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Finite Partially Ordered Sets of Cohomological Dimension One

CHARLES CHING-AN CHENG

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

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INTRODUCTION

Let $\mathbb{C}$ be a small category and R a ring with identity. We shall denote the category of unitary left R-modules by mod R and the category of covariant functors $\mathbb{C} \to \text{mod } R$ by $(\text{mod } R)^{\mathbb{C}}$. The *R-cohomological dimension* of $\mathbb{C}$ is defined by

$$\text{cd}_R\mathbb{C} = \sup\{k \mid \lim^k \neq 0\}$$

where $\lim^k$ is the kth right derived functor of the (inverse) limit functor $\lim: (\text{mod } R)^{\mathbb{C}} \to \text{mod } R$. There is a natural isomorphism $\lim M = \text{Hom}(\Delta R, M)$ where ΔR denotes the constant R-valued functor. It follows that $\text{cd}_R\mathbb{C} = \text{pd}_{\mathbb{C}}\, \Delta R$ where $\text{pd}_{\mathbb{C}}$ denotes projective dimension in $(\text{mod } R)^{\mathbb{C}}$. We shall denote $\text{cd}_{\mathbb{Z}}\mathbb{C}$ simply by $\text{cd}\, \mathbb{C}$. It is not difficult to show that $\text{cd}_R\mathbb{C} \leqslant \text{cd}\, \mathbb{C}$ for all $R \neq 0$.

Laudal characterized all small categories $\mathbb{C}$ with $\text{cd}\, \mathbb{C} = 0$ in [2]. Stallings [7] and Swan [8] characterized nontrivial free groups as those satisfying $\text{cd}\, \mathbb{C} = 1$. Mitchell [5] proved that if $\mathbb{C}$ is a directed set and $\aleph_n$ is the smallest cardinal number of a cofinal subset, then $\text{cd}_R\mathbb{C} = n + 1$ for any nonzero ring R. In this paper, we shall characterize those finite posets $\mathbb{C}$ such that $\text{cd}_R\mathbb{C} \leqslant 1$.

1. MAIN THEOREM

Throughout $\mathbb{C}$ will be a finite poset unless otherwise specified. Subsets of $\mathbb{C}$ will be considered as full subcategories. We shall denote the category of abelian groups by Ab. If $p, q \in \mathbb{C}$, $p < q$, are such that there exists no $k \in \mathbb{C}$ with $p < k < q$, then we say that q is a *cover* for p and p a *cocover* for q.

Let $S_p : \text{Ab} \to \text{Ab}^{\mathbb{C}}$ be the left adjoint of the pth evaluation functor. Then

$$\begin{aligned} S_p(A)(q) &= A \qquad && q \geqslant p \\ &= 0 && \text{otherwise.} \end{aligned}$$

The functor S_p preserves projectives since it is exact and its right adjoint is exact. Consider the exact sequence

$$0 \to K \to \bigoplus_m S_m(\mathbb{Z}) \xrightarrow{\epsilon} \Delta\mathbb{Z} \to 0 \tag{1}$$

in $\mathrm{Ab}^{\mathbb{C}}$, where the coproduct is indexed by minimal elements m of $\mathbb{C}$, and where the mth coordinate of ϵ is induced by the identity on $\mathbb{Z}$. Since the middle term P of (1) is projective, cd $\mathbb{C} \leqslant 1$ if and only if K is projective.

LEMMA 1.1. *Let e be a minimal element of $\mathbb{C}$ which has only one cover p. Let $\mathbb{C}' = \mathbb{C} - \{e\}$. Then* cd $\mathbb{C} \leqslant 1$ *if and only if* cd $\mathbb{C}' \leqslant 1$.

