Applications of Ky Fan’s inequality on $\sigma$-compact set to variational inclusion and $n$-person game theory

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Abstract

In this paper, Ky Fan’s inequality on $\sigma$-compact set is applied to variational inclusions and $n$-person game theory. We give results of some variational inclusions and existence of non-cooperative equilibrium in $n$-person game on $\sigma$-compact set.

Keywords: Ky Fan’s inequality; Set-valued map; Variational inclusion; $n$-Person game; Cooperative equilibrium point

1. Introduction

As is well known, Ky Fan’s inequality is used widely in many aspects of nonlinear analysis (e.g., see [1–5]). Recently, some set-valued versions of Ky Fan’s inequality for set-
valued mappings are obtained and also applied to variational inclusion theory, fixed point theory, contingent derivative, optimal and equilibrium theory, etc. (see [7–9]). The classical Ky Fan’s inequality demands domain of the function involved is a convex compact set. A kind of non-compact sets called σ-compact set is defined and some generalized KKM theorems are obtained in [10]. A generalized Ky Fan’s inequality on a closed set is given in [6].

Inspired and motivated by the results mentioned above, in this paper, we will discuss some variational inclusion problems and verify the existence of non-cooperative equilibrium in \( n \)-person game theory on a σ-compact set by using the version of Ky Fan’s inequality on the set. A fixed point theorem and some results about variational inclusions are obtained in Section 3 after preliminaries in Section 2. We gave an existence theorem for non-cooperative equilibrium in \( n \)-person games in which the set of multistrategies is a σ-compact convex subset in Section 4. The corresponding results in [3] are generalized.

2. Preliminaries

Let \( X \) be a real normed vector space with topological dual \( X^* \), and \( \langle \cdot, \cdot \rangle \) be the duality pairing between \( X \) and \( X^* \). If \( K \) is a nonempty subset of \( X \), we always assume that \( \varphi : K \times K \to R \) is a given function. For \( x_0 \in X \) and \( \eta > 0 \), we denote the closed ball centered at \( x_0 \) with radius \( \eta \) by

\[
B_X(x_0, \eta) = B(x_0, \eta) = \{ x \in X : \| x - x_0 \| \leq \eta \}.
\]

We recall that the domain of an extended-real-valued function \( f : X \to \overline{R} := R \cup \{+\infty\} \) is the set where it is finite and is denoted by \( \text{dom}(f) := \{ x \in X : f(x) < +\infty \} \). A function \( f : X \to \overline{R} \) is said to be lower semi-continuous (lsc) at \( x_0 \in \text{dom}(f) \) if for all \( \lambda < f(x_0) \), there exists \( \eta > 0 \) such that

\[
\lambda \leq f(x), \quad \forall x \in B(x_0, \eta).
\]

We say that \( f \) is lsc if it is lsc everywhere in its domain. A function \( f \) is said to be upper semi-continuous (usc) if \(-f \) is lsc.

**Proposition 2.1.** [3] A function \( f : X \to \overline{R} \) is lower semi-continuous at \( x_0 \) if and only if

\[
f(x_0) \leq \liminf_{x \to x_0} f(x).
\]

Let \( X \) and \( Y \) be two normed vector spaces and \( F : X \to 2^Y \) be a set-valued map. We write \( \text{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\} \) for the graph of \( F \) and \( \text{dom}(F) := \{ x \in X : F(x) \neq \emptyset \} \) for the domain of \( F \). If \( A \) is a subset of \( X \) then \( F(A) := \bigcup_{x \in A} F(x) \) and if \( B \subset Y \), \( \text{F}^{-1}(B) := \{ x \in X : B \cap F(x) \neq \emptyset \} \).

