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Homotopical smallness and closeness ☆

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The aim of this paper is to introduce the concepts of homotopical smallness and closeness. These are the properties of homotopical classes of maps that are related to recent developments in homotopy theory and to the construction of universal covering spaces for non-semi-locally simply connected spaces, in particular to the properties of being homotopically Hausdorff and homotopically path Hausdorff. The definitions of notions in question and their role in homotopy theory are supplemented by examples, extensional classifications, universal constructions and known applications.

1. Introduction

The concepts of homotopical smallness and closeness are related to various versions of the property of being homotopically Hausdorff, which have been introduced and studied in [5,6,20].

Definition 1. A space X is called:

(i) (weakly) **homotopically Hausdorff** if for every $x_0 \in X$ and for every non-trivial $\gamma \in \pi_1(X, x_0)$ there exists a neighborhood U of x_0 such that no loop in U is homotopic (in X) to γ rel. x_0;

(ii) **strongly homotopically Hausdorff** if for every $x_0 \in X$ and for every essential closed curve $\gamma \in X$ there is a neighborhood of x_0 that contains no closed curve freely homotopic (in X) to γ;

(iii) **homotopically path Hausdorff** if for every path $w:[0,1] \to X$ with $w(0) = P$ and $w(1) = Q$ and every non-trivial homotopy class $\alpha \in \pi_1(X, P)$ there exist finitely many open sets $U(P_1), \dots, U(P_k)$ ($P_1 = P$ and $P_k = Q$) covering $w([0,1])$ such that for a suitable partition $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_k = 1$, $U(P_j)$ covers $w([t_{j-1}, t_j])$, $P_j \in w([t_{j-1}, t_j])$ and such that for any path $v:[0,1] \to X$ that satisfies $v(0) = Q$, $v(1) = P$, $P_{k-j+1} \in v([t_{j-1}, t_j])$ and $v([t_{j-1}, t_j]) \subset U(P_{k-j+1})$, $\forall j$ the concatenation of w and v does not belong to the homotopy class α.

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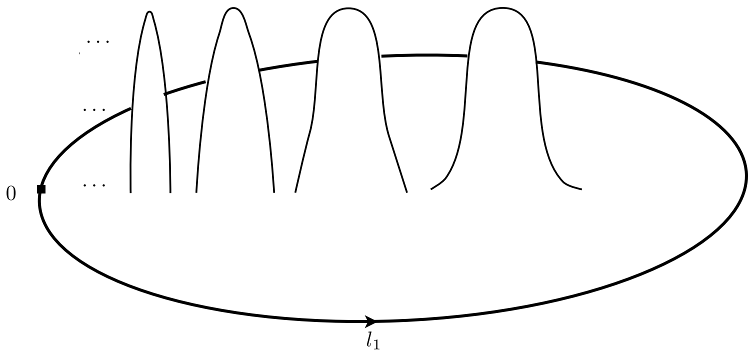


Fig. 1. Harmonic archipelago.

Note that the property of being homotopically Hausdorff is weaker than both the property of being strongly homotopically Hausdorff and the property of being homotopically path Hausdorff. These are separation properties for homotopical classes of maps and play a significant role in homotopy theory for locally wild spaces, for example, spaces which are not semi-locally simply connected, etc. Good examples of such spaces are Hawaiian earring (denoted by *HE*) and Harmonic archipelago (denoted by *HA*).

The *HE* is a countable metric wedge of circles (circular loops) whose diameters tend to zero, i.e.,

$$HE := \bigcup_{i\in\mathbb{Z}_+} S^1\left(\left(\frac{1}{i},0\right),\frac{1}{i}\right) \subset \mathbb{R}^2$$

where $S^1(C,r) \cong S^1$ is the circle in $\mathbb{R}^2$ with center C and radius r. Circles $S^1((\frac{1}{i},0),\frac{1}{i})$ are equipped with the positive (respectively negative) orientation and denoted by l_i (respectively l_i^-). The intersection of all circles is denoted by 0. It turns out that *HE* is homotopically Hausdorff but not semi-locally simply connected.

The Harmonic archipelago was defined in [2] and studied in [8]. In order to construct it begin with $HE \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. For each pair of consecutive loops (l_i, l_{i+1}) attach the disc B_i^2 in the following way: identify the boundary ∂B_i^2 with the loop $l_i * l_{i+1}^-$ and stretch the interior of B_i^2 up so that one of its interior points (called the peak point of B_i^2) is at height 1. The situation is presented in Fig. 1 where discs B_i^2 are represented by "bumps". It is easy to see that *HA* is not homotopically Hausdorff.

A detailed study of relationship between properties of Definition 1 is presented in [6] and [9]. The distinction between them is demonstrated by spaces Y, Y', Z, Z' of [9]. Essential parts of these spaces turn out to be a generic spaces where certain properties of smallness and closeness occur.

Properties of Definition 1 arise in connection to the universal path space.

Definition 2. Let (X, x_0) be a pointed path connected space. The **universal path space** $\widehat{X}$ is the set of equivalence classes of paths $\alpha : [0,1] \to X, \alpha(0) = x_0$ under the following equivalence relation: $\alpha \sim \beta$ iff $\alpha(1) = \beta(1)$ and the concatenation $\alpha * \beta^-$ (where $\beta^-(t) := \beta(1-t)$) is homotopic to a constant path at x_0, denoted by 1_{x_0}. The space $\widehat{X}$ is given a topology generated by the sets

$$N(U,\alpha) := \{\beta \mid \beta \simeq \alpha * \varepsilon,\ \varepsilon : ([0,1],0) \to (U,\alpha(1))\}$$

where U is an open neighborhood of $\alpha(1) \in X$. The natural endpoint projection $\hat{p} : \widehat{X} \to X$ is called the **endpoint map**.

Universal path space is called a **universal covering space** if the endpoint projection has the unique path lifting property.

The following are well-known facts that appear in [5,6,9,18,20,1,3,10].

Proposition 3. *Let (X, x_0) be a path connected space.*

(i) *X is semi-locally simply connected iff the fibers $\hat{p}^{-1}(x) \subset \widehat{X}$ of the endpoint projection are discrete subspaces for all $x \in X$.*
(ii) *X is homotopically Hausdorff iff the fibers $\hat{p}^{-1}(x) \subset \widehat{X}$ of the endpoint projection are Hausdorff subspaces for all $x \in X$.*
(iii) *If $\widehat{X}$ is a universal covering space then X is homotopically path Hausdorff.*
(iv) *If X is homotopically Hausdorff and $\pi_1(X, x_0)$ is countable then $\widehat{X}$ is a universal covering space.*
(v) *If X is homotopically path Hausdorff then $\widehat{X}$ is a universal covering space.*

The property of being homotopically Hausdorff is closely related to small loops, which were introduced and studied in [18]. In this paper we extend the approach of [18] in order to define smallness and closeness for a wider class of maps and relate new concepts to existing examples and properties. As a result we obtain the following classification.

Theorem 4. *Let X be a path connected space.*

(i) *A space X is homotopically Hausdorff if it contains no non-trivial pointed small loop.*
(ii) *A space X is strongly homotopically Hausdorff if it contains no non-trivial free small loop.*
(iii) *A locally path connected space X is homotopically path Hausdorff if there is no pair of paths in X that are close relatively to the endpoints of the interval.*

Statements (i) and (ii) are apparent from Definitions 31 and 16. Statement (iii) is the content of Proposition 48.

2. Technical preliminaries

We introduce several notions that will be used in the course of the paper. The following definition is a generalization of a classical concept of an absolute extensor which will be used to classify certain cases of smallness and closeness.

Definition 5. Let $A \subseteq X$ be a closed subspace and let Y be any topological space. Space Y is an **absolute extensor** for the inclusion $A \hookrightarrow X$ [notation: $(A \hookrightarrow X)\tau Y$ or $Y \in AE(A \hookrightarrow X)$] if every map $A \to Y$ extends over X.

Note that a path connected space Y is simply connected iff it is an absolute extensor for the inclusion $\partial B^2 \hookrightarrow B^2$.

The notion of an m-stratified space as defined in [18] is a description of construction rather than the property of a space as every space is m-stratified. It mimics the structure of *CW*-complexes by building spaces through attachment of smaller pieces via quotient maps.

Definition 6. Let $\{Y_i, A_i\}_{i \geqslant 0}$ be a countable collection of pairs of spaces where $A_i \subseteq Y_i$ is closed for every i. Topological space X is an **m-stratified** (map stratified) space with parameters $\{Y_i, A_i\}_i$ if it is homeomorphic to the direct limit of spaces $\{X_i\}_{i \geqslant 0}$ where spaces X_i are defined inductively as

- $X_0 := Y_0$,
- $X_i := X_{i-1} \cup_{f_i} Y_i$ for some maps $f_i : A_i \to X_{i-1}$.

The sets Y_i are called **m-strata**.

When applying the construction of an m-stratification we will usually adopt the notation of Definition 6. Lemma 7 has origins in the theory of *CW*-complexes presented in [13]. It describes the behavior of compact subsets with respect to an m-stratification.

Lemma 7. *([18]) Suppose Y is an m-stratified space so that m-strata Y_i can be decomposed as $Y_i = \coprod_j Y_i^j$ where $Y_i^j \subset Y_i$ are open regular subspaces (i.e. open subspaces which are regular topological spaces). Let $K \subset Y$ be a compact space. Define X_i^j to be the image of Y_i^j in Y. Then K is contained in a finite union of subsets $X_i^j \subset Y$.*

Another important property is related to extensions of maps. Any synchronized collection of maps on m-stratas induces a continuous map on Y.

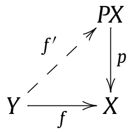
Lemma 8. *Let Y be an m-stratified space and let $g_i : Y_i \to Z$ be a collection of maps satisfying $g_i|_{A_i} = g_i|_{f_i(A_i)} \circ f_i$. Then maps g_i induce a continuous map on Y.*

The notion of a universal Peano space was defined in [4]. It allows us to study certain properties of a non-locally path connected space.