Proof. The left adjoint G of the restriction functor $T\colon \mathrm{Ab}^{\mathbb{C}} \to \mathrm{Ab}^{\mathbb{C}'}$ extends $D \in \mathrm{Ab}^{\mathbb{C}'}$ by adding 0 at e, and so is exact. The right adjoint R of T extends $D \in \mathrm{Ab}^{\mathbb{C}'}$ by adding $D(p)$ at e, and so is also exact. Since T is exact and has an exact right adjoint, it preserves projective resolutions. Applying T to (1), we get an exact sequence

$$0 \to T(K) \to T(P) \to T(\Delta\mathbb{Z}) \to 0 \tag{2}$$

in $\mathrm{Ab}^{\mathbb{C}'}$ with the middle term projective. Since $K(m) = 0$ for m minimal, we have

$$K = GT(K). \tag{3}$$

Since $TG = \mathrm{id}$, it follows easily (see [4, Corollary 1.2]) that

$$\mathrm{pd}_{\mathbb{C}} GT(K) = \mathrm{pd}_{\mathbb{C}'} T(K). \tag{4}$$

Since $T(\Delta\mathbb{Z})$ is constant $\mathbb{Z}$-valued over $\mathbb{C}'$, we have

$$\mathrm{pd}_{\mathbb{C}'} T(\Delta\mathbb{Z}) = \mathrm{cd}\,\mathbb{C}'.$$

Hence cd $\mathbb{C}' \leqslant 1$ if and only if $\mathrm{pd}_{\mathbb{C}'} T(\Delta\mathbb{Z}) \leqslant 1$, which, by (2), is true if and only if pd $T(K) = 0$. But by (3) and (4), $\mathrm{pd}_{\mathbb{C}} K = \mathrm{pd}_{\mathbb{C}'} T(K)$. Therefore it follows from (1) that cd $\mathbb{C}' \leqslant 1$ if and only if $\mathrm{pd}_{\mathbb{C}}\, \Delta\mathbb{Z} \leqslant 1$, that is, if and only if cd $\mathbb{C} \leqslant 1$.

LEMMA 1.2. *Let p be an element of $\mathbb{C}$ which has only one cocover e. Let $\mathbb{C}' = \mathbb{C} - \{p\}$. Then* cd $\mathbb{C}$ = cd $\mathbb{C}'$.

Proof. The left adjoint G of $T\colon \mathrm{Ab}^{\mathbb{C}} \to \mathrm{Ab}^{\mathbb{C}'}$ extends $D \in \mathrm{Ab}^{\mathbb{C}'}$ by adding $D(e)$ at p, and so is exact. Also $TG = \mathrm{id}$. Hence $\mathrm{pd}_{\mathbb{C}'}\, \Delta\mathbb{Z} = \mathrm{pd}_{\mathbb{C}} G(\Delta\mathbb{Z}) = \mathrm{pd}_{\mathbb{C}}\, \Delta\mathbb{Z}$, i.e., cd $\mathbb{C}'$ = cd $\mathbb{C}$.

Recall that the *height* of an element x in a finite poset is the greatest integer n such that there is a chain $x_0 < x_1 < \cdots < x_n = x$. Define an element of $\mathbb{C}$

to be *superfluous* if it is of height 0 (minimal) with only one cover or if it has height one and only one cocover. Iterating as many times as possible the process of eliminating a superfluous element, we obtain a finite poset $E(\mathbb{C})$. We shall show that, up to isomorphism, $E(\mathbb{C})$ is independent of the order in which superfluous elements are removed. Let $S(\mathbb{C})$ denote the set of all superfluous elements of $\mathbb{C}$. The following lemma follows easily from the definition of superfluous elements.

LEMMA 1.3. *If* $x, y \in S(\mathbb{C})$, *then either*

(a) $x \in S(\mathbb{C} - \{y\})$ *and* $y \in S(\mathbb{C} - \{x\})$, *or*
(b) $\mathbb{C} - \{x\} \simeq \mathbb{C} - \{y\}$.

PROPOSITION 1.4. *Let* $\mathbb{C}$ *be a finite poset, and suppose that* $\mathbb{C} - \{a_1, a_2, ..., a_n\}$ *has no superfluous elements, where* $a_i \in S(\mathbb{C} - \{a_1, a_2, ..., a_{i-1}\})$, $i = 1, ..., n$. *If also* $\mathbb{C} - \{b_1, b_2, ..., b_m\}$ *has no superfluous elements and*

$$b_j \in S(\mathbb{C} - \{b_1, b_2, ..., b_{j-1}\}), \qquad j = 1, ..., m,$$

then $\mathbb{C} - \{a_1, a_2, ..., a_n\} \simeq \mathbb{C} - \{b_1, b_2, ..., b_m\}$.