A set-valued map \( F : X \to 2^Y \) is said to be lower semi-continuous (lsc) at \( x \in \text{dom}(F) \) if and only if for any \( y \in F(x) \) and for any sequence of elements \( x_n \) in \( X \) converging to \( x \), there exists a sequence of elements \( y_n \in F(x_n) \) converging to \( y \). The set-valued map \( F : X \to 2^Y \) is said to be upper semi-continuous (usc) at \( x_0 \in X \) if and only if for any
neighborhood $U$ of $F(x_0)$, there exists $\eta > 0$ such that for every $x \in B_X(x_0, \eta)$, we have $F(x) \subset U$. $F$ is said to be lsc (respectively usc) on a subset $A \subset X$ if it is lsc (respectively usc) at every point in $A \cap \text{dom}(F)$.

The support function of the subset $K$ of the normed vector space $X$ is defined by

$$\sigma_K(p) := \sigma(K, p) := \sup_{x \in K} \langle p, x \rangle,$$

where $p \in X^*$.

A set-valued map $C : X \to 2^Y$ is said to be upper hemi-continuous at $x_0 \in K \subset X$ if and only if for all $p \in Y^*$, the function $\sigma(C(x), p)$ is upper semi-continuous at $x_0$. It is said to be upper hemi-continuous on $K$ if it is upper hemi-continuous at every point $x_0 \in K$.

Remark 2.2. Any upper semi-continuous mapping is upper hemi-continuous. The graph of an upper hemi-continuous set-valued map with convex closed values is closed (see [3]).

For more discussion about the lsc, usc, hemi-continuous mappings and the support function, we refer to [1–4].

In the following, we need also the next definition and theorem which can be find in [7].

**Definition 2.3.** A nonempty set $K \subset X$ is called $\sigma$-compact if there is a sequence $\{K_n\}$ of compact subsets of $X$ such that $K = \bigcup_{n=1}^{\infty} K_n$.

**Theorem 2.4.** [7, Theorem 3.2] Let $X$ be a finite-dimensional real normed vector space, $K$ a convex and $\sigma$-compact subset of $X$, $Y$ a real normed vector space, $M$ a nonempty subset in $Y$ and $F : K \times K \to 2^Y$ a set-valued map. Suppose that

(i) the set-valued map $U : K \to 2^K$, defined by $U(x) := \{u \in K : (x, u) \in F^{-1}(Y \setminus M)\}$, is lsc on $K$;
(ii) for each finite set $\{u_1, u_2, \ldots, u_n\} \subset K$, $\text{conv}\{u_1, u_2, \ldots, u_n\} \subset \{x \in K : \exists i = 1, \ldots, n, F(x, u_i) \subset M\}$;
(iii) there is a nonempty compact subset $A$ of $K$ such that for every $x \in K \setminus A$ there exists $u \in A$ satisfying $F(x, u) \cap (Y \setminus M) \neq \emptyset$.

Then there exists $\bar{x} \in K$ such that $F(\bar{x}, u) \subset M$ for every $u \in K$.

3. Fixed point theorem and variational inclusions

We first introduce a lemma which is important for our discussion in the sequel, and present a fixed point theorem and then three results about variational inclusions.

**Lemma 3.1.** Let $K$ be a nonempty $\sigma$-compact convex subset of the finite-dimensional real normed vector space $X$ and let $f : K \times K \to \bar{R}$ be a function such that

(i) for every $y \in K$, $x \to f(x, y)$ is lsc;
(ii) for every \( x \in K, \ y \to f(x, y) \) is concave;
(iii) there is a nonempty compact subset \( A \) of \( K \) such that for every \( x \in K \setminus A \) there exists \( u \in A \) satisfying \( f(x, u) > \sup_{x \in K} f(x, x) \).

Then there exists \( \bar{x} \in K \) such that \( \sup_{y \in K} f(\bar{x}, y) \leq \sup_{x \in K} f(x, x) \).