Definition 9. Let X be a path connected space. The **universal Peano space** (or Peanification) *PX* of X is the set X equipped with a new topology, generated by all path components of all open subsets of the existing topology on X. The **universal Peano map** is the natural bijection $p : PX \to X$.

Note that *PX* is locally path connected. As an example, the Peanification of the Warsaw circle is a semi-open interval. The name "universal Peano map" refers to the universal map lifting property for locally path connected spaces.

Proposition 10. *([4]) Let Y be a locally path connected space. Then every map $f : Y \to X$ uniquely lifts to a map $f' : Y \to PX$.*



Proof. Since p is bijection the only possible choice for f' is pf. Let us prove it is continuous. Choose $y \in Y$ and let $x = f(y)$. Every open neighborhood $U' \subset PX$ of $x' = p(x) \in PX$ is a path component of an open neighborhood $U \subset X$ of $x \in X$. The preimage $f(U)$ is an open neighborhood of y which contains an open path connected neighborhood W of y as Y is locally path connected. Then $f(W) \subset U$ is path connected and contains x hence $f'(W)$ is contained in U'. □

If Y is locally path connected then so is $Y \times [0, 1]$ which yields the following corollary.

Corollary 11. *Let Y be a locally path connected space and let X be a path connected space.*

(i) *The set of homotopy classes of maps $[Y, X]$ is in natural bijection with $[Y, PX]$.*
(ii) *The set of homotopy classes of maps $[Y, X]_\bullet$ in the pointed category is in natural bijection with $[Y, PX]_\bullet$.*
(iii) *$\pi_k(X) = \pi_k(PX)$, for all $k \in \mathbb{Z}^+$.*
(iv) *$H_k(X) = H_k(PX)$, for all $k \in \mathbb{Z}^+$.*

Proposition 10 implies that paths and homotopies between paths, on the base of which the universal path space is defined, are the same in X and PX.

Corollary 12. *The universal path spaces $\widehat{X}$ and $\widehat{PX}$ are homeomorphic for every path connected space (X, x_0).*

Given a path connected space X which is not locally path connected, Corollary 12 describes the information about the space which is lost in the construction of the universal path space. Essentially it is the same information that is lost in the construction of the universal Peano space. In particular, the universal Peano space PX has the same homotopy and homology groups as X but may have different shape group. On the other hand, spaces $\widehat{X}$ and PX are locally path connected even if the space X is not.

3. Homotopical smallness

The definition of small maps first appeared in [18] in the form of small loops.

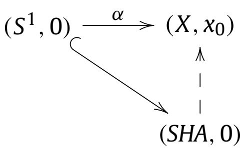
Definition 13. A loop $\alpha : (S^1, 0) \to (X, x_0)$ is **small** iff there exists a representative of the homotopy class $[\alpha]_{x_0} \in \pi_1(X, x_0)$ in every open neighborhood U of x_0. A small loop is a **non-trivial small loop** if it is not homotopically trivial.

Griffiths' space of [12] and HA of [2] are well-known spaces with non-trivial small loops. Another example is the strong Harmonic Archipelago SHA. The topology of SHA can be described in terms of m-stratified spaces with the following parameters (using the notation of Definition 6):

$$Y_0 = HE, \qquad Y_i = B_i^2, \qquad A_i = \partial B_i^2 = S_i^1, \qquad f_i = l_i l_{i+1}^- : S_i^1 \to HE.$$

Both HA and SHA are obtained from HE by attaching discs B_i^2 along loops $l_i l_{i+1}^-$. The difference is that in the case of SHA an infinite collection of discs $\{B_i^2\}$ is attached to HE by the quotient map (making it more natural as suggested by the proof of Proposition 14), while in the case of HA attachment is carried on in $\mathbb{R}^3$ so that the resulting space HA is metric. SHA is a generic example of a non-trivial small loop in a first countable in the sense of the following proposition, which can be proved using Lemma 8. Its generalization will be proven later.

Proposition 14. *Assume that $x_0 \in X$ has a countable basis of neighborhoods. A loop $\alpha : (S^1, 0) \to (X, x_0)$ is small iff it extends to $F : (SHA, 0) \to (X, x_0)$ where $l_1 : (S^1, 0) \hookrightarrow (SHA, 0)$ is the boundary loop.*



The same proposition can be proven for *HA* instead of *SHA* as well but the proof is somewhat more complicated. Another construction related to small loops are small loop spaces as defined, constructed and studied in [18].

Definition 15. A non-simply connected space X is a **small loop space** if for every $x \in X$, every loop $\alpha : (S^1, 0) \to (X, x)$ is small.

The following subsections generalize the notion of smallness and accompanying constructions to a general case in various categories.

3.1. Homotopical smallness in unpointed category

This subsection is devoted to homotopical smallness of arbitrary spaces in unpointed category. All homotopies and maps are considered to be unpointed (i.e. spaces have no basepoint and homotopies need not preserve any point).

We start with a definition of smallness in the unpointed category. The absence of a basepoint implies that we should specify a point at which we would like to consider smallness. By smallness we mean the property of being able to find a homotopic representative of a map in every neighborhood of a point.

Definition 16. A map $f : Y \to X$ is (homotopically freely) **small** at $x \in X$ (in unpointed topological category) if for each open neighborhood U of x there is a (free) homotopy $H : Y \times [0, 1] \to X$ so that $H|_{Y\times\{0\}} = f$ and $H|_{Y\times\{1\}}(Y) \subset U$. A small map is a **non-trivial small map** if it is not homotopically trivial.

Proposition 17. *Suppose $f : Y \to X$ is a small map at $x \in X$ and $g : f(Y) \to Z$ is a map. If g extends over X then $gf : Y \to Z$ is a small map at $g(x)$.*

For the rest of this section we will assume S to be a directed set with no maximal element (hence S is infinite) and the smallest (initial) element s_0, unless otherwise stated. Definition 18 introduces a generic examples of small maps which classify all small maps in terms of extension theory.

Definition 18. The **Sydney opera space** of S with respect to the space Y (in the topological category) [notation: $FSO_Y(S)$] is a space constructed in the following way.

Take a disjoint union $\coprod_{s\in S} Y_s$ of copies of space Y, one copy for each element in S. Upon this union attach spaces $W_s := Y \times [0, 1]$ for each $s \in S\backslash\{s_0\}$, so that $Y \times \{0\} \subset W_s$ is identified with Y_{s_0} and $Y \times \{1\} \subset W_s$ is identified with Y_s. Add another point $\{0\}$ to obtain the space $FSO_Y(S) := \bigcup_{s\in S\backslash\{s_0\}} W_s \cup \{0\}$ and define the following topology. The subset $U \subset FSO_Y(S)$ is open if either of the following is true:

(i) $0 \notin U$ and U is open in W_s, $\forall s \in S\backslash\{s_0\}$,
(ii) $0 \in U$, U is open in W_s, $\forall s \in S\backslash\{s_0\}$ and there exists $t_0 \in S$ such that $Y_t \subset U$, $\forall t \geqslant t_0$.

For a fixed directed set S with the initial element s_0 the rule $Y \mapsto FSO_Y(S)$ is a functor on the category of the topological spaces. Space $FSO_Y(S)$ can be given various structures of an m-stratified space. The simplest one would start with $\{0\} \cup \coprod_{s\in S} Y_s$ (with appropriate topology as described in Definition 18) upon which we attach homotopies W_s. Using the notation of Definition 6 the topology of $FSO_Y(S)$ can be expressed by the following parameters: $Y_0 = \{0\} \cup \coprod_{s\in S} Y_s$ (with topology described in Definition 18),

$$Y_1 = \coprod_{s\in S\backslash\{s_0\}} \big(Y \times [0,1]\big)_s, \qquad A_1 = \coprod_{s\in S\backslash\{s_0\}} \big(Y \times \{0,1\}\big)_s,$$

$$f_1|_{(Y\times\{0\})_s} = 1_{Y_0}, \qquad f_1|_{(Y\times\{1\})_s} 1_{Y_s}.$$

Note that $0 \in FSO_Y(S)$ is not path connected to Y_{s_0}.

Lemma 19. *The natural inclusion $Y \to Y_{s_0} \subset FSO_Y(S)$ is small at* 0.

Proof. Using homotopies W_s we can homotope the inclusion into arbitrary neighborhood of 0. □

If Y is contractible then the inclusion $Y \to Y_{s_0} \subset FSO_Y(S)$ is homotopically trivial. A necessary condition for such inclusion to be homotopically non-trivial is homotopical non-triviality of Y. Sufficient condition is given by Corollary 22.

Lemma 20. *Let $f : K \to FSO_Y(S)$ be a map from a compact space K to a regular space Y. Then $f(K)$ is contained in the subspace*

$$\bigcup_{s\in T} W_s \cup \bigcup_{s\in S} Y_s \cup \{0\}$$

where $T \subseteq S$ is some finite subset. Furthermore, such f factors over $\bigcup_{s\in S} Y_s \cup \{0\} \hookrightarrow FSO_Y(S)$ up to homotopy.

Proof. The first part follows by Lemma 7. To prove the second part consider a strong deformation retraction

$$\bigcup_{s\in T} W_s \cup \bigcup_{s\in S} Y_s \cup \{0\} \to \bigcup_{s\in S-T} Y_s \cup Y_{s_0} \cup \{0\}. \qquad \square$$

Lemma 21. *Let $f: K \to Y_{s_0} \subset FSO_Y(S)$ be a map from a compact space K to a regular space Y and suppose $H: K\times[0,1] \to FSO_Y(S)$ is a homotopy so that $H|_{K\times\{0\}} = f$. Then $H(K\times[0,1])$ is contained in the subspace*

$$\bigcup_{s\in T} W_s$$

where $T \subseteq S$ is some finite subset.