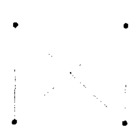
Proof. We shall prove it by induction on n. If $n = 1$, then $m \geq 1$. Hence $a_1, b_1 \in S(\mathbb{C})$. By Lemma 1.3, we have $\mathbb{C} - \{a_1\} \simeq \mathbb{C} - \{b_1\}$. Now suppose the proposition is true for all $k < n$. If $\mathbb{C} - \{a_1\} \simeq \mathbb{C} - \{b_1\}$, then we are done by induction. If $a_1 \in S(\mathbb{C} - \{b_1\})$ and $b_1 \in S(\mathbb{C} - \{a_1\})$, then suppose $\mathbb{C} - \{a_1, b_1, x_1, ..., x_r\}$ has no superfluous elements, where

$$x_i \in S(\mathbb{C} - \{a_1, b_1, x_1, ..., x_{i-1}\}), \qquad 1 \leq i \leq r.$$

By induction, $r = n - 2$, and we have

$$\begin{aligned} \mathbb{C} - \{a_1, ..., a_n\} &\simeq \mathbb{C} - \{a_1, b_1, x_1, ..., x_r\} \\ &\simeq \mathbb{C} - \{b_1, a_1, x_1, ..., x_r\} \\ &\simeq \mathbb{C} - \{b_1, ..., b_m\}. \end{aligned}$$

From Lemmas 1.1 and 1.2 it follows that $\operatorname{cd} \mathbb{C} \leq 1$ if and only if $\operatorname{cd} E(\mathbb{C}) \leq 1$. In particular, if $E(\mathbb{C}) = \mathbb{1}$, then $\operatorname{cd} \mathbb{C} \leq 1$. The converse is not true in general. For example, if $\mathbb{C}$ is the following poset



(all arrows are going down), then $\operatorname{cd} \mathbb{C} \leq 1$. But $E(\mathbb{C}) = \mathbb{C} \neq \mathbb{1}$. However, we shall show that if $\mathbb{C}$ has a terminal element, then the converse holds.

Let M be the set of all minimal elements of $\mathbb{C}$. For each $q \in \mathbb{C}$, define the poset $\mathbb{C}_q = \{p \in \mathbb{C} \mid p \leqslant q\}$ and the poset ${}_q\mathbb{C} = \{p \in C \mid p \geqslant q\}$. We shall denote the number of elements in a finite set S by $\#S$.

LEMMA 1.5. *Define $n(q)$ inductively by the formula*

$$\sum_{p \leqslant q} n(p) = \#(M \cap \mathbb{C}_q) - 1,$$

for all $q \in \mathbb{C}$. If cd $\mathbb{C} \leqslant 1$, *then $n(q) \geqslant 0$ for all q.*

Proof. Consider the exact sequence (1). Since cd $\mathbb{C} \leqslant 1$, K is projective. Hence there exists, for each $p \in \mathbb{C}$, a unique nonnegative integer $m(p)$ such that

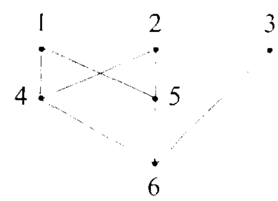
$$K = \bigoplus_{p \in \mathbb{C}} S_p(\mathbb{Z}^{m(p)}),$$

[6, Corollary 23.4]. Hence $K(q) = \oplus_{p \leqslant q} \mathbb{Z}^{m(p)}$ for all $q \in \mathbb{C}$. But from the exact sequence (1),

$$K(q) = \Big(\bigoplus_{M \cap \mathbb{C}_q} \mathbb{Z}\Big)\Big/\mathbb{Z}.$$

Taking ranks, we see that $\sum_{p \leqslant q} m(p) = \#(M \cap \mathbb{C}_q) - 1$. Hence $m(p) = n(p)$, and so the result follows.

