Periodic solutions for $p$-Laplacian neutral functional differential equation with deviating arguments

Yanling Zhu*, Shiping Lu

Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China

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Abstract

By using the theory of coincidence degree, we study a kind of periodic solutions to $p$-Laplacian neutral functional differential equation with deviating arguments such as $(\varphi_p(x(t) - c x(t - \sigma)))' + g(t, x(t - \tau(t))) = p(t)$, a result on the existence of periodic solutions is obtained.

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1. Introduction

The problem of periodic solutions of ordinary differential equation was extensively studied, see Refs. [1–4]. In recent years, there are many results about periodic solutions to second-order scalar differential equations with deviating arguments [5–10]. For example, in [10] the authors studied the following equation with a deviating argument:

$$x''(t) + f(x(t))x'(t) + g(x(t - \tau(t), x(t))) = e(t).$$

(1.1)

By using Mawhin’s continuation theorem, some results on the existence of periodic solution are obtained. But the corresponding problem of $p$-Laplacian differential equation with deviating arguments has been studied far less often. The reason for this is that the differential operator

* Corresponding author.

E-mail addresses: zhuyanling990723@ sina.com (Y. Zhu), lushiping26@ sohu.com (S. Lu).
(\varphi_p(u))' = (|u|^{p-2}u)' (p \neq 2) is no longer linear, so the theory of coincidence degree cannot be applied directly. In [11], we found that the authors studied periodic solutions for the following $p$-Laplacian Liénard equation with a deviating argument:

$$
(\varphi_p(x'(t)))' + f(t,x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t).
$$

(1.2)

The condition imposed on $g(x)$ is either $\lim_{|x| \to +\infty} \sup |\frac{g(x)}{x}| < r$ or $|g(u) - g(v)| \leq l|u - v|$. In this paper, we study the existence of periodic solutions for $p$-Laplacian neutral functional differential equation with deviating arguments

$$
(\varphi_p(x(t) - cx(t - \sigma)))' + g(t,x(t - \tau(t))) = p(t),
$$

(1.3)

where $\varphi_p : R \to R$, $\varphi_p(u) = |u|^{p-2}u$, $g \in C(R^2, R)$, $\tau(t)$, $p(t)$ are continuous periodic functions defined on $R$ with period $T > 0$, $\sigma, c \in R$ are constants such that $|c| \neq 1$. By using the theory of coincidence degree, we obtain a new result to guarantee the existence of periodic solutions. The significance is that the condition imposed on $g(t, x)$ is very weak and we only need $|c| \neq 1$. Furthermore, an example is given to demonstrate our result.

2. Main lemmas

We first rewrite Eq. (1.3) in the following form:

$$
\begin{cases}
(Ax_1)'(t) = \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t), \\
x_2'(t) = -g(t,x_1(t - \tau(t))) + p(t),
\end{cases}
$$

(2.1)

where $1/p + 1/q = 1$. We can easily see if $x(t) = (x_1(t), x_2(t))^T$ is a $T$-periodic solution of Eq. (2.1), then $x_1(t)$ is a $T$-periodic solution of Eq. (1.3).

We set the following notations: $T > 0$ is a constant, $C_T = \{\varphi \in C(R, R): \varphi(t + T) = \varphi(t)\}$ with the norm $||\varphi|| = \max_{t \in [0, T]} |\varphi(t)|$, $X = Y = \{x = (x_1(\cdot), x_2(\cdot))^T \in C(R, R^2): x(t) \equiv x(t + T)\}$ with the norm $||x|| = \max(|x_1|_0, |x_2|_0)$, $|x|_p = \int_0^T |x(t)|^p dt$. Clearly, $X$ and $Y$ are Banach spaces. We also defined operators $A$ and $L$ in the following form:

$$
A : C_T \to C_T, \quad (Ax)(t) = x(t) - cx(t - \sigma),
$$

$$
L : D(L) \subset C_T \to C_T, \quad Lx = \left( \begin{array}{c} (Ax_1)' \\ x_2' \end{array} \right),
$$

where $x = (x_1, x_2)$.

Lemma 2.1. [12] If $|c| \neq 1$, then $A$ has continuous bounded inverse on $C_T$, and

1. $||A^{-1}x|| \leq \frac{||x||}{||c||-1}$, $\forall x \in C_T$;
2. $\int_0^T |(A^{-1}f)(t)| dt \leq \frac{1}{||c||} \int_0^T |f(s)| ds$, $\forall f \in C_T$;
3. $\int_0^T |(A^{-1}f)(t)|^2 dt \leq \frac{1}{(||c||)^2} \int_0^T |f(s)|^2 ds$, $\forall f \in C_T$. 