Proof. By Lemma 20 the compact set $H(K\times[0,1])$ is contained in

$$A := \bigcup_{s\in T} W_s \cup \bigcup_{s\in S} Y_s \cup \{0\}$$

for some finite $T \subseteq S$. Note that $H(K\times[0,1])$ is connected to Y_{s_0} by paths as $H|_{K\times\{0\}} \subset Y_{s_0}$. Sets $\{0\}$ and $Y_t \cap (K\times[0,1])$ for $t \in S-T$ are not path connected to Y_{s_0} in A hence $H(K\times[0,1]) \subseteq \bigcup_{s\in T} W_s$. $\square$

Corollary 22. *Let Y be a compact Hausdorff space which is not homotopically trivial. Then the natural inclusion $i: Y \to Y_{s_0} \subset FSO_Y(S)$ is a homotopically non-trivial small map at 0.*

Proof. Suppose there is a homotopy H in $FSO_Y(S)$ between the inclusion i and a constant map. Because Y is compact such homotopy is a compact map therefore its image is contained in $\bigcup_{t\in T} W_s$ where $T \subseteq S$ is a finite subset. Space $\bigcup_{t\in T} W_t$ can be naturally retracted to Y_{s_0}. Composing homotopy H with such retraction we contradict the fact that Y is homotopically non-trivial. $\square$

Similarly as in the case of small loops we can classify small maps in terms of extension theory.

Proposition 23. *Let S be a directed set with the smallest element s_0 so that for a point $x \in X$ there is a basis $\{U_s\}_{s\in S}$ that satisfies $(U_s \subseteq U_t)$ iff $(s \geqslant t)$. Map $f: Y_{s_0} \to X$ is small at $x \in X$ iff it extends over $FSO_Y(S)$ to a map F so that $F(0) = x$.*

Proof. We only have to prove one direction. Suppose the map $f: Y_{s_0} \to X$ is small at $x \in X$. For each $s \in S$ there is a homotopy H between f and a map whose image is contained in U_s. Use such homotopy to naturally define the map F on W_s and additionally define $F(0) := x$. This rule defines a continuous map on $FSO_Y(S) - \{0\}$ as the topology on it is quotient. The preimage $F^{-1}(U_t)$ of any basic open neighborhood U_t of x is open in $W_s, \forall s \in S$, and contains entire Y_s for all $s \geqslant t$ therefore it is open in $FSO_Y(S)$. Hence the extension F is continuous. $\square$

Definition 24. Let Y be a topological space. Space X is called a **small Y-space** if the following conditions hold:

(i) There exists a map $Y \to X$ which is not homotopically trivial.
(ii) Every map $Y \to X$ is small at every $x \in X$.

Corollary 25. *Let S be a directed set with the smallest element s_0 so that for $x \in X$ there is a basis $\{U_s\}_{s\in S}$ that satisfies $(U_s \subseteq U_t)$ iff $(s \geqslant t)$. Suppose there exists a map $Y \to X$ which is not homotopically trivial. Space X is a small Y-space iff $((Y_{s_0} \cup \{0\}) \hookrightarrow FSO_Y(S))\tau X$.*

The rest of this subsection is devoted to the existence of a small Y-space for non-contractible compact Hausdorff spaces. With these properties we can imitate the construction of a small loop space of [18]. The Hausdorff property of Y implies that Y and $FSO_Y(\mathbb{N})$ are regular spaces which allows Lemma 7 to be used in the case of $FS(Y)$. The following definition introduces a generic example of a small Y-space.

Definition 26. Let Y be a topological space. The space $\boldsymbol{FS(Y)}$ is an m-stratified space with

$$Y_0 = FSO_Y(\mathbb{N}), \qquad S_i := \{(g,x);\ g: Y \to X_{i-1},\ x \in X_{i-1}\},$$
$$Y_i := \coprod_{(g,x)\in S_i} FSO_Y(\mathbb{N})_{g,x}, \qquad A_i := \coprod_{(g,x)\in S_i} \big(Y_0 \cup \{0\}\big)_{g,x},$$
$$f_i(0_{g,x}) = x, \qquad f_i|_{(Y_0)_{g,x}} = g,$$

where $(Y_0)_{g,x} \subset FSO_Y(\mathbb{N})_{g,x}$ is the initial copy of Y in $FSO_Y(\mathbb{N})_{g,x}$.

Lemma 27. *Suppose Y is a compact Hausdorff space. Every map $f : Y \to FS(Y)$ is small.*

Proof. Choose any $x \in FS(Y)$. By Lemma 7 there is $k \in \mathbb{N}$ so that $f(Y) \subset X_k$ and $x \in X_k$ according to m-stratification of $FS(Y)$. Then f can be made small in X_{k+1} via the attached space $FSO_Y(\mathbb{N})_{f,x}$. □

Proposition 28. *Suppose Y is a compact Hausdorff space which is not homotopically trivial. Then the natural inclusion $f : Y \to Y_{s_0} \subset X_0 \subset FS(Y)$ is homotopically non-trivial.*

Proof. Suppose there is a homotopy H taking f to a constant map. Using Lemmas 7 and 21 one can construct a retraction of $H(Y \times I)$ to Y_{s_0}. Composing such retraction with H would imply that Y is homotopically trivial, which is a contradiction. □

Corollary 29. *If the space Y is compact Hausdorff and homotopically non-trivial then $FS(Y)$ is a small Y-space.*

Space $FS(Y)$ is a universal example of a small Y-space in the following way.

Proposition 30. *Suppose $f : Y_{s_0} \to X$ is a map to a small Y-space X where $Y_{s_0} \subset X_0 \subset FS(Y)$. Then f extends over $FS(Y)$.*

Proof. Follows from Lemma 8. □

3.2. Homotopical smallness in pointed category

The aim of this subsection is to develop similar results for smallness in the pointed category. Recall that pointed homotopy is a homotopy that fixes the base point. All spaces, maps and homotopies of this section are considered to be in the pointed topological category.

Definition 31. A map $f : (Y, y_0) \to (X, x_0)$ between pointed topological spaces is (homotopically) **small** (in the pointed topological category) if for each open neighborhood U of x_0 there exists a (pointed) homotopy (i.e. $H : (Y \times [0,1], (y_0, 0)) \to (X, x_0)$, $H(y_0, t) = x_0$, $\forall t \in [0,1]$) so that $H|_{Y\times\{0\}} = f$ and $H|_{Y\times\{1\}}(Y \times \{1\}) \subset U$. A small map is a **non-trivial small map** if it is not homotopically trivial.

Proposition 32. *Let $f : (Y, y_0) \to (X, x_0)$ be a small map and let $g : (f(Y), x_0) \to (Z, z_0)$ be a map. If g extends over X then $gf : (Y, y_0) \to (Z, z_0)$ is small.*

The following definition introduces an analogue of the *FSO* spaces in the pointed category. Recall that S is assumed to be a directed set with no maximal element and the smallest element s_0.

Definition 33. The **Sydney opera space** [notation: $SO_Y(S)$] of S with respect to the space (Y, y_0) (in the pointed topological category) is a topological space constructed in the following way.

Consider the wedge $\bigvee_{s\in S}(Y_s, y_0)$ of $|S|$ copies of the space Y, one copy for each element of S, obtained by identifying the base points y_0 of the spaces (Y_s, y_0). Define its basepoint to be the wedge point and denote it by y_0 as well. On this wedge attach the spaces $W_s := Y \times [0,1]$ for each $s \in S\backslash\{s_0\}$, so that $Y \times \{0\} \subset W_s$ is identified with Y_{s_0}, $Y \times \{1\} \subset W_s$ is identified with Y_s and $\{y_0\} \times I$ is identified with $y_0 \in \bigvee_{s\in S\backslash\{s_0\}}(Y_s, y_0)$. Define $SO_Y(S)$ to be the set $\bigcup_s W_s$ with the following topology. A subset $U \subset SO_Y(S)$ is open if either of the following is true:

(i) $y_0 \notin U$ and U is open in W_s, $\forall s \in S\backslash\{s_0\}$,
(ii) $y_0 \in U$, U is open in W_s, $\forall s \in S\backslash\{s_0\}$ and there is $t_0 \in S$ so that $Y_t \subset U$, $\forall t \geqslant t_0$.

For a fixed directed set S with the initial element s_0 the rule $Y \mapsto SO_Y(S)$ is a functor in the category of the pointed topological spaces. The space $SO_Y(S)$ is a natural quotient of $FSO_Y(S)$ and can be given various structures of an m-stratified space. The simplest one is almost identical to the one of $FSO_Y(S)$. Start with the wedge $\bigvee_{s\in S}(Y_s, y_0)$ (with appropriate topology as described in Definition 33) upon which we attach homotopies W_s. Using the notation of Definition 6 the topology of $SO_Y(S)$ can be expressed by the following parameters: $Y_0 = \coprod_{s\in S} Y_s$ (with topology described in Definition 33),

$$Y_1 = \coprod_{s\in S\backslash\{s_0\}} \big(Y \times [0,1]\big)_s, \qquad A_1 = \coprod_{s\in S\backslash\{s_0\}} \big(Y \times \{0,1\} \cup \{y_0\} \times [0,1]\big)_s,$$

$$f_1|_{(Y\times\{0\})_s} = 1_{Y_0}, \qquad f_1|_{(Y\times\{1\})_s} = 1_{Y_s} \qquad f_1\big(\big(\{y_0\} \times [0,1]\big)_s\big) = \{y_0\}.$$

The topology of an m-stratified space implies that the natural inclusion $(Y, y_0) \cong (Y_{s_0}, y_0) \subset (SO_Y(S), y_0)$ is small. The nature of compact subsets implies that such inclusion homotopically non-trivial if Y is not contractible.

Lemma 34. *Let S be a directed set with the smallest element s_0. The natural inclusion $(Y, y_0) \to (Y_{s_0}, y_0) \subset (SO_Y(S), y_0)$ is small.*

Proof. Use homotopies W_s. □

Lemma 35. *Suppose $f:(K,k_0) \to (SO_Y(S), y_0)$ is a map defined on a compact Hausdorff space (K,k_0). Then $f(K)$ is contained in the subspace*

$$\bigcup_{s\in T} W_s \cup \bigcup_{s\in S} Y_s$$

where $T \subset S$ is a finite subset. Furthermore, such f factors over $(\bigcup_{s\in S} Y_s, y_0) \hookrightarrow (SO_Y(S), y_0)$ up to homotopy.