Remark. The converse of the above lemma is not true. For example if $\mathbb{C}$ is the poset



then $n(1) = n(2) = n(3) = 0$, $n(4) = n(5) = 1$ and $n(6) = 0$. But our main theorem will show (or one can show directly) that cd $\mathbb{C} = 2$.

A poset $\mathbb{C}$ is *initially finite* if $\mathbb{C}_q$ is finite for all $q \in \mathbb{C}$. It is known [6, Corollary 23.6] that if $\mathbb{C}$ is initially finite, then

$$\mathrm{cd}_R\, \mathbb{C} = \sup_q \mathrm{cd}_R\, \mathbb{C}_q\,. \tag{5}$$

LEMMA 1.6. *Suppose that* cd $\mathbb{C} \leqslant 1$ *and let $p \in \mathbb{C}$ be such that height $p > 1$. Then $n(p) \geqslant \#M_p$ where $M_p = \{m \mid m$ is minimal and p covers $m\}$.*

Proof. By (5), cd $\mathbb{C}_p \leqslant$ cd $\mathbb{C}$. Hence cd $\mathbb{C}_p \leqslant 1$. By Lemma 1.1,

$\mathrm{cd}(\mathbb{C}_p - M_p) \leqslant 1$, so the unique number $n'(p)$(with respect to $\mathbb{C}_p - M_p$) is nonnegative. Since height $p > 1$, $n(p) \geqslant \#M_p$ where $n(p)$ is the unique number associated with p in Lemma 1.5 with respect to $\mathbb{C}_p$, hence $\mathbb{C}$.

LEMMA 1.7. *Let $\mathbb{C}$ be a finite poset with a terminal object. Then* $\mathrm{cd}\,\mathbb{C} \leqslant 1$ *if and only if* $E(\mathbb{C}) = \mathbb{1}$.

Proof. One direction is clear as mentioned before. Suppose $\mathrm{cd}\,\mathbb{C} \leqslant 1$ and $E(C) \neq \mathbb{1}$. Then $E(\mathbb{C})$ has the property that each minimal element has at least two covers and that each element of height one has at least two co-covers. Let the set of all covers of the minimal elements be P and let the set of all such elements of height one be Q. Then, by Lemma 1.5,

$$\#M - 1 = \sum_{p\in\mathbb{C}} n(p) \geqslant \sum_{p\in P-Q} n(p) + \sum_{p\in Q} n(p). \tag{6}$$

Since elements of $P - Q$ satisfies the condition of Lemma 1.6, $n(p) \geqslant \#M_p \geqslant 1$, for all $p \in P - Q$. Also, as $p \in Q$ covers at least two minimal elements, $n(p) \geqslant 1$ according to Lemma 1.5. Hence, from (6), $\#M - 1 \geqslant \#P \geqslant \#Q$. Using (6) again, we see that

$$\begin{aligned}\#M - 1 &\geqslant \sum_{p\in P-Q} \#M_p + \sum_{p\in Q} (\#M_p - 1)\\ &= \sum_{p\in P-Q} \#M_p + \sum_{p\in Q} \#M_p - \#Q.\end{aligned}$$

The first two terms must be greater than $2\#M$, as every element of M has at least two covers. Hence $\#M - 1 \geqslant 2\#M - \#Q$, so $\#M \leqslant \#Q - 1 < \#Q$, a contradiction.

Combining Lemma 1.7 and (5) we deduce our main theorem:

THEOREM 1.8. *Let $\mathbb{C}$ be an initially finite poset. Then* $\mathrm{cd}\,\mathbb{C} \leqslant 1$ *if and only if $E(\mathbb{C}_q) = \mathbb{1}$ for all $q \in \mathbb{C}$.*

Actually Theorem 1.8 is valid for arbitrary coefficient rings. To see this, we recall from [4] that an object $D \in \mathrm{Ab}^{\mathbb{C}}$ is *split* if the morphism $\sum_{i\leqslant q} \mathrm{im}(D_{iq}) \to D(q)$ is a coretraction for each $q \in \mathbb{C}$, where D_{iq} is the morphism induced by $i \leqslant q$. Also D is *pointwise free* if $D(q)$ is free for all $q \in \mathbb{C}$.

