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Note

On edge domination numbers of graphs

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Abstract

Let $\gamma_s'(G)$ and $\gamma_{ss}'(G)$ be the signed edge domination number and signed star domination number of G, respectively. We prove that $2n-4\geqslant\gamma_{ss}'(G)\geqslant\gamma_s'(G)\geqslant n-m$ holds for all graphs G without isolated vertices, where $n=|V(G)|\geqslant 4$ and $m=|E(G)|$, and pose some problems and conjectures.

Keywords: Signed edge domination function; Signed edge domination number; Signed star domination function; Signed star domination number

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider simple graphs only.

Let $G=(V,E)$ be a graph. For $u\in V$, then $N_G(u)$ and $N_G[u]$ denote the open and closed neighborhoods of u in G, resp. $d_G(u)=|N_G(u)|$ is the degree of u in G, and $\delta(G)$ denotes the minimum degree of G. For $S\subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S. For $v\in V$, the symbol $G-v=G[V(G)\backslash\{v\}]$. If H is an induced subgraph of G, we write $H\leqslant G$. For $e=uv\in E(G)$, $N_G(e)=\{e'\in E(G)|e'$ is adjacent to $e\}$ is called the open edge-neighborhood of e in G, and $N_G[e]=N_G(e)\cup\{e\}$ is called the closed one. If $v\in V$, then $E_G(v)=\{uv\in E|u\in V\}$ is called the edge-neighborhood of v in G. For simplicity, $N_G[e]$ and $E_G(v)$ are denoted by $N[e]$ and $E(v)$, respectively.

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In recent years, several kinds of domination problems in graphs have been investigated [2–4,6–8], most of these belonging to the vertex domination. In [5] we introduced the signed edge domination in graphs.

Definition 1.1 (*Xu [5]*). Let $G=(V(G), E(G))$ be a graph. A function $f: E(G) \to \{+1, -1\}$ is called the signed edge domination function (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geqslant 1$ for every $e \in E(G)$. The signed edge domination number $\gamma'_s(G)$ of G is defined as $\gamma'_s(G) = \min\{\sum_{e \in E(G)} f(e) | f$ is an SEDF of $G\}$.

For any totally disconnected graph $G = \overline{K}_n$, then define $\gamma'_s(G) = 0$.

Definition 1.2. Let $G=(V, E)$ be a graph without isolated vertices. A function $f: E \to \{+1 -1\}$ is called the signed star domination function (SSDF) of G if $\sum_{e \in E(v)} f(e) \geqslant 1$ for every $v \in V(G)$. The signed star domination number of G is defined as $\gamma'_{ss}(G) = \min\{\sum_{e \in E} f(e) | f$ is an SSDF of $G\}$.

We define $\gamma'_s(G) = 0$ for all totally disconnected graphs $G = \overline{K}_n$.

By Definitions 1.1 and 1.2 we have

Lemma 1.3. *For any two vertex-disjoint graphs G_1 and G_2, we have*

$$\gamma'_s(G_1 \cup G_2) = \gamma'_s(G_1) + \gamma'_s(G_2) \quad \textit{and} \quad \gamma'_{ss}(G_1 \cup G_2) = \gamma'_{ss}(G_1) + \gamma'_{ss}(G_2).$$

Obviously, an SSDF is an SEDF of G; thus we have the following.

Lemma 1.4. *For any graph G without isolated vertices, $\gamma'_{ss} G \geqslant \gamma'_s(G)$.*

By Definition 3, it is easy to see the following.

Lemma 1.5. *For all graphs G, if $v \in V(G)$, then $\gamma'_{ss}(G) \leqslant \gamma'_{ss}(G-v) + d_G(v)$.*

A graph G is said to be a θ*-graph* if G is a connected graph with degree sequence $d = (2, 2, \ldots, 2, 3, 3)$. That is, a θ-graph consists of a cycle and a path whose two end-vertices are on the cycle.

Lemma 1.6. *Any θ-graph contains a cycle of even length* (*even cycle*).

Proof. It is obvious.

2. Signed edge domination

The following terminology and notation are useful to prove our main results.

A graph G with an SEDF f of G, denoted by (G, f), is called *a signed graph*. For a signed graph (G, f), we know from Definition 1.1 that $\gamma'_s(G) = \sum_{e \in E(G)} f(e)$.

For simplicity, given a signed graph (G, f), an edge $e \in E(G)$ is said to be +1 edge of (G, f) if $f(e) = +1$, analogously, an edge $e \in E(G)$ is said to be -1 edge of (G, f) if $f(e) = -1$. Write $E^+(G, f) = \{e \in E(G) | f(e) = +1\}$ and $E^-(G, f) = \{e \in E(G) | f(e) = -1\}$.