**Proof.** Let \( m = \sup_{x \in K} f(x, x) \), \( M = (-\infty, m] \) and \( Y = \mathbb{R}^1 \). We observe that assumptions (ii) and (iii) ensure that the conditions (ii) and (iii) of Theorem 2.4 hold. The assumption (i) ensures that \( U : K \to 2^K, U(x) = \{ y \in K : f(x, y) > m \} \), is lsc on \( K \). Indeed, let \( V \) be an open subset of \( X \) and \( x_0 \in K \) such that \( U(x_0) \cap V \neq \emptyset \), then take any fixed point \( y_0 \in U(x_0) \cap V \), we have \( f(x_0, y_0) > m \). From assumption (i), there exists a neighborhood \( V_0 \) of \( x_0 \) such that \( f(x_0, y_0) > m \) for all \( x \in V_0 \). Hence \( y_0 \in U(x) \cap V \) for all \( x \in V_0 \), i.e., \( U(x) \cap V \neq \emptyset \). So \( U \) is lsc on \( K \). On the other hand, we have \( U(x) = \{ y \in K : (x, y) \in f^{-1}(Y \setminus M) \} \). This means the condition (i) of Theorem 2.4 holds. So the result holds. \( \square \)

**Theorem 3.2.** Let \( K \) be a \( \sigma \)-compact convex subset of a finite-dimensional real normed vector space \( X \) and let \( f : K \to K \) be a continuous mapping. If there is a nonempty compact subset \( A \) of \( K \) such that for every \( x \in K \setminus A \), there exists \( y \in A \) satisfying

\[
\| x - f(x) \| > \| y - f(x) \|,
\]

then there exists \( \bar{x} \in K \) such that \( \bar{x} = f(\bar{x}) \).

**Proof.** We set

\[
\varphi(x, y) = -\| y - f(x) \| + \| x - f(x) \|, \quad \forall x, y \in K.
\]

Then for every \( (x, y) \in K \times K \), \( \varphi(x, y) \) is continuous and for every \( x \in K \), \( \varphi(x, y) \) is concave with respect to \( y \in K \), i.e., the conditions (i) and (ii) of Lemma 3.1 are satisfied. From the assumptions, we know that the condition (iii) of Lemma 3.1 is also satisfied. So there exists \( \bar{x} \in K \) such that

\[
\varphi(\bar{x}, y) = -\| y - f(\bar{x}) \| + \| \bar{x} - f(\bar{x}) \| \leq \sup_{x \in K} \varphi(x, x) = 0, \quad \forall y \in K.
\]

We take \( y = f(\bar{x}) \), then \( \| \bar{x} - f(\bar{x}) \| \leq 0 \), i.e., \( \bar{x} = f(\bar{x}) \). \( \square \)

**Theorem 3.3.** Let \( K \) be a \( \sigma \)-compact convex subset of \( \mathbb{R}^n \) and let \( C : K \to 2^{\mathbb{R}^n} \) be a set-valued mapping with nonempty values. If

(i) \( \varphi(x, y) := -\sigma(C(x), y) \) is lsc in \( x \) on \( K \);
(ii) \( \forall x \in K, C(x) - R^n_+ \) is a convex closed subset;
(iii) there is a nonempty compact subset \( A \) of \( K \) such that for every \( x \in K \setminus A \), there exists \( y \in A \) satisfying \( \sigma(C(x), y) < 0 \); and
(iv) \( \forall x \in K, \sigma(C(x), x) \geq 0 \).

Then there exists \( \bar{x} \in K \) such that \( C(\bar{x}) \cap R^n_+ \neq \emptyset \).
Proof. We consider the function
\[ \varphi(x, y) = -\sigma(C(x), y), \quad \forall (x, y) \in K \times K. \]
This function is concave in \( y \in K \) (since \( y \rightarrow \sigma(C(x), y) \) is convex) and is lsc in \( x \) on \( K \).
It follows from the conditions (iii) and (iv) that, for each \( x \in K \setminus A \), there exists \( y \in A \) satisfying
\[ \varphi(x, y) = -\sigma(C(x), y) > 0 \geq \sup_{x \in K} -\sigma(C(x), x) = \sup_{x \in K} \varphi(x, x). \]
By Lemma 3.1, there exists \( \bar{x} \in K \) such that
\[ \sup_{y \in K} \varphi(\bar{x}, y) \leq \sup_{x \in K} \varphi(x, x) \leq 0, \]
i.e., \( \sigma(C(\bar{x}), y) \geq 0, \forall y \in K \). Since \( \sigma(-R^+_n, y) = 0 \) whenever \( y \in R^+_n \) and \( \sigma(-R^+_n, y) = +\infty \) whenever \( y \notin R^+_n \), it is equivalent to
\[ 0 \leq \sigma(C(\bar{x}) - R^+_n, y), \quad \forall y \in R^n. \]
Since \( C(\bar{x}) - R^+_n \) is a convex and closed subset, it follows from the separation theorem that if \( 0 \notin C(\bar{x}) - R^+_n \) then there exists \( y \in R^n \), such that \( \sigma(C(\bar{x}) - R^+_n, y) < 0 \), which is a contradiction. This implies \( C(\bar{x}) \cap R^+_n \neq \emptyset. \)