By Hale’s terminology [13], a solution $x(t)$ of Eq. (1.3) is that $x \in C(R, R)$ such that $Ax \in C^1(R, R)$ and Eq. (1.3) is satisfied on $R$. In general, $x$ is not $C^1(R, R)$. But from Lemma 2.1, it is easy to see that $(Ax)' = Ax'$. So a $T$-periodic solution $x$ of Eq. (1.3) must be $C^1(R, R)$. According to the first part of Lemma 2.1, we can easily obtain that $\ker L = R^2$, $\text{Im } L = \{ y \in Y: \int_0^T y(s) \, ds = 0 \}$. So $L$ is a Fredholm operator with index zero. Let project operators $P, Q$ be as follows:

$$P : X \to \ker L, \quad Px = \frac{1}{T} \int_0^T x(s) \, ds,$$

$$Q : Y \to \text{Im } Q \subset R^2, \quad Qy = \frac{1}{T} \int_0^T y(s) \, ds,$$

then $\text{Im } P = \ker L$, $\ker Q = \text{Im } L$. Set $L_p = L|_{D(L) \cap \ker P}$ and $L^{-1}_p : \text{Im } L \to D(L)$ denotes the inverse of $L_p$, then

$$[L^{-1}_p y](t) = \left( \begin{array}{c} (A^{-1}Fy_1)(t) \\ (Fy_2)(t) \end{array} \right),$$

$$[Fy](t) = -\int_t^T y(s) \, ds + \int_0^s \frac{\theta}{T} y(s) \, ds,$$

where $y(t) = (y_1(t), y_2(t))$.

**Lemma 2.2.** [13] Let $X$ and $Y$ be two Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \overline{\Omega} \to Y$ be $L$-compact on $\overline{\Omega}$. If all the following conditions hold:

1. $Lx \neq \lambda Nx$, $\forall x \in \partial \Omega \cap D(L)$, $\forall \lambda \in (0, 1)$;
2. $Nx \notin \text{Im } L$, $\forall x \in \partial \Omega \cap \ker L$;
3. $\deg \{ JQN, \Omega \cap \ker L, 0 \} \neq 0$, where $J : \text{Im } Q \to \ker L$ is an isomorphism,

then equation $Lx = Nx$ has a solution on $\Omega \cap D(L)$.

### 3. Main result

**Theorem.** Suppose that $p > 2$ and there exist positive constants $D$ and $r \geq 0$ such that

- $[H_1]$ $x[g(t, x) - p(t)] > 0$, $\forall t \in R, |x| > D$;
- $[H_2]$ $\lim_{x \to -\infty} \sup_{t \in [0, T]} \frac{|g(t, x) - p(t)|}{|x|^{p-1}} \leq r$.

Then Eq. (1.2) has at least one $T$-periodic solution, if $\frac{2T^p r (1 + |c|)}{(1 - |c|)^2} < 1$.

**Proof.** We easily see that Eq. (2.1) has a $T$-periodic solution if and only if the following operator equation

$$Lx = Nx,$$

has a $T$-periodic solution, where $N : C_T \to C_T$, 

$$Lx = Nx,$$
\[(N x)(t) = \left( \begin{array}{c} \varphi_q(x_2(t)) \\ -g(t, x_1(t - \tau(t))) + p(t) \end{array} \right). \]

From (2.2), we see that \( N \) is \( L \)-compact on \( \Omega \), where \( \Omega \) is any open, bounded subset of \( \mathbb{C}_T \).

Take \( \Omega_1 = \{ x: x \in \mathbb{C}_T, Lx = \lambda Nx, \lambda \in [0, T] \} \).

\( x \in (x_1(t), x_2(t)) \in \Omega_1 \), then \( x \) must satisfy

\[
\begin{aligned}
(Ax_1)'(t) &= \lambda \varphi_q(x_2(t)) = \lambda |x_2(t)|^{q-2}x_2(t), \\
x_2'(t) &= -\lambda g(t, x_1(t - \tau(t))) + \lambda p(t).
\end{aligned}
\]

(3.1)

From the first equation of (3.1), we can know \( x_2(t) = \varphi_p\left( \frac{1}{\lambda} (Ax_1)'(t) \right) \), which together with the second equation of (3.1) yields

\[
\begin{aligned}
\left( \varphi_p\left( \frac{1}{\lambda} (Ax_1)'(t) \right) \right)' + \lambda g(t, x_1(t - \tau(t))) &= \lambda p(t), \\
\left( \varphi_p((Ax_1)'(t)) \right)' + \lambda^p g(t, x_1(t - \tau(t))) &= \lambda^p p(t).
\end{aligned}
\]