Proof. The first part follows by Lemma 7. To prove the second part consider the strong deformation retraction

$$\bigcup_{s\in T} W_s \cup \bigcup_{s\in S} Y_s \to \bigcup_{s\in S-T} Y_s \cup Y_{s_0}. \qquad \square$$

Corollary 36. *Let (Y, y_0) be a compact Hausdorff, homotopically non-trivial space. The natural inclusion $i:(Y, y_0) \to (Y_{s_0}, y_0) \subset (SO_Y(S), y_0)$ is a homotopically non-trivial small map.*

Proof. Suppose there is a pointed homotopy in $SO_Y(S)$ between i and a constant map. The space (Y, y_0) is compact hence the image of such homotopy is contained in $\bigcup_{s\in T} W_s \cup \bigcup_{s\in S-T} Y_s$ where $T \subseteq S$ is some finite subset. The subspace $\bigcup_{t\in T} W_T \subset SO_Y(S)$ can be retracted to Y_{s_0} and the subspace $\bigcup_{s\in S-T} Y_s \subset SO_Y(S)$ can be retracted to y_0. Composing the homotopy with these retractions we obtain a contraction of (Y, y_0), a contradiction. □

Proposition 37. *Let S be a directed set with the smallest element s_0 so that $x \in X$ has a basis of neighborhoods $\{U_s\}_{s\in S}$ that satisfies $(U_s \subseteq U_t)$ iff $(s \geqslant t)$. The map $f:(Y_{s_0}, y_0) \to (X, x_0)$ is small iff it extends over $SO_Y(S)$.*

Proof. We only have to prove one direction. Suppose the map $f:(Y_{s_0}, y_0) \to (X, x_0)$ is small. For each $s \in S$ there is a homotopy H between f and a map with its image contained in U_s. Use such homotopy to naturally define a map on W_s. With this rule we have defined a continuous map on $SO_Y(S) - \{y_0\}$ by the definition of topology. The preimage of any basic open neighborhood U_t, $t \in S$ of x is open in W_s, $\forall s \in S$, and contains all W_s, $s \geqslant t$ therefore it is open. Hence the extension is continuous. □

Definition 38 introduces a small Y-space. It is followed by the extensional classification and a construction of such space. Small Y-space is a generalization of a small loop space which was shown to have interesting properties in [18].

Definition 38. Let (Y, y_0) be a topological space. We call X a **small Y-space** if the following conditions hold:

(i) There exists a map $f:(Y, y_0) \to (X, f(y_0))$ which is not homotopically trivial.
(ii) Every map $f:(Y, y_0) \to (X, f(y_0))$ is small.

Corollary 39. *Let X be a topological space and suppose S is a directed set with the smallest element s_0 so that for every $x \in X$ there exists a basis $\{U_s\}_{s\in S}$ of open neighborhoods of x satisfying $(U_s \subseteq U_t)$ iff $(s \geqslant t)$. Suppose there exists a homotopically non-trivial map $f:(Y, y_0) \to (X, f(y_0))$. Space X is a small Y-space iff $(Y_{s_0} \hookrightarrow SO_Y(S))\tau X$.*

Definition 40. Let (Y, y_0) be a topological space. The space $\boldsymbol{S(Y)}$ is an m-stratified space with

$$Y_0 = SO_Y(\mathbb{N}), \qquad S_i := \{g;\ g: Y \to X_{i-1}\},$$
$$Y_i := \coprod_{g\in S_i} SO_Y(\mathbb{N})_g, \qquad A_i := \coprod_{g\in S_i} (Y_0)_g, \qquad f_i = \coprod_{g\in S_i} g,$$

where $(Y_0)_{g,x} \subset SO_Y(\mathbb{N})_{g,x}$ is the initial copy of Y in $SO_Y(\mathbb{N})_g$.

Lemma 41. *If (Y, y_0) is a compact Hausdorff space then every $f:(Y, y_0) \to (S(Y), f(y_0))$ is small.*

Proof. By Lemma 7 there exists $k \in \mathbb{N}$ so that $f(Y) \subset X_k$. Pointed map $f:(Y, y_0) \to (S(Y), f(y_0))$ is small in X_{k+1} due to attached space $SO_Y(\mathbb{N})_f$. □

Proposition 42. *The natural inclusion $f:(Y, y_0) \to (Y_{s_0}, y_0) \subset (X_0, y_0) \subset (S(Y), y_0)$ is homotopically non-trivial in $S(Y)$ if Y is compact Hausdorff and homotopically non-trivial.*

Proof. Suppose there exists a homotopy H between f and a constant map. By Lemma 35 the homotopy H factors over $\bigcup_{s\in T} W_s \cup \bigcup_{s\in S-T} Y_s$ for some finite subset $T \subset S$. Observe that there exists a retraction of $\bigcup_{s\in T} W_s \cup \bigcup_{s\in S-T} Y_s$ to Y_{s_0}: retract W_s to Y_{s_0} for $s \in T$ and contract the rest to y_0. The composition of H with such retraction implies that Y is homotopically trivial, a contradiction. □

Corollary 43. *Space $S(Y)$ is a small Y-space if Y is compact Hausdorff and homotopically non-trivial.*

Space $S(Y)$ is a universal example of a small Y-space in the following way.

Proposition 44. *Suppose $f:(Y_{s_0}, y_0) \to (X, x_0)$ is a map to a small Y-space (X, x_0) where $Y_{s_0} \subset X_0 \subset FS(Y)$. Then f extends over $(FS(Y), y_0)$.*

4. Homotopical closeness

Homotopical closeness is a concept which generalizes homotopical smallness. Its development is motivated by considering how close the two loops are in some space. Roughly speaking, loop α is close to loop $\beta \not\simeq \alpha$ if for each $\varepsilon > 0$ there exists a homotopic representative α_ε of α so that $d(\alpha_\varepsilon(t), \beta(t)) < \varepsilon$, $\forall t$. In other words, there is no pair of close loops if the following condition holds: whenever there are homotopic loops α_ε so that $\alpha_\varepsilon(t) \xrightarrow{\varepsilon\to 0} \alpha(t)$ then $\alpha_\varepsilon \simeq \alpha$. This condition is related to the property of being π_1-shape injective (or just shape injective) due to [5] and to the property of being homotopically path Hausdorff.

4.1. On homotopical smallness and closeness

The aim of this section is to discuss some issues concerning the relationship between homotopical smallness and closeness. The following is the definition of closeness we employ for the future use.

Definition 45. Let $A \subset Y$ be a closed subspace of Y and let (X, d) be a metric space. Map $f:Y \to X$ is (homotopically) **close** to the map $g:Y \to X$ relatively to A (denoted by $\operatorname{rel} A$) if the following conditions hold:

(a) f is not homotopic to g, $\operatorname{rel} A$ (i.e. there exists no homotopy between f and g that fixes all points of A);
(b) for each $\varepsilon > 0$ there exists a homotopy $H_\varepsilon : Y \times [0,1] \to X$ so that
  (i) $H_\varepsilon|_{Y\times\{0\}} = f$;
  (ii) $H_\varepsilon(a, t) = g(a)$, $\forall a \in A$, $\forall t \in [0,1]$;
  (iii) $d(H_\varepsilon(y,1), g(y)) < \varepsilon$, $\forall y \in Y$.

The first observation is that (homotopical) closeness is only considered in metric spaces. The reason is that, roughly speaking, we want to obtain homotopically equivalent maps $f_n : Y \to X$ that point-wise uniformly converge to a map $f:Y \to X$ which is not homotopically equivalent to any f_n. The structure of a metric space was not required in the case of smallness (i.e. closeness to a constant map) as we were only considering convergence towards one point. On the other hand the definition of closeness contains convergence of sequences with potentially different limit points. This generalization also allows us to consider closeness relatively to a subset A. Smallness could only be considered relatively to $\emptyset$ or a basepoint (yielding pointed and unpointed smallness).

Another issue is the invariance of closeness and smallness. Every small map $Y \to X$ is topologically invariant (i.e. smallness is preserved by homeomorphisms of X). On the other hand the closeness is not preserved by homeomorphisms as suggested by the following example. Consider planar spaces X and Z (see Fig. 2) defined by the following rule

$$X := \{x > 0,\ y = 0\} \cup \{x = 0,\ y > 1\} \cup \bigcup_{n\in\mathbb{Z}_+} \left\{x = \frac{1}{n},\ y > 0\right\},$$

$$Z := \{x > 0,\ y = 0\} \cup \{x = 0,\ y > 1\} \cup \bigcup_{n\in\mathbb{Z}_+} \left\{y = nx - 1;\ x > \frac{1}{n}\right\}.$$

Observe that there exists a homeomorphism $h: X \to Z$ which fixes the subset $\{x > 0,\ y = 0\} \cup \{x = 0,\ y > 1\}$ and linearly maps $\{x = \frac{1}{n},\ y > 0\}$ to $\{y = nx - 1;\ x > \frac{1}{n}\}$. Consider the map $f:\mathbb{R}_+ \to X$ defined by $f(t) := (1, 1+t)$. Note that f is close (but not homotopic) to the map g defined by $g(t) := (0, 1+t)$. On the other hand hf is not close to hg.

However, closeness is Lipschitz invariant and closeness of maps in a compact space is topologically invariant as proved by the following statements.