For any signed graph (H, g), two spanning subgraphs $H^+(g)$ and $H^-(g)$ of H are defined as $V(H^+(g)) = V(H^-(g)) = V(H)$, $E(H^+(g)) = E^+(H, g)$ and $E(H^-(g)) = E^-(H, g)$. $H^+(g)$ and $H^-(g)$ are called *the positive subgraph and negative subgraph* of (H, g), resp. For every vertex $u \in V(H)$, we define *the degree difference* $d^*(H, g, u)$ of u in (H, g) as $d^*(H, g, u) = d_{H^+(g)}(u) - d_{H^-(g)}(u)$. And further, if $e = uv \in E(H)$, since g is an SEDF of H, by Definition 1.1, we have

$$\sum_{e' \in N[e]} g(e') = d^*(H, g, u) + d^*(H, g, v) - g(uv) \geqslant 1. \tag{1}$$

For two signed graphs (G, f) and (H, g), then $(G, f) = (H, g)$ if and only if $G = H$ and $f = g$.

In [6], we have shown that $\gamma'_s(T) \geqslant 1$ for all trees $T \neq K_1$, and the following theorem generalizes this result.

Theorem 2.1. *Let G be a graph with $\delta(G) \geqslant 1$, then $\gamma'_s(G) \geqslant |V(G)| - |E(G)|$ and this bound is sharp.*

Proof. For convenience, we define $T(H) = |V(H)| - |E(H)|$ for all graphs H. So, our aim is to prove that $\gamma'_s(G) \geqslant T(G)$.

By Lemma 1.3 and noting that $T(G_1 \cup G_2) = T(G_1) + T(G_2)$, we may suppose that G is the connected graph.

Let $A = \{u \in V(G) | d_G(u) \geqslant 2\}$ and $B = \{u \in V(G) | d_G(u) = 1\}$. Note that $\delta(G) \geqslant 1$. We have $V(G) = A \cup B$ and $A \cap B = \phi$. When $|A| \leqslant 1$, then G is a star, this theorem is obvious. Next, we can suppose $|A| \geqslant 2$. Write $G_A = G[A]$.

Let f be such an SEDF that $\gamma'_s(G) = \sum_{e \in E(G)} f(e)$, based on the signed graph (G, f), we define a signed graph (G^*, f^*) which satisfies the following three properties:

(a) $\gamma'_s(G) = \sum_{e \in E(G^*)} f^*(e)$,
(b) $T(G) = T(G^*)$,
(c) $G_A = G^*[A]$ and G_A contains no -1 edge of (G^*, f^*).

Let $s = |E^-(G, f) \cap E(G_A)|$.

If $s = 0$, we define $G^* = G$ and $f^* = f$. Obviously, the signed graph (G^*, f^*) satisfies the above three properties. Thus, we can suppose $s \geqslant 1$.

Let $E^-(G, f) \cap E(G_A) = \{e_1, e_2, \ldots, e_s\}$, where $e_j = u_j v_j \, (j = 1, 2, \ldots, s)$.

Next we define one by one s signed graphs $(G^{(1)}, f^{(1)}), (G^{(2)}, f^{(2)}), \ldots, (G^{(s)}, f^{(s)})$.

Let $(G^{(0)}, f^{(0)}) = (G, f)$, from $i = 1$ to s. We define one by one $G^{(i)}$ from $G^{(i-1)}$ by adding two pendant edges and define $f^{(i)}$ by $f^{(i-1)}$ as the following Cases 1–2 such that e_i is a +1 edge of $(G^{(i)}, f^{(i)})$ (note that e_i is a -1 edge of $(G^{(i-1)}, f^{(i-1)})$), and hence e_i is a +1 edge of (G^*, f^*).

We may suppose $e_i = u_i v_i$ is a -1 edge of $(G^{(i-1)}, f^{(i-1)})$. By (1), we have
$d^*(G^{(i-1)}, f^{(i-1)}, u_i) + d^*(G^{(i-1)}, f^{(i-1)}, v_i) - f^{(i-1)}(u_i v_i) \geqslant 1$, that is,

$d^*(G^{(i-1)}, f^{(i-1)}, u_i) + d^*(G^{(i-1)}, f^{(i-1)}, v_i) \geqslant 0$. Without loss of generality, we may suppose $d^*(G^{(i-1)}, f^{(i-1)}, u_i) \geqslant d^*(G^{(i-1)}, f^{(i-1)}, v_i)$.