COROLLARY 1.9. *Let $\mathbb{C}$ be a finite poset and R any nonzero ring. Then* $\mathrm{cd}_R\mathbb{C} \leqslant 1$ *if and only if $E(\mathbb{C}_q) = \mathbb{1}$ for all $q \in \mathbb{C}$.*

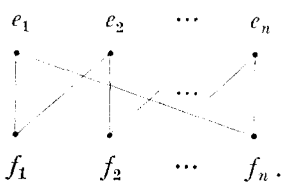
Proof. By the theorem it suffices to prove that $\mathrm{cd}_R\mathbb{C} \leqslant 1$ if and only if $\mathrm{cd}\,\mathbb{C} \leqslant 1$. One direction is clear, since $\mathrm{cd}_R\mathbb{C} \leqslant \mathrm{cd}\,\mathbb{C}$ is true for any small category. Now suppose cd $\mathbb{C} \nleqslant 1$. Tensoring the exact sequence (1) with R, we get an exact sequence

$$0 \to R \otimes K \to \bigoplus_{m \in M} S_m(R) \to \Delta R \to 0$$

in (mod $R)^{\mathbb{C}}$. By [4, Lemma 3.7], K is split and pointwise free. Since K is not projective, by [4, Theorem 3.4], $\mathrm{cd}_R \mathbb{C} = \mathrm{pd}_{\mathbb{C}} \Delta R \geqslant 2 + \mathrm{pd}\, R = 2$.

2. Applications

The *n-crown* $C_n (n \geqslant 2)$ is the poset

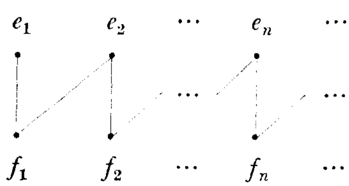


(1)

If we add an initial element and a terminal element to C_n, we obtain the *suspended n-crown* $\hat{C}_n$. Following Mitchell [6], we say that $\mathbb{C}$ *contains a (suspended) crown* if it contains $C_n(\hat{C}_n)$ as a full subcategory for some $n \geqslant 2$ with the further condition in case $n = 2$ that there exists no $k \in \mathbb{C}$ such that $e_i < k < f_i$, $i = 1, 2$.

Lemma 2.1. *Let* $\mathbb{C}$ *be a finite poset. Then* $E(\mathbb{C})$ *contains a crown if and only if* $E(\mathbb{C}) \neq \mathbb{1}$.

Proof. If $E(\mathbb{C})$ contains a crown, then certainly $E(\mathbb{C}) \neq \mathbb{1}$. Conversely suppose $E(\mathbb{C}) \neq \mathbb{1}$. Choose a minimal element of $E(\mathbb{C})$, say e_1. Let f_1 be a cover for e_1. There exists a minimal element, say e_2, such that $f_1 > e_2$ and $e_1 \neq e_2$. Let f_2 be a cover for e_2 distinct from f_1. Continuing in this way, we obtain a diagram in $E(\mathbb{C})$ of form



where $e_{i+1} \neq e_i$, $f_{i+1} \neq f_i$, f_i is a cover of e_i, and e_i is minimal. Since $E(\mathbb{C})$ is finite, there exists i, j, $i \neq j$ such that $e_i = e_j$ or $f_i = f_j$. In either case, by renumbering the subscripts, we obtain a (not necessarily full) subcategory (1) which satisfies the following:

(1) e_i is minimal, $i = 1,..., n$,

(2) f_i is a cover for e_i, $i = 1,..., n$,

(3) $e_{i+1} \neq e_i$ and $f_{i+1} \neq f_i$ for all i (subscripts mod n).