Proposition 46. *Let $f:Y \to X$ be close to $g:Y \to X$ (in a metric space X) and suppose a map $h: X \to Z$ is uniformly continuous. Then hf is either close or homotopic to hg.*

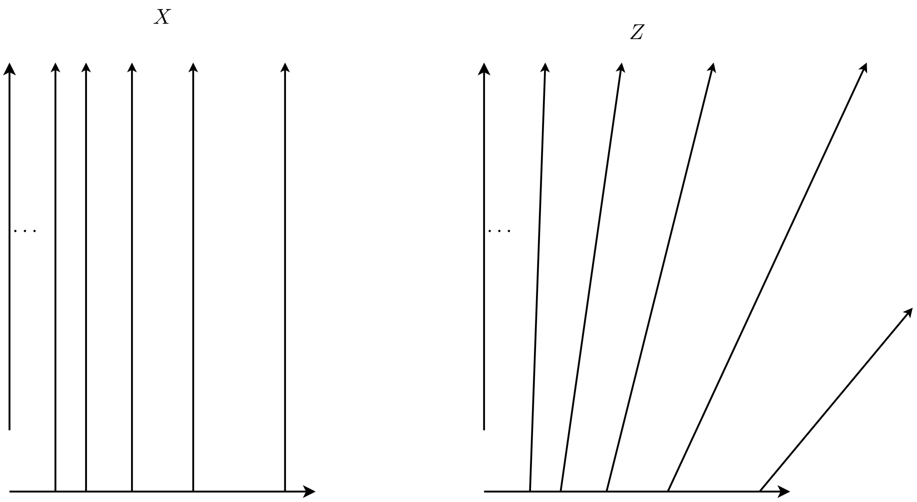


Fig. 2. Spaces used to disprove the topological invariance of closeness.

Proof. For every $\varepsilon > 0$ let δ_ε denote a positive number so that if $d_X(x, y) < \delta_\varepsilon$ then $d_Z(f(x), f(y)) < \varepsilon$. Consider homotopies H_ε according to Definition 45. Given a homeomorphism h the homotopies $\widetilde{H}_\varepsilon := hH_{\delta_\varepsilon}$ satisfy condition (ii) of Definition 45. □

Corollary 47. *Let $f : Y \to X$ be close to $g : Y \to X$ (in a metric space X) and let $h : X \to Z$ be a map. If X is compact Hausdorff or h is Lipschitz then hf is either close or homotopic to hg.*

Another observation is related to the nature of closeness as a relation, i.e. the absence of symmetry. Note that f being close to g does not imply g being close to f. The reason is that the definition of closeness (f being close to g) requires homotopic representative of f converging to the map g and not only to homotopic representative of g. The relaxation of condition of closeness as suggested by the last sentence would yield a symmetric (and transitive) relation of closeness. However, such relaxation would change the nature of closeness (and smallness) drastically. In particular, every two maps to a graph of function $f(x) = x^{-2}$ would be either homotopic or close. Such relaxation would redefine small loops (within the unpointed category) in the following way: a loop is small iff it has a homotopic representative of diameter at most ε for every $\varepsilon > 0$. This would mean that the punctured open disc is a small loop space but an open annulus is not hence smallness would not be a topological invariant. Also, the absence of close paths relatively to the endpoints would not coincide with the concept of the property of being homotopically path Hausdorff as proved by Proposition 48. For these reasons the definition of closeness is not symmetric.

The notion of small loops (in the pointed category) is closely related to the property of being homotopically Hausdorff (i.e. it is equivalent to the absence of small loops). In a similar fashion the closeness of paths relatively to the endpoints is related to the property of being homotopically path Hausdorff in a locally path connected space.

Proposition 48. *A locally path connected metric space X has the property of being homotopically path Hausdorff iff there are no close paths $[0, 1] \to X$ relatively to the endpoints of the interval.*

Proof. Suppose X is not homotopically path Hausdorff. According to Definition 1 there exist paths $w, v : [0, 1] \to X$, $v \not\simeq w \operatorname{rel}\{0, 1\}$ with the following property: for any chosen $n \in \mathbb{Z}_+$ and a cover of $w([0, 1])$ by open sets of diameter at most $\frac{1}{n}$ the conditions of Definition 1 are not satisfied due to some path v_n homotopic to v relatively to $\{0, 1\}$. In particular, $d(w(t), v_n(t)) < \frac{1}{n}$, $\forall t \in [0, 1]$ as $w(t)$ and $v_n(t)$ are contained in a set of diameter at most $\frac{1}{n}$. This implies that v is close to w relatively to $\{0, 1\}$.

To prove the other direction we use local path connectedness of X. Suppose the path $v : [0, 1] \to X$ is close to the path w relatively to $\{0, 1\}$. Choose any cover of $w([0, 1])$ by finitely many path connected open sets $U(P_1), \dots, U(P_k)$ and any partition $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_k = 1$ so that:

(i) $U(P_j)$ covers $w([t_{j-1}, t_j])$;
(ii) $P_j \in w([t_{j-1}, t_j])$.

There exists $\varepsilon > 0$ so that the ε-neighborhood of $w([t_{j-1}, t_j])$ is contained in $U(P_j)$, $\forall j$. The closeness of v to w relatively to $\{0, 1\}$ allows us to choose a path $v' \simeq v \operatorname{rel}\{0, 1\}$ so that $d_X(v'(t), w(t)) < \varepsilon$ hence $U(P_j)$ covers $v'([t_{j-1}, t_j])$. Since the sets

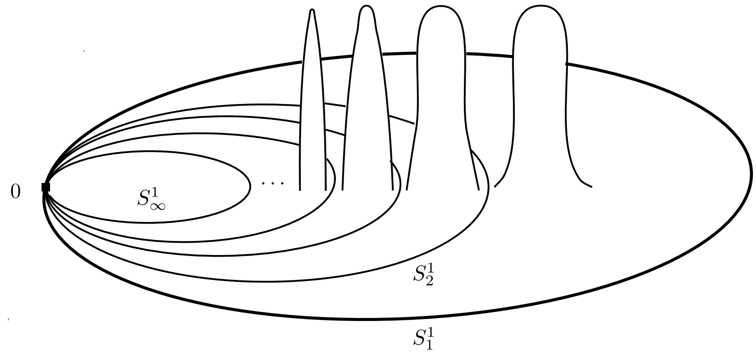


Fig. 3. The space $C(S^1,\{0\})$.

$U(P_j)$ are path connected we can (for each j) redefine $v'|_{[t_{j-1},t_j]}$ so that we do not change the homotopy type relatively to $\{t_{j-1},t_j\}$, $P_j \in v'([t_{j-1},t_j])$ and $U(P_j) \supset v'([t_{j-1},t_j])$. Such path v' contradicts Definition 1 for the loop $\alpha = w * v^-$ hence X is not homotopically path Hausdorff. □

If the space X is not locally path connected then close paths need not contradict the property of being homotopically path Hausdorff. This fact is connected to the following observation. If a path $f:[0,1]\to X$ is close to the path g (relatively to $\{0,1\}$) then:

(i) f is close to g in the Peanification *PX* if X is locally path connected.
(ii) f may not be close to g in the Peanification *PX* (for some metric on *PX*) if X is not locally path connected.

Statement (i) is obvious as $X = PX$ in the case of a locally path connected space. Note that statement (ii) requires a structure of metric space on *PX* in order to consider closeness. To prove statement (ii) we construct the space $C(S^1,\{0\})$ which is a modification of *HA*. Recall that *HA* is constructed with the aim to create a small loop. In order to do this we attach big homotopies along the loops converging to a point. The construction of $C(S^1,\{0\})$ follows the same philosophy for closeness. We attach big homotopies along loops converging to another loop (rather than a point). Recall that $0 = (0,0) \in \mathbb{R}^2$ and $S^1(S,r)$ denotes a circle in $\mathbb{R}^2$ with center S and radius r. Define

$$S^1_n := S^1\left(\left(1+\frac{1}{n},1\right),1+\frac{1}{n}\right) \quad \text{for } n\in\mathbb{Z}_+, \qquad S^1_\infty := S^1\big((1,0),1\big).$$

Naturally embed $\bigcup_{n\in\mathbb{Z}_+} S^1_n \cup S^1_\infty$ in $\mathbb{R}^3$ and attach spaces $A_n = (S^1\times[0,1])_n$ for all $n\in\mathbb{Z}_+$ so that:

(i) we identify $(S^1\times\{0\})_n$ with S^1_n;
(ii) we identify $(S^1\times\{1\})_n$ with S^1_{n+1};
(iii) we identify $(\{0\}\times[0,1])_n$ with 0;
(iv) the rest of A_n is stretched up so that it reaches height (z-coordinate) 1 (i.e. A_n has a point which is of distance at least 1 from S^1_n and S^1_{n+1}).

In other words, we attach big homotopies between loops S^1_n as suggested by Fig. 3. Since closeness is only defined in a metric space we cannot attach all A_n by quotient maps (as in *SHA*) but rather within a metric space $\mathbb{R}^3$.

Remark. The conditions above about the nature of the attached homotopies A_n do not uniquely define the space $C(S^1,\{0\})$. The reason is, roughly speaking, that the homotopies A_n approach the loop S^1_∞ rather than just a point. For example, the homotopies A_n may be chosen so that for any given $x\in S^1_\infty - \{0\}$ the space either is or is not locally path connected at x. Fig. 3 suggests that $C(S^1,\{0\})$ is locally path connected everywhere and that there are small loops at $(1,0,0)\in S^1_\infty$ as the humps (i.e. subspaces of A_n with z-coordinate at least 1) of A_n converge to that point. In order to comply with later definitions we demand the humps of $C(S^1,\{0\})$ to converge to entire S^1_∞ so that $C(S^1,\{0\})$ is not locally path connected at any point of $S^1_\infty - \{0\}$. For an alternative description of $C(S^1,\{0\})$ see Definition 54.

Note that the loops S^1_n are homotopic to each other relatively to 0 via homotopies A_n but these homotopies cannot be combined to obtain a homotopy to the limit loop S^1_∞. Similarly as in the case of *HA* we can prove that the map $(S^1,0)\to(S^1_1,0)\subset C(S^1,\{0\})$ is close to the map $(S^1,0)\to(S^1_\infty,0)\subset C(S^1,\{0\})$ relatively to 0. In both cases we consider a map of the form $e^{i\varphi}\mapsto(A+Be^{i\varphi},0)$. However, the Peanification of $C(S^1,\{0\})$ is a wedge of S^1 and $(S^1\times[0,1)\cup\{0\}\times\{1\})$ hence it contains no close loops. The space $C(S^1,\{0\})$ is an example of a homotopically path Hausdorff space with close loops.