Case 1: When $d^*(G^{(i-1)}, f^{(i-1)}, u_i) \geqslant 1$, $G^{(i)}$ can be obtained from $G^{(i-1)}$ by adding two pendant edges $u_i w_i$ and $u_i w_i'$ in $G^{(i-1)}$; this also adds two pendant vertices w_i and w_i'. Define an SEDF $f^{(i)}$ of $G^{(i)}$ as follows:

$$f^{(i)}(e) = \begin{cases} f^{(i-1)}(e) & \text{when } e \in E(G^{(i)}) \backslash \{u_i v_i, u_i w_i, u_i w_i'\}, \\ +1 & \text{when } e = u_i v_i, \\ -1 & \text{when } e \in \{u_i w_i, u_i w_i'\}. \end{cases}$$

Case 2: When $d^*(G^{(i-1)}, f^{(i-1)}, u_i) = d^*(G^{(i-1)}, f^{(i-1)}, v_i) = 0$, $G^{(i)}$ can be obtained from $G^{(i-1)}$ by adding two pendant edges $u_i w_i$ and $v_i w_i'$ in vertices u_i and v_i, resp. Analogously, we define an SEDF $f^{(i)}$ of $G^{(i)}$ as follows:

$$f^{(i)}(e) = \begin{cases} f^{(i-1)}(e) & \text{when } e \in E(G^{(i)}) \backslash \{u_i v_i, u_i w_i, v_i w_i'\}, \\ +1 & \text{when } e = u_i v_i, \\ -1 & \text{when } e \in \{u_i w_i, v_i w_i'\}. \end{cases}$$

Combining Cases 1 and 2, we have obtained $(G^{(i)}, f^{(i)})$ from $(G^{(i-1)}, f^{(i-1)})$ $(i = 1, 2, \ldots, s)$.

Let $(G^*, f^*) = (G^{(s)}, f^{(s)})$, note that $\gamma_s'(G) = \sum_{e \in E(G)} f(e)$, it is easy to see that $G_A = G^*[A]$ and G_A contains no -1 edge of (G^*, f^*). And further, $T(G) = T(G^*)$ and $\gamma_s'(G) = \sum_{e \in E(G)} f(e) = \sum_{e \in E(G^{(i)})} f^{(i)}(e) = \sum_{e \in E(G^*)} f^*(e)$ holds for each i $(i = 1, 2, \ldots, s)$. Next we prove that $\sum_{e \in E(G^*)} f^*(e) \geqslant T(G)$.

Let C be the set of all pendant edges in G^*, p and q denote the numbers of all +1 edges and -1 edge in C, resp. This implies that G^* has $p + q$ pendant vertices.

For any vertex $u \in A$, if u is not adjacent to any pendant vertex in G^*, then $d^*(G^*, f^*, u) = d_{G_A}(u) \geqslant 2$; if u is adjacent to a pendant vertex v in G^*, by (1) we have $d^*(G^*, f^*, u) + d^*(G^*, f^*, v) - f^*(uv) \geqslant 1$ (note that $d^*(G^*, f^*, v) = f^*(uv)$), it implies that $d^*(G^*, f^*, u) \geqslant 1$ holds for every vertex $u \in A$. So, $2|E(G_A)| + p - q = \sum_{u \in A} d_{G_A}(u) + p - q = \sum_{u \in A} d^*(G^*, f^*, u) \geqslant |A|$. Note that G_A contains no -1 edge of (G^*, f^*), and thus we have $\sum_{e \in E(G^*)} f^*(e) = |E(G_A)| + p - q \geqslant |A| - |E(G_A)| = (|A| + p + q) - (|E(G_A)| + p + q) = |V(G^*)| - |E(G^*)| = T(G^*) = T(G)$.

We have proved that

$$\gamma_s'(G) \geqslant |V(G)| - |E(G)| \tag{2}$$

holds for any graph G with $\delta(G) \geqslant 1$. By the following statement, we have completed the proof of Theorem 2.1. □

Statement. Given a graph H with $\delta(H) \geqslant 1$, there exists a graph G such that $H \leqslant G$ and $\gamma_s'(G) = |V(G)| - |E(G)|$.

Proof. Let G be the graph obtained from H by adding $d_H(u) - 1$ pendant edges in vertex u for every $u \in V(H)$. Define an SEDF f of G as follows:

when $e \in E(H)$, $f(e) = +1$; when $e \in E(G) \backslash E(H)$, $f(e) = -1$. We get a signed graph (G, f), and so $\gamma_s'(G) \leqslant \sum_{e \in E(G)} f(e) = |E(H)| - \sum_{u \in V(H)} [d(u) - 1] = |V(H)| -$

$|E(H)| = |V(G)| - |E(G)|$. By (2), we have $\gamma'_s(G) = |V(G)| - |E(G)|$ and this proof is complete. □

3. Signed star domination

Lemma 3.1. *For any graph G, if $\delta(G) \geqslant 3$, then G contains a θ-graph as subgraph, and hence G contains an even cycle.*