(3.2)

Integrating both sides of (3.2) over \([0, T]\), we have

\[
\int_0^T \left| (Ax_1)'(t) \right|^p dt = \lambda^p \int_0^T \left| g(t, x_1(t - \tau(t))) - p(t) \right| dt = 0.
\]

(3.3)

By integral mean value theorem, there is a constant \( \xi \in [0, T] \) such that \( g(\xi, x_1(\xi - \tau(\xi))) = p(\xi) = 0 \). So from assumption [\( H_1 \)] we can get \( |x_1(\xi - \tau(\xi))| \leq D \). So

\[
|x_1|_0 \leq D + \int_0^T \left| x_1'(t) \right| dt.
\]

(3.4)

On the other hand, multiplying the two sides of Eq. (3.2) by \((Ax_1)(t)\) and integrating them over \([0, T]\), we get

\[
\begin{aligned}
-\int_0^T \left| (Ax_1)'(t) \right|^p dt &= -\int_0^T \left| Ax_1'(t) \right|^p dt \\
&= -\lambda^p \int_0^T \left[ g(t, x_1(t - \tau(t))) - p(t) \right] [x_1(t) - cx_1(t - \sigma)] dt,
\end{aligned}
\]

(3.5)

i.e.,

\[
\int_0^T \left| (Ax_1)'(t) \right|^p dt = \lambda^p \int_0^T \left[ g(t, x_1(t - \tau(t))) - p(t) \right] [x_1(t) - cx_1(t - \sigma)] dt
\]

\[
\leq |x_1|_0 \int_0^T \left| g(t, x_1(t - \tau(t))) - p(t) \right| dt
\]
\[ + |c| \int_0^T |g(t, x_1(t - \tau(t))) - p(t)| x_1(t - \sigma) \, dt \]
\[ \leq (1 + |c|) |x_1|_0 \int_0^T |g(t, x_1(t - \tau(t))) - p(t)| \, dt. \]  
(3.6)

In view of \( \frac{2TP^r(1+|c|)}{(1-|c|)^2} < 1 \), there exists a constant \( \varepsilon > 0 \) such that \( \frac{2TP^r(1+|c|)(r+\varepsilon)}{(1-|c|)^2} < 1 \).

From assumption \([H2]\), we get that there exists a constant \( \rho > 0 \) such that
\[ |g(t, x) - p(t)| \leq (r + \varepsilon)|x|^{p-1}, \quad \forall t \in \mathbb{R}, x < -\rho. \]  
(3.7)

Let \( E_1 = \{ t \in [0, T]: x_1(t - \tau(t)) < -\rho \}, \ E_2 = \{ t \in [0, T]: |x_1(t - \tau(t))| \leq \rho \}, \ E_3 = \{ t \in [0, T]: x_1(t - \tau(t)) > \rho \}. \) By the second equation of (3.1) it is easy to see that
\[ \left( \int_{E_1} + \int_{E_2} + \int_{E_3} \right) \left[ g(t, x_1(t - \tau(t))) - p(t) \right] \, dt = 0. \]  
(3.8)

Hence
\[ \int_{E_3} |g(t, x_1(t - \tau(t))) - p(t)| \, dt = \int_{E_3} \left[ g(t, x_1(t - \tau(t))) - p(t) \right] \, dt \]
\[ = - \left( \int_{E_1} + \int_{E_2} \right) \left[ g(t, x_1(t - \tau(t))) - p(t) \right] \, dt \]
\[ \leq \left( \int_{E_1} + \int_{E_2} \right) |g(t, x_1(t - \tau(t))) - p(t)| \, dt. \]  
(3.9)

Therefore, by (3.7) and (3.9) we get
\[ \int_0^T |g(t, x_1(t - \tau(t))) - p(t)| \, dt = \left( \int_{E_1} + \int_{E_2} + \int_{E_3} \right) |g(t, x_1(t - \tau(t))) - p(t)| \, dt \]
\[ \leq 2 \left( \int_{E_1} + \int_{E_2} \right) |g(t, x_1(t - \tau(t))) - p(t)| \, dt \]
\[ \leq 2 \int_{E_1} (r + \varepsilon)|x_1(t - \tau(t))|^{p-1} \, dt + 2\tilde{g}_\rho T \]
\[ \leq 2(r + \varepsilon)T|x_1|_0^{p-1} + 2\tilde{g}_\rho T, \]  
(3.10)

where \( \tilde{g}_\rho = \max_{t \in E_2} |g(t, x_1(t - \tau(t))) - p(t)|. \) From (3.6) and (3.10), we know
\[ \int_0^T |(Ax'_1)(t)|^p \, dt \leq 2(1 + |c|)T[r + \varepsilon]x_1|_0^p + 2\tilde{g}_\rho |x|_0. \]  
(3.11)