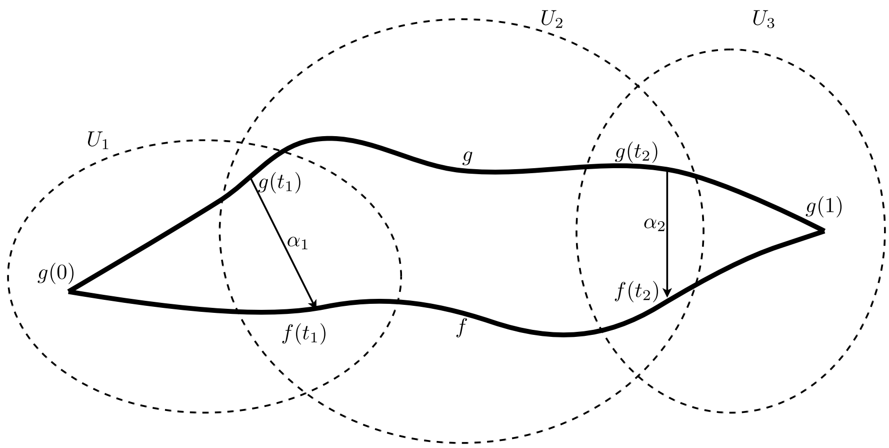


Fig. 4. Notation concerning close paths f and g.

The last observation is related to the Spanier group of a space. Groups π^s and π^{sg} are generated by small loops and defined in [18]. Close loops have no influence on these groups but may interfere with the Spanier group. The following is the definition of a Spanier group for locally path connected spaces as presented in [9] and [16].

Definition 49. Let (X, x_0) be a locally path connected space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of X by open neighborhoods. Define $\pi_1(\mathcal{U}, x_0)$ as the subgroup of $\pi_1(X, x_0)$ consisting of the homotopy classes of loops that can be represented by a product (concatenation) of the following type:

$$\prod_{j=1}^{n} u_j * v_j * u_j^-,$$

where u_j are paths that run from x_0 to a point in some U_i and each v_j is a closed path inside the corresponding U_i based at the endpoint of u_j. We call $\pi_1(\mathcal{U}, x_0)$ the **Spanier group of** (X, x_0) **with respect to** $\mathcal{U}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be an open covers of X and let $\mathcal{U}$ be a refinement of $\mathcal{V}$. Then $\pi_1(\mathcal{U}, x_0) \subset \pi_1(\mathcal{V}, x_0)$. This inclusion relation induces an inverse limit defined via the directed system of all covers with respect to refinement. We will call such limit the **Spanier group** of the space X and denote it by $\pi_1^{sp}(X, x_0)$.

Proposition 50. *Let (X, x_0) be a locally path connected space.*

(i) $\pi_1^{sg}(X, x_0) \subset \pi_1^{sp}(X, x_0)$.
(ii) *If $f : ([0, 1], 0) \to (X, x_0)$ is close to g relatively to $\{0, 1\}$ in a metric space (X, d) then $[f * g^-] \in \pi_1^{sp}(X, x_0)$.*

Proof. Claim (i) is true as every element of $\pi_1^{sg}(X, x_0)$ is contained in each $\pi_1(\mathcal{U}, x_0)$ by the definition.

To prove claim (ii) we partially imitate the proof of Proposition 48. Fix a cover $\mathcal{U}$ of X and choose a finite subfamily $U_1, \dots, U_k \subset \mathcal{U}$ covering $g([0, 1])$ so that for some partition $0 = t_0 < t_1 < \cdots < t_k = 1$ the set U_j contains $g([t_{j-1}, t_j])$, $\forall j$. There exists $\varepsilon > 0$ so that for every j:

- ε-neighborhood of $g([t_{j-1}, t_j])$ is contained in U_j;
- every point of the ε-neighborhood of $g(t_j)$ is connected to $g(t_j)$ by a path in $U_j \cap U_{j+1}$.

We can assume $d_X(f(t), g(t)) < \varepsilon$, $\forall t$. For each j let α_j denote an oriented path in $U_j \cap U_{j+1}$ between $g(t_j)$ and $f(t_j)$ as denoted by Fig. 4. We can assume α_0 and α_k to be constant paths. Observe that the oriented loop Q_j defined as a concatenation

$$\alpha_{j-1} * f|_{[t_{j-1},t_j]} * \alpha_j^- * (g|_{[t_{j-1},t_j]})^-$$

is based at $g(t_j)$ and contained in U_j. The class $[f * g^-]$ is contained in $\pi_1(\mathcal{U}, x_0)$ because it can be expressed as

$$\prod_{j=1}^{k} g|_{[0,t_{j-1}]} * Q_j * (g|_{[0,t_{j-1}]})^-.$$

Hence $[f * g^-] \in \pi_1^{sp}(X, x_0)$. $\square$

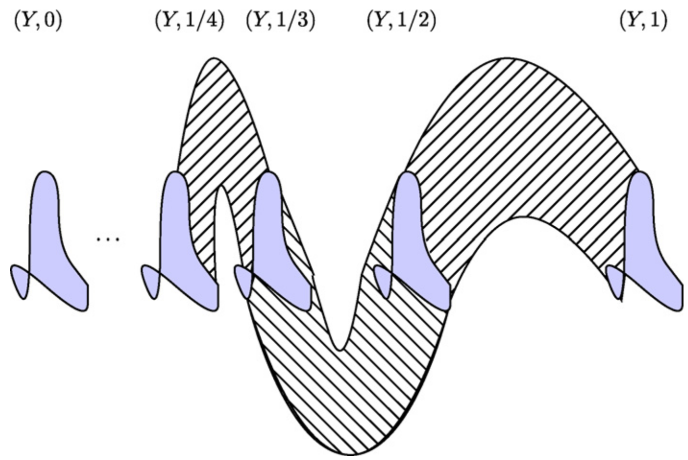


Fig. 5. Space $C(Y)$.

4.2. Standard constructions

In this subsection we present some aspects of closeness which are motivated by similar results on smallness. Since the closeness is only defined in metric spaces the construction of an m-stratified space and some other features of smallness are not applicable. The absence of these obstruct the generalization of some constructions including the small loop space. However it is possible to construct the space $C(Y,A)$ with maps $f,g:Y\to C(Y,A)$ for which the map f is close to g relatively to A.

Given a metric space X its metric will be denoted by d or d_X. The metric on a product of metric spaces is defined by $d_{X\times Y}((x_1,y_1),(x_2,y_2)):=d_X(x_1,x_2)+d_Y(y_1,y_2)$.

4.2.1. Free closeness

We first consider free closeness, i.e. closeness relatively to ∅. The following is a generalization of the *SO* spaces.

Definition 51. Let Y be a metric space. Metric space $C(Y)$ is the subspace of $Y\times[0,1]\times[-1,1]$ defined as

$$\left\{\left(y,t,\sin\frac{\pi}{t}\right);\ (y,t)\in Y\times(0,1]\right\}\cup Y\times\{0\}\times\{0\}.$$

Fig. 5 schematically represents space $C(Y)$: copies of space Y connected by big (dashed) homotopies converge to $(Y,0,0)$. In the case of Y being a single point we obtain $C(Y)=\{(0,0)\}\cup\{(x,\sin\frac{1}{x});\ x\in(0,1]\}$. For any point $(y,t,s)\in C(Y)$ we refer to y, t, s as the first, the second and third coordinate respectively. The role of these coordinates is the following:

- the first coordinate allows space Y to be embedded;
- the second coordinate represents homotopies between converging embeddings;
- the third coordinate makes homotopies via the second coordinate big so that they cannot extend over $(Y,0,0)$.

Note that $(Y,0,0)$ is not path connected to $(Y,1,0)$ which yields the following result.

Proposition 52. *The map $f:Y\to C(Y)$ defined by $y\mapsto(y,1,0)$ is close to the map $g:Y\to C(Y)$ defined by $y\mapsto(y,0,0)$.*

Proposition 53. *Suppose Y is a compact metric space and the map $f:Y\to X$ is not homotopic to $g:Y\to X$. The map f is close to g iff there exists a map $F:C(Y)\to X$ so that $F|_{(Y,1,0)}=f$ and $F|_{(Y,0,0)}=g$.*

Proof. The existence of an extension F implies that f is close to g by Corollary 47.

To prove the other direction assume that f is close to g. Hence for all $n\in\mathbb{Z}_+$ there exist maps

$$H_n:Y\times\left[\frac{1}{n+1},\frac{1}{n}\right]\to X;\qquad H_n|_{Y\times\{n+1\}}=H_{n+1}|_{Y\times\{n+1\}};$$

$$d\left(H_n\left(y,\frac{1}{n}\right),g(y)\right)<\frac{1}{n};\qquad H_1|_{Y\times\{1\}}=f.$$

We have to adjust the maps H_n in order to construct a continuous map on $C(Y)$. The idea is to adjust the maps H_n so that they only depend on Y-coordinate in appropriate neighborhoods of $(Y, n+1)$ and (Y, n).

Given any $n \in \mathbb{Z}_+$ and any $\delta < (\frac{1}{n} - \frac{1}{n+1})/2$ we can assume $H_n(y,t) = H_n(y, \frac{1}{n+1})$ if $|t - \frac{1}{n+1}| < \delta$ and $H_n(y,t) = H_n(y, \frac{1}{n})$ if $|t - \frac{1}{n}| < \delta$. A required modification can be obtained as follows. Extend H_n to a map $Y \times [a,b] \to X$ so that

$$H_n(y,t) := H_n\left(y, \frac{1}{n+1}\right) \quad \text{if } t < \frac{1}{n+1};$$
$$H_n(y,t) := H_n\left(y, \frac{1}{n}\right) \quad \text{if } t > \frac{1}{n}.$$

The linear contraction $c : [a,b] \to [\frac{1}{n+1}, \frac{1}{n}]$ for appropriate a and b induces a map

$$Y \times \left[\frac{1}{n+1}, \frac{1}{n}\right] \xrightarrow{1\times c} Y \times [a,b] \xrightarrow{H_n} X$$

which satisfies the required condition.