Proof. Without loss of generality, we may suppose that G is a connected graph. Let T be a spanning tree of G. v is a pendant vertex of T. That is, $d_T(v) = 1$. Since $\delta(G) \geqslant 3$, there exist at least two vertices u and w such that $uv, wv \in E(G) \backslash E(T)$. Define $H = T + \{uv, wv\}$. Obviously, H contains a θ-graph as subgraph, which is the maximum 2-connected subgraph of H. Note that $H \subseteq G$ and by Lemma 1.6, G contains an even cycle. We have completed the proof of Lemma 3.1. □

Theorem 3.2. *For any graph G of order n $(n \geqslant 4)$, then $\gamma'_{ss}(G) \leqslant 2n - 4$, and this bound is sharp.*

Proof. We use the induction on $m = |E(G)|$. The result is clearly true for $m \leqslant 3$ (note that $n \geqslant 4$);

Suppose that the theorem is true for all graphs G_1 with $|E(G_1)| \leqslant m - 1$ and $4 \leqslant |V(G_1)| \leqslant n$. Now we consider a graph G with $|E(G)| = m$. Without loss of generality, we may suppose that $\delta(G) \geqslant 1$.

Case 1: $1 \leqslant \delta(G) \leqslant 2$. There exists a vertex $v \in V(G)$ such that $d_G(v) = \delta(G) \leqslant 2$. Note that $|E(G - v)| \leqslant m - 1$. When $|V(G - v)| = n - 1 = 3$, that is, $|V(G)| = n = 4$, it is easy to check that $\gamma'_{ss}(G) \leqslant 4 = 2n - 4$. When $|V(G - v)| = n - 1 \geqslant 4$; by the induction hypothesis, we have $\gamma'_{ss}(G - v) \leqslant 2(n - 1) - 4 = 2n - 6$. By Lemma 1.5, we have $\gamma'_{ss}(G) \leqslant \gamma'_{ss}(G - v) + d_G(v) \leqslant 2n - 6 + 2 = 2n - 4$.

Case 2: $\delta(G) \geqslant 3$. We see from Lemma 3.1 that G contains an even cycle C. Let $H = G - E(C)$. By the induction hypothesis, H has an SSDF f with $\sum_{e \in E(H)} f(e) \leqslant 2n - 4$. Extending f from H by signing $+1$ and -1 alternatively along C, we obtain an SSDF for G, and hence $\gamma'_{ss}(G) \leqslant 2n - 4$.

Since $\gamma'_{ss}(K_{2,n-2}) = 2n - 4$ $(n \geqslant 4)$, the upper bound given in Theorem 3.2 is sharp. We have completed the proof of Theorem 3.2. □

Corollary 3.3. *For all graphs G of order n, if $\delta(G) \geqslant 1$, then $\gamma'_{ss}(G) \geqslant \lceil n/2 \rceil$.*

Proof. Let f be an SSDF of G such that $\gamma'_{ss}(G) = \sum_{e \in E(G)} f(e)$. For every edge $e = uv \in E(G)$, $e \in E(u)$ and $e \in E(v)$. Thus, we have

$$\gamma'_{ss}(G) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in E(v)} f(e) \geqslant \frac{1}{2} \sum_{v \in V(G)} 1 = \frac{n}{2}.$$

Note that $\gamma'_{ss}(G)$ is an integer. The proof is complete. □

4. Some open problems and conjectures

We know from Lemma 1.4 that $\gamma'_{ss}(G) \geqslant \gamma'_s(G)$ holds for any graph G, and so we have the following.

Problem 4.1. Characterize the graphs G which satisfy the equality $\gamma'_s(G) = \gamma'_{ss}(\mathbf{G})$.

We know from Theorem 2.1 that $\gamma'_s(G) \geqslant |V(G)| - |E(G)|$ holds for any graph G with $\delta(G) \geqslant 1$. A natural problem is the following.

Problem 4.2. Characterize the graphs G which satisfy the equality $\gamma'_s(G) = |V(G)| - |E(G)|$.

Although in [5] we have determine the exact value of $\psi(m) = \min\{\gamma'_s(G)|G$ is a graph of size $m\}$ for all positive integers m, it seems more difficult to solve the following:

Problem 4.3 (*Xu [5]*). Determine the exact value of $g(n) = \min\{\gamma'_s(G)|G$ is a graph of order $n\}$ for every positive integer n.

Conjecture 4.4. For any graph G of order n $(n \geqslant 1)$, $\gamma'_s(G) \leqslant n - 1$.

If true, the upper bound is the best possible for odd n. For example, let G be the subdivision of the star $K_{1,(n-1)/2}$. Clearly, $\gamma'_s(G) = |E(G)| = n - 1$ (the subdivision of a graph G is the graph obtained from G by subdividing each edge of G exactly once).

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