Substituting (3.4) into (3.11), we have
\[
\int_0^T |(Ax'_1(t)|^p dt \leq 2(1 + |c|)T(r + \varepsilon) \left( D + \int_0^T |x'_1(t)| dt \right)^p
\]

+ 2(1 + |c|)T \tilde{g}_\rho \left( D + \int_0^T |x'_1(t)| dt \right).
\]  
(3.12)

Case (1). If \( \int_0^T |x'_1(t)| dt = 0 \), from (3.4) we see \( |x_1|_0 < D \).

Case (2). If \( \int_0^T |x'_1(t)| dt > 0 \), then we know

\[
\left( D + \int_0^T |x'_1(t)| dt \right)^p = \left( \int_0^T |x'_1(t)| dt \right)^p \left( 1 + \frac{D}{\int_0^T |x'_1(t)| dt} \right)^p.
\]  
(3.13)

By the knowledge of mathematical analysis, there is a constant \( \delta > 0 \) such that

\[(1 + x)^p < 1 + (1 + p)x, \quad \forall x \in [0, \delta].\]  
(3.14)

If \( D/\int_0^T |x'_1(t)| dt > \delta \), then \( \int_0^T |x'_1(t)| dt < D/\delta \), so from (3.4) we have \( |x_1|_0 < D/\delta + D \).

If \( D/\int_0^T |x'_1(t)| dt \leq \delta \), then by (3.14) we know

\[
\left( D + \int_0^T |x'_1(t)| dt \right)^p \leq \left( \int_0^T |x'_1(t)| dt \right)^p \left( 1 + \frac{(p + 1)D}{\int_0^T |x'_1(t)| dt} \right)^{p-1}
\]

\[
\leq T^{p/q} \int_0^T |x'_1(t)|^p dt + (p + 1)D T^{p-1} \left( \int_0^T |x'_1(t)|^p dt \right)^{1/q}.
\]

By (3.12) we obtain

\[
\int_0^T |Ax'_1(t)|^2 dt \leq T^{p-2/p} \left[ \int_0^T |Ax'_1(t)|^p dt \right]^{2/p} \leq T^{p-2/p} \int_0^T |Ax'_1(t)|^p dt
\]

\[
\leq 2T^p (1 + |c|) (r + \varepsilon) \int_0^T |x'_1(t)|^p dt
\]

\[
+ 2T^{p-1} \left( 1 + |c| \right) (p + 1)D(r + \varepsilon) \left( \int_0^T |x'_1(t)|^p dt \right)^{1/q}
\]

\[
+ 2T (1 + |c|) \tilde{g}_\rho \left[ D + T^{1/q} \left( \int_0^T |x'_1(t)|^p dt \right)^{1/p} \right].
\]  
(3.15)
By applying the third part of Lemma 2.1, we get
\[ \int_0^T |x'_1(t)|^2\,dt = \int_0^T |(A^{-1}Ax'_1)(t)|^2\,dt \leq \frac{\int_0^T |(Ax'_1)(t)|^2\,dt}{(1-|c|)^2}. \]

So it follows from (3.15) that
\[ \int_0^T |x'_1(t)|^2\,dt \leq \frac{2T^p(1+|c|(r+\varepsilon))}{(1-|c|)^2} \int_0^T |x'_1(t)|^p\,dt + \frac{2(1+|c|)(p+1)D(r+\varepsilon)}{(1-|c|)^2} T^{\frac{p-1}{q}} \left( \int_0^T |x'_1(t)|^p\,dt \right)^{1/q} + \frac{2T(1+|c|)\bar{g}\rho}{(1-|c|)^2} \left[ D + T^{1/q} \left( \int_0^T |x'_1(t)|^p\,dt \right)^{1/p} \right]. \]

As \( q > 1 \), \( \frac{2T^p(1+|c|(r+\varepsilon))}{(1-|c|)^2} < 1 \), there is a constant \( M_1 > 0 \) such that \( \int_0^T |x'_1(t)|^2\,dt \leq M_1 \). It follows from (3.4) that
\[ |x_1|_0 \leq D + T^{1/2}M_1^{1/2} := M_2. \]