Define the map $F : C(Y) \to X$ by the rule $(y,t,s) \mapsto H_n(y,t)$ if $t \in [\frac{1}{n+1}, \frac{1}{n}]$ and $F(y,0,0) = f(y)$. We claim that for a suitable choice of maps H_n the map F is continuous. Note that $C(Y) \cap (Y \times (0,1] \times [-1/2, 1/2])$ is a disjoint union of closed neighborhoods of $(Y, 1/n, 0)$ which are all homeomorphic to $Y \times [-1,1]$. We can assume (by applying the modification above) that the maps H_n are appropriately modified to ensure that $F(y,s,t)$ only depends on y on each of these sets. In particular, any sequence $a_n = F(y, t_i, s(t_i))$ with fixed y and t_i converging to $1/n$ is eventually constant. The stabilization occurs (if not before) for i with the property that for all successive indexes $j > i$ we have $|s(t_j)| < 1/2$ and $|t_j - 1/n| < 1/(4n^2)$.

Note that F is continuous at every point (y,t,s) with $t > 0$ as the maps H_n are continuous and agree on the intersection of their domains. To prove that F is continuous consider a convergent sequence $(y_i, t_i, s_i) \to (y_0, 0, 0)$ in $C(Y)$. Given any $\varepsilon > 0$ choose i_0 so that for every $i > i_0$:

- $d(f(y_i), f(y_0)) < \varepsilon/2$;
- $|s_i| < 1/2$;
- $t_i < 1/n_\varepsilon < \varepsilon/2$ for some $n_\varepsilon \in \mathbb{Z}_+$ (i.e. $d(f(y,t_i,s_i), f(y,0,0)) < \frac{1}{n_\varepsilon}$, $\forall y \in Y$).

Then

$$d\big(F(y_i,t_i,s_i), F(y_0,0,0)\big) < d\big(F(y_i,t_i,s_i), F(y_i,0,0)\big) + d\big(F(y_i,0,0), F(y_0,0,0)\big) < 1/n_\varepsilon + \varepsilon/2 < \varepsilon,$$

since $F(y_i,t_i,s_i) \in \{F(y_i, 1/n, 0)\}_{n \geqslant n_\varepsilon}$ hence F is continuous. □

4.2.2. Relative closeness

Definition 54. Let Y be a metric space and let $A \subset Y$ be a closed subspace. Choose a map $\varphi : Y \to [0,1]$ so that $A = \varphi^{-1}(\{0\})$. The metric space $C(Y,A)$ is a subspace of $Y \times [0,1] \times [-1,1]$ defined as

$$\left\{\left(y, \varphi(y)t, \varphi(y)\sin\frac{\pi}{t}\right);\ (y,t) \in (Y \backslash A) \times (0,1]\right\} \cup Y \times \{0\} \times \{0\}.$$

The space $C(Y,A)$ depends on a choice of map φ. Nevertheless we omit φ form the notation of $C(Y,A)$ as the properties of our interest do not depend on the choice of a map φ. Note that $C(Y,A)$ is not locally path connected at any point of $(Y \backslash A, 0, 0)$. In the case of $(Y,A) = ([0,1], \{1\})$ the space $C(Y,A)$ is a cone over the space

$$C\big(\{0\}\big) = \big\{(0,0)\big\} \cup \left\{\left(x, \sin\frac{1}{x}\right);\ x \in (0,1]\right\}.$$

The following proposition provides some examples of a relatively close maps.

Proposition 55. *Suppose $A \subset Y$ is a closed subspace of a metric space Y. The inclusion $i_1 : Y \hookrightarrow C(Y,A)$ defined by $y \mapsto (y, \varphi(y), \varphi(y))$ is homotopic or close to the inclusion $i_2 : Y \hookrightarrow C(Y,A)$ rel A, where $i_2 : y \mapsto (y,0,0)$.*

Proof. The homotopies $H : Y \times [1/n, 1] \to C(Y,A)$ defined by the rule

$$(y,t) \mapsto \left(y, \varphi(y)t, \varphi(y)\sin\frac{\pi}{t}\right)$$

homotope i_1 arbitrarily close to i_2. □

In the case of free closeness the inclusions of Proposition 52 were not homotopic due to an argument on path connectedness. Such argument cannot be employed for the inclusions i_1 and i_2 of Proposition 55 as $C(Y, A)$ is path connected if Y is path connected and $A \neq \emptyset$. In order to find a condition for the inclusions i_1 and i_2 not to be homotopic relatively to A consider the space $W := Y_1 \cup_{a\sim(a,(0,1]);\, a\in A} (Y_2 \times (0, 1])$ where Y_1 and Y_2 are isomorphic to a locally path connected metric space Y. Using the argument of Peanification for the points of $(Y - A, 0, 0)$ observe that the natural bijection $W \leftrightarrow C(Y, A)$ induces a natural bijection on maps and homotopies from Y, $\operatorname{rel} A$ if Y is locally path connected. The space W is homotopic to $Y_1 \cup_A Y_2$ with the natural inclusions of Y_1 and Y_2 corresponding to the inclusions i_1 and i_2 of Proposition 55. Hence the inclusions i_1 and i_2 of Proposition 55 are homotopic $\operatorname{rel} A$ iff the inclusions of Y_1 and Y_2 into $Y_1 \cup_A Y_2$ are homotopic $\operatorname{rel} A$. The later of these conditions is equivalent (via contraction of Y_2 in $Y_1 \cup_A Y_2$) to Y/A being contractible which is the case if $A \hookrightarrow X$ is a cofibration and a homotopy equivalence.

Corollary 56. *Suppose the cofibration inclusion $A \hookrightarrow Y$ of a closed subspace into a locally path connected metric space Y is not a homotopy equivalence. Then the inclusion $i_1 : Y \hookrightarrow C(Y, A)$ defined by $y \mapsto (y, \varphi(y), \varphi(y))$ is close to the inclusion $i_2 : Y \hookrightarrow C(Y, A) \operatorname{rel} A$, where $i_2 : y \mapsto (y, 0, 0)$.*

Combining Corollary 56 and Proposition 53 we obtain the extensional classification of certain types of relative close maps.

Proposition 57. *Consider the following situation*:

- *Y is a compact, locally path connected metric space*;
- *$A \subset Y$ is a closed subspace*;
- *the natural inclusion $A \hookrightarrow Y$ is a cofibration which is not a homotopy equivalence*;
- *the map $f : Y \to X$ is not homotopic to the map $g : Y \to X$, $\operatorname{rel} A$*;
- *the inclusions $i_1, i_2 : Y \to C(Y, A)$ are defined by $i_1(y) = (y, \varphi(y), \varphi(y))$ and $i_2(y) = (y, 0, 0)$.*

Then the map f is close to g, $\operatorname{rel} A$ iff there exists a map $F : C(Y, A) \to X$ so that $Fi_1 = f$ and $Fi_2 = g$.

Proof. The existence of an extension F implies that f is close to g by Corollary 47. To prove the other direction assume that f is close to g, $\operatorname{rel} A$. By Proposition 53 there exists an appropriate extension F over $C(Y)$. Note that for every $a \in A$ the closed subset $W_a := C(Y) \cap (\{a\} \times [0, 1] \times [-1, 1])$ is mapped by F to a. The map F induces an extension over the quotient space $C(Y)/_{W_a;\, a\in A} = C(Y, A)$. $\square$

4.3. Closeness and compactness

Closeness of maps due to Definition 45 is considered within metric spaces in order to enforce a uniform continuouity of the approaching maps. On the other hand the idea of close maps appears in (iii) of Definition 1 where closeness of paths is considered in a non-metric space. The aim of this section is to introduce the notion of closeness for maps with compact domain and possibly non-metric range.

Definition 58. Suppose $f, g : K \to Y$ are maps defined on a compact Hausdorff space K so that $f \not\simeq g$, $\operatorname{rel} A$ for some closed subspace $A \subset K$. The map f is close to g if for every finite open cover $U_1, \ldots, U_k$ of $g(K)$ there exist:

- a collection $B_1, \ldots, B_k$ of closed subsets of K so that $K = \bigcup_i B_i$ and $g(B_i) \subset U_i$;
- the map $f' \simeq f$, $\operatorname{rel} A$ so that $f'(B_i) \subset U_i$, $\forall i$.

Let us prove that both definitions of closeness agree if considered in a metric space Y. Suppose the map $f : K \to Y$ is close $\operatorname{rel} A$ to the map g in terms of Definition 58 where $A \subset K$ is a close subset of a compact Hausdorff space K and Y is a metric space. Given any $\varepsilon > 0$ we can cover $g(K)$ by a collection of open sets of diameter at most ε. The map f' referred to such cover by Definition 58 is homotopic to f, $\operatorname{rel} A$ and satisfies the condition $d(f'(x), g(x)) < \varepsilon$, $\forall x \in K$, hence f is close to g in terms of Definition 45.

To prove the opposite implication assume that the map $f : K \to Y$ is close $\operatorname{rel} A$ to the map g in terms of Definition 45. Given any finite open cover $U_1, \ldots, U_k$ of $g(K)$ choose a collection $B_1, \ldots, B_k$ of closed subsets of K so that $K = \bigcup_i B_i$ and $B_i \subset U_i$. There exists an $\varepsilon > 0$ so that for all i the ε-neighborhood of B_i is also in U_i. A map $f' \simeq f$, $\operatorname{rel} A$ with the property of $d(f'(x), g(x)) < \varepsilon$, $\forall x \in K$ satisfies the conditions of Definition 58 hence the definitions are equivalent.

