space. J Geom 2005;83:10–21.

[8] Kasap E, Ayyıldız FT. Surfaces with common geodesics in Minkowski 3-space. Appl Math Comput 2006;177:260–70.

[9] Kim YH, Yoon WD. Classification of ruled surfaces in Minkowski 3-space. J Geom Phys 2004;49:89–100.

[10] Küçük A. On the developable timelike trajectory ruled surfaces in Lorentz 3-space IR^3_1. Appl Math Comput 2004;157:483–9.

[11] Wang DL, Liu J, Xiao DZ. Kinematic differential geometry of a rigid body in spatial motion I. A new adjoin approach and instantaneous properties of a point trajectory in spatial kinematics. Mech Mach Theory 1997;32(4):419–32.

[12] O'Neill B. Semi-Riemannian geometry with applications to relativity. London: Academic Press; 1983.

[13] Uğurlu HH, Çalışkan A. Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi, Celal Bayar Üniversitesi Yayınları, Yayın No: 0006; 2012.

[14] Birman G, Nomizo K. Trigonometry in Lorentzian geometry. Ann Math Mont 1984;91(9):543–9.

[15] Beem JK, Ehrlich PE. Global Lorentzian geometry. New York: Marcel Dekker; 1981.

[16] Uğurlu HH, Önder M. Instantaneous rotation vectors of skew timelike ruled surfaces in Minkowski 3-space, VI. Geometry symposium, 01–04 July 2008, Bursa, Turkey; 2008.

[17] Uğurlu HH, Yakut NN, Öztunç S. The concept of angle in Minkowski 3-space, X. Geometry symposium, Balıkesir, Turkey, 13–16 July 2012; 2012.

Mehmet Önder, Assistant Professor Doctor of Mathematics, Faculty of Arts and Sciences, Mathematics Department, Celal Bayar University, Muradiye, Manisa, Turkey. The author graduated from Celal Bayar University and received his M.Sc. Degree in 2002 and 2004, respectively. He obtained his Ph. D. in 2012 from the same university. He is a Faculty of Arts and Sciences Member at Celal Bayar University since 2004. His research interests comprise: Differential geometry, line geometry, dual and dual Lorentzian geometry.

H. Hüseyin Ugurlu, Full Professor of Mathematics, Faculty of Education, Department of Secondary Education Science and Mathematics Teaching, Mathematics Teaching Program, Gazi University, Ankara, Turkey. The author graduated from Ankara University in 1983. He received his M.Sc. degree and Ph.D. degree from Gazi University in 1987 and 1992, respectively. He worked at Gazi University as a Faculty of Science member from 1984 to 1993. Also, he worked at Celal Bayar University as a Faculty of Arts and Sciences member from 1993 to 2004. Now, he is a Faculty of Education Member at Gazi University since 2004. His research interests comprise: Differential geometry, line geometry, dual and dual Lorentzian geometry.