By the first equation of (3.1) we have \( \int_0^T |x_2(t)|^{q-2}x_2(t)\,dt = 0 \), which implies there is a constant \( t_1 \in [0, T] \) such that \( x_2(t_1) = 0 \). So \( |x_2|_0 \leq \int_0^T |x'_2(t)|\,dt \). By the second equation of (3.1) we obtain
\[ \int_0^T |x'_2(t)|\,dt \leq \int_0^T |g(t, x_1(t-\tau(t)))|\,dt + \int_0^T |p(t)|\,dt \leq TgM_2 + |p|_1, \]

where \( gM_2 = \max_{|x| \leq M_2, t \in [0, T]} |g(t, x)| \). So we have
\[ |x_2|_0 \leq TgM_2 + |p|_1 := M_3. \]

Let \( M = \sqrt{M_2^2 + M_3^2} + 1 \), \( \Omega = \{ x = (x_1, x_2)^T : |x_1|_0 < M, |x_2|_0 < M \} \) and \( \Omega_2 = \{ x \in \partial\Omega : x \in \ker L \} \), then
\[ QNx = \frac{1}{T} \int_0^T \left( \varphi_q(x_2(t)) \left(-g(t, x_1(t-\tau(t))) + p(t) \right) \right)\,dt = \left( |x_2|^{q-2}x_2 \left(-g(t, x_1) + p(t) \right) \right). \]

If \( QNx = 0 \), then \( x_2 = 0, x_1 = M \) or \(-M\). But when \( x_1 = M \), we know \(-g(t, x_1) + p(t) < 0\), which yields a contradiction. Similarly when \( x_1 = -M \), we also have \( QNx \neq 0 \), i.e., \( \forall x \in \Omega, x \notin \ker L \). So conditions (1) and (2) of Lemma 2.2 are both satisfied. Next we show that condition (3) of Lemma 2.2 is also satisfied. Define the isomorphism \( J : \ker Q \rightarrow \ker L \) as follows:
\[ J(x_1, x_2)^T = (x_2, x_1)^T. \]

Let \( H(\mu, x) = \mu x + \frac{1-\mu}{T} JQNx, (\mu, x) \in \Omega \times [0, 1] \), then we have
\[ H(\mu, x) = \left( \mu x_1 + \frac{1-\mu}{T} \int_0^T \left[ -g(t, x_1) + p(t) \right] dt \right) \left( \mu + \frac{1-\mu}{T} |x_2|^q \right) x_2, \]
\[
\forall (x, \mu) \in (\partial \Omega \cap \ker L) \times [0, 1].
\]
If \( H(\mu, x) = 0 \), then \( x_2 = 0, x_1 = M \) or \(-M\). Similar to the above proof we can see that \( H(\mu, x) \neq 0 \). Hence
\[
\text{deg} \{ J Q N, \Omega \cap \ker L, 0 \} = \text{deg} \{ H(0, x), \Omega \cap \ker L, 0 \} = \text{deg} \{ H(1, x), \Omega \cap \ker L, 0 \} \neq 0.
\]
So condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation \( Lx = Nx \) has a solution \( x(t) = (x_1(t), x_2(t))^T \) on \( \overline{\Omega} \cap D(L) \), i.e., Eq. (1.3) has a \( T \)-periodic solution \( x_1(t) \).

**Corollary.** Suppose that \( p > 2 \) and there exist positive constants \( D \) and \( r \) such that

\[ [H_1^+] \; x [g(t, x) - p(t)] > 0, \forall t \in R, |x| > D; \]
\[ [H_2^+] \; \lim_{x \to +\infty} \sup_{t \in [0, T]} \frac{|g(t(x), p(t))|}{x} \leq r. \]

Then Eq. (1.3) has at least one \( T \)-periodic solution, if \[ \frac{2Pr(1+|c|)}{(1-|c|)^2} < 1. \]

As an application, we consider the following equation
\[ \varphi_3 \left( x(t) - 5x(t - \pi) \right)' + g(t, x(t - \sin t)) = e^{\cos^2 t}, \tag{3.16} \]
where
\[ g(t, x) = \begin{cases} e^{\sin^2 t} x, & x \geq 0, \\ \frac{x}{18\pi^2} e^{\sin^2 t}, & x < 0. \end{cases} \]

According to the theorem, we have \( p = 3, c = 5, r = \frac{1}{18\pi^2} \), so \[ \frac{2Pr(1+|c|)}{(1-|c|)^2} < 1. \] Hence, by using our theorem we know Eq. (3.16) has at least one \( 2\pi \)-periodic solution.

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