5. Applications

Other that the classification of Theorem 4, homotopical smallness and closeness can efficiently be used in construction of certain spaces. The simplest case is the topologist's sine curve, which is equivalent to $C(\{0\})$. It can also be considered

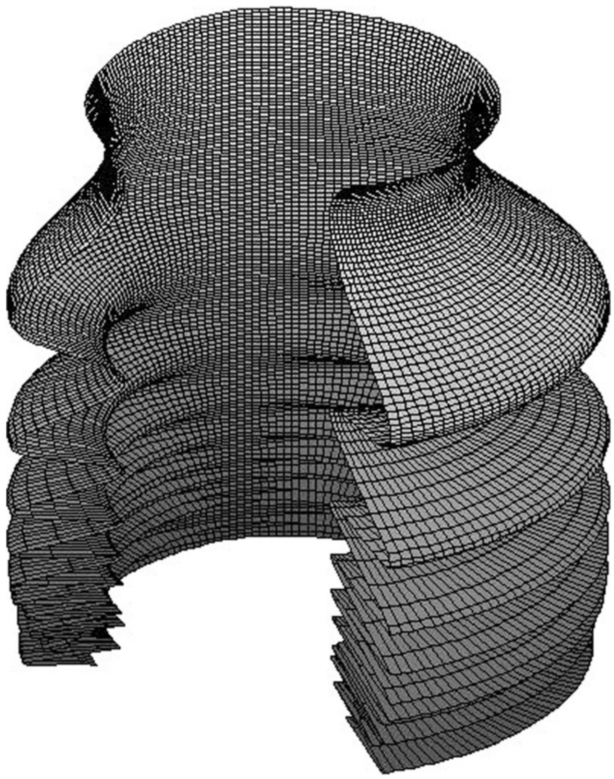


Fig. 6. The surface portion of the harmonic vase.

as a one-dimensional harmonic archipelago HA^1. The harmonic archipelago of dimension n [denoted by HA^n] is a subset of $\mathbb{R}^{n+1}$ is constructed from a wedge of spheres $\{S_i^n\}_{i\in\mathbb{Z}_+}$ radii $1/i$ by attaching big homotopies (i.e. of diameter at least 1) between each pair of consecutive spheres (S_i^n, S_{i+1}^n).

The harmonic vase was defined in [19] as a subset of $\mathbb{R}^3$. It has an essential role in the proof of Theorem 63. It consists of a disc

$$B^2 = \{(x, y, 0) \in \mathbb{R}^3;\ x^2 + y^2 \leqslant 9\},$$

and a surface portion (see Fig. 6)

$$r := \frac{|\varphi|}{\pi} \sin\frac{\pi}{z} + 2, \quad z \in (0, 1],\ \varphi \in [-\pi, \pi],$$

where (r, φ) are polar coordinates in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and z is the coordinate of $\{0\}^2 \times \mathbb{R}$ so that (r, φ, z) are cylindric coordinates in $\mathbb{R}^3$. The motivation for HV is a construction of a loop f which is close to a homotopically trivial embedding g of S^1 in a compact space. The quotient space $HV/\{\varphi = 0\}$ is homeomorphic to a compact space $C(S^1, \{0\}) \cup B^2$ where the disc B^2 is attached in appropriate way as described above. Furthermore, the union of the surface portion of HV and ∂B^2 is equivalent to $C(S^1, \{0\})$.

Space A as defined in [6] was developed as an example of a space which is homotopically Hausdorff but not strongly homotopically Hausdorff, i.e., it has no small loops but has free small loops. It is a subspace of $\mathbb{R}^3$ consisting of three parts:

- the surface portion which is obtained by rotating the topologist's sine curve

  $$\{(0,0,0)\} \cup \left\{\left(x, 0, \sin\frac{1}{x}\right);\ x \in (0, 1]\right\}$$

  around the z-axis, as suggested by Fig. 7;
- the central limit arc $\{0\} \times \{0\} \times [-1, 1]$;
- connecting arcs, i.e. a system of countably many closed radial arcs emerging from the central limit arc so that A is compact and locally path connected.

The surface portion of A can be considered as an unpointed version of HA, the quotient $C(S^1)/_{(S^1,0,0)}$ or as a modified version (in the same way as HA is a modified version of $SO_{S^1}(\mathbb{N})$) of $FSO_{S^1}(\mathbb{N})$.

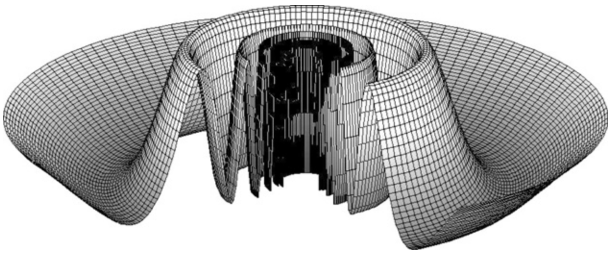


Fig. 7. The surface portion of space A.

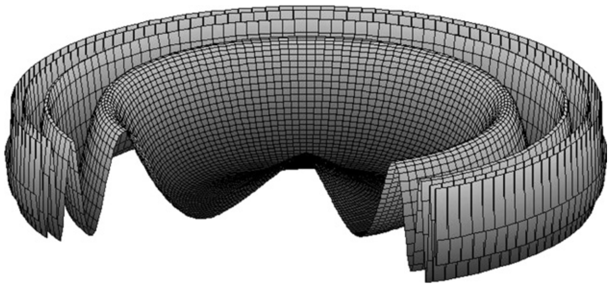


Fig. 8. The surface portion of space B.

Space A and its properties were studied in [9] under the name of space Y'. A similar space called Y is defined and studied in the same paper. It consists of the same surface portion and the same central limit arc, but instead of connecting arcs there is a single simple arc connecting the central limit arc with the surface portion. The space Y is not locally path connected but the connecting arc makes it path connected.

Space Y distinguishes between two definitions of semi-local simple connectedness that appear in the literature. According to [9]:

- space X is based semi-locally simply connected iff every point $x \in X$ has a neighborhood $U \subset X$ so that $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.
- space X is unbased semi-locally simply connected iff every point $x \in X$ has a neighborhood $U \subset X$ so that every loop in U is contractible.

Both definitions of semi-local simple connectedness agree if the space is locally path connected. It turns out that the space Y is based but not unbased semi-locally simply connected due to its topology at the central limit arc.

Space B as defined in [6] was developed as an example of a space which is strongly homotopically Hausdorff but not shape injective. It is a subspace of $\mathbb{R}^3$ consisting of three parts:

- the surface portion which is obtained by rotating the topologist's sine curve
$$\{(0,0,0)\} \cup \left\{\left(x, 0, \sin\frac{1}{x}\right);\ x \in (0,1]\right\}$$
around the axis $\{1\} \times \{0\} \times \mathbb{R}$, as suggested by Fig. 8;
- the outer annulus obtained by rotating $\{0\} \times \{0\} \times [-1,1]$ around the axis $\{1\} \times \{0\} \times \mathbb{R}$;
- connecting arcs, i.e. a system of countably many closed radial arcs emerging from the outer annulus so that B is compact and locally path connected.

The surface portion of B is essentially the same as $C(S^1)$. Space B and its properties were studied in [9] under the name of space Z'. A similar space called Z is defined and studied in the same paper.

5.1. Realization theorems

One of the basic problems in homotopy theory is the realization of various groups as a homotopy invariants of certain spaces. In particular, we are interested in the following question: given a group G when can we realize it as a fundamental group of a path connected space X which possesses the following properties:

(i) X is compact;
(ii) X is metric;
(iii) X is locally path connected?

It turns out that these three conditions are too restrictive for the realization of all countable groups.

Theorem 59. *([15]) Let X be a compact metric space which is path connected and locally path connected. If the fundamental group of X is not finitely generated then it has the power of the continuum.*

An improvement to this theorem has been made in [5] and [7].

Theorem 60. *Let X be a compact metric space which is path connected and locally path connected. If the fundamental group of X is not finitely presented then it has the power of the continuum.*

However, if we omit any of the three properties mentioned above we can realize all countable groups. It is well known (see [11]) that every group can be realized as a fundamental group of a path connected *CW* complex of dimension two and every countable group can be realized as a fundamental group of a countable path connected *CW* complex of dimension two. Every countable *CW* complex is homotopy equivalent to a locally finite (hence metrizable) *CW* complex of the same dimension which yields the realization in terms of metric, locally path connected spaces. Since the metric space of such realization is a two-dimensional *CW* complex it can be embedded in $\mathbb{R}^5$.

Theorem 61. *Let G be a countable group. Then G can be realized as a fundamental group of a two-dimensional metric space X which is path connected and locally path connected.*

This result implies that, omitting the compactness from the list above, we can realize all countable groups as a fundamental groups of a space with prescribed properties. Similarly we can omit metrizability in order to obtain a realization in terms of compact locally path connected space.

Theorem 62. *([14]) Let G be a countable group. Then G can be realized as a fundamental group of a compact space X which is path connected and locally path connected.*

The realization in terms of a compact metric space was proven in [19] using the techniques of homotopical closeness (i.e. the harmonic vase and its variation: the braided harmonic vase) and the universal Peano space. It turns out that given a locally path connected space X in certain circumstances, one can construct a compact space Y so that $PY \simeq X$, i.e. the spaces have the same fundamental group.

Theorem 63. *Let G be a countable group. Then G can be realized as a fundamental group of a two-dimensional compact metric space $X \subset \mathbb{R}^4$ which is path connected.*

The approach of [19] in terms of homotopical smallness was generalized in [17] in order to obtain a wider class of realization theorems. This improvement includes the realization of appropriately prescribed groups as a homotopy or homology groups of a space. The realization results are implied by the following fact.

Proposition 64. *([17]) For every countable CW complex K there is a compact metric space X such that PX is homotopy equivalent to K.*

The proof of the above (and similar results of [17]) is motivated by our construction of spaces possessing homotopical smallness and homotopical closeness.

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