Suppose n is the smallest integer for which such a subset (1) exists ($n \geqslant 2$). If $n > 2$, then $\mathbb{C}$ contains (1) as a full subcategory, for otherwise we would contradict the minimality of n. In case $n = 2$, if there exists $k \in \mathbb{C}$ such that $e_i < k < f_i$, $i = 1, 2$, then (2) is contradicted. Hence $E(\mathbb{C})$ contains a crown.

A poset $\mathbb{C}$ is *locally finite* if ${}_p\mathbb{C}_q = \{x \in \mathbb{C} \mid p \leqslant x \leqslant q\}$ is finite for all $p, q \in \mathbb{C}$. The following theorem is due to Mitchell [4, Theorem 4.6]. Using Lemma 2.1, we shall simplify the proof of the more difficult of its implications.

THEOREM 2.2. *Let $\mathbb{C}$ be a locally finite poset, and let $\mathscr{A}$ be any abelian category with finite global dimension. Then $\mathbb{C}$ contains a suspended crown if and only if*

$$\text{gl dim } \mathscr{A}^{\mathbb{C}} \geqslant 3 + \text{gl dim } \mathscr{A}. \tag{2}$$

Proof. The "only if" direction is easily proved using the fact that if $\mathbb{C}$ contains a suspended crown, then it contains it as a retract [6, Lemma 35.6]. For the other direction, consider the exact sequence

$$0 \to K \to S_p(\mathbb{Z}) \to L_p(\mathbb{Z}) \to 0 \tag{3}$$

in $\text{Ab}^{\mathbb{C}}$, where

$$\begin{aligned} L_p(\mathbb{Z})(q) &= \mathbb{Z} \quad &&\text{if} \quad p = q \\ &= 0 \quad &&\text{if} \quad p \neq q. \end{aligned}$$

Then

$$\begin{aligned} K(q) &= \mathbb{Z} \quad &&\text{if} \quad q > p \\ &= 0 \quad &&\text{otherwise}, \end{aligned}$$

and it follows that

$$\text{cd } {}_p\mathbb{C} - \{p\} = \text{pd } K. \tag{4}$$

But from (3) we have

$$\text{pd } K = \text{pd } L_p(\mathbb{Z}) - 1. \tag{5}$$

Now suppose (2) holds. By [4, Lemma 3.1], $\text{pd } L_p(\mathbb{Z}) \geqslant 3$, for some $p \in \mathbb{C}$. Hence $\text{cd } {}_p\mathbb{C} - \{p\} \geqslant 2$. Using Eq. (5) of Section 1, we see that there exists

$q \in {}_p\mathbb{C} - \{p\}$ such that $\operatorname{cd}({}_p\mathbb{C} - \{p\})_q \geqslant 2$. Hence, by Lemma 2.1, $E({}_p\mathbb{C} - \{p\})_q$ contains a crown. Since it is easy to see that $\mathbb{C}$ contains a crown whenever $E(\mathbb{C})$ does, $({}_p\mathbb{C} - \{p\})_q$ contains a crown, and therefore $\mathbb{C}$ contains a suspended crown.

Define the *R-homological dimension* of $\mathbb{C}$ by

$$\operatorname{hd}_R\mathbb{C} = \sup\{k \mid \operatorname{colim}^k \neq 0\}$$

where colim^k is the kth left derived functor of the colimit (direct limit) functor colim: $(\operatorname{mod} R)^{\mathbb{C}} \to \operatorname{mod} R$. Latch and Mitchell have shown [1] that if $\mathbb{C}$ is a finite category, then $\operatorname{cd}_R\mathbb{C}^{\mathrm{op}} = \operatorname{hd}_R\mathbb{C}$. Hence the main theorem gives a characterization of the finite posets $\mathbb{C}$ such that $\operatorname{hd}_R\mathbb{C} \leqslant 1$.

In contrast to the situation for cohomological dimension one, the class of finite posets $\mathbb{C}$ satisfying $\operatorname{cd}_R\mathbb{C} \leqslant 2$ depends on the ring R. Examples which are obtained by ordering the cells of certain cell complexes under the inclusion relation are presented in [6].

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