Remark 3.4. Under the condition that \( K \) is a compact subset of \( R^n \), Debruy, Gale and Nikaïdo obtained a theorem which is known as Debruy–Gale–Nikaïdo Theorem. Here we give the same result of the Debruy–Gale–Nikaïdo Theorem under different conditions. Also it is well known that when \( C \) is upper hemi-continuous, the function \( \varphi(x, y) := -\sigma(C(x), y) \) is lsc in \( x \) on \( K \) (see [3]).

Theorem 3.5. Let \( K \) be a \( \sigma \)-compact and convex subset of a finite-dimensional normed vector space \( X \) and let \( S : K \to 2^X \) be a lsc set-valued map. Suppose that

(i) there exists a nonempty compact subset \( A \subset K \) such that for every \( x \in K \setminus A \), there exist \( t \in S(x) \) and \( u \in A \) satisfying
\[ \langle t, u - x \rangle < 0; \]
(ii) for any finite subset \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset [0, 1], \{u_1, u_2, \ldots, u_n\} \subset K, \ u = \sum_{i=1}^{n} \alpha_i u_i, \)
we have
\[ \sum_{i=1}^{n} \alpha_i \langle t_i, u_i - u \rangle \geq 0, \quad \text{for all } t_i \in S(u). \]
Then there exists \( \bar{x} \in K \) such that \( \langle t, u - \bar{x} \rangle \geq 0, \) for all \( t \in S(\bar{x}) \) and \( u \in A \).

Proof. Let \( Y = R, M = [0, +\infty), \) we introduce the following set-valued mappings:
\[ F : K \times K \to 2^R, \quad F(x, y) = \{ a \in R : \langle t, y - x \rangle = a, \ t \in S(x) \}, \]
and
\[ U : K \to 2^K, \quad U(x) = \{ u \in K : \exists t \in S(x), \ \langle t, u - x \rangle < 0 \}. \]
We first show that \( U \) is lsc on \( K \). Since \( S \) is lsc on \( K \), i.e., for any \( x \in K \), \( t \in S(x) \) and \( \{ x_n \} \subset K \), \( x_n \to x \) as \( n \to +\infty \), there exists \( t_n \in S(x_n) \) such that \( t_n \to t \). For sufficient large \( n \), we have \( \langle t_n, u - x_n \rangle < 0 \) whenever \( u \in U(x) \). This is to say \( u \in U(x_n) \) for sufficient large \( n \). So \( U \) is lsc on \( K \). On the other hand, \( U(x) = \{ u \in K : F(x, u) \cap (-\infty, 0) \neq \emptyset \} = \{ u \in K : (x, u) \in F^{-1}(Y - M) \} \), \( \forall x \in K \). So the condition (i) of Theorem 2.4 is satisfied.

Secondly, we show that the condition (ii) of Theorem 2.4 is satisfied. Suppose, on the contrary, that there exist a finite subset \( \{ u_1, u_2, \ldots, u_n \} \subset K \) and \( u_0 \in \text{conv}\{u_1, u_2, \ldots, u_n\} \) such that \( F(u_0, u_i) \cap (-\infty, 0) \neq \emptyset \) for every \( i = 1, \ldots, n \). Hence \( u_0 = \sum_{i=1}^{n} \alpha_i u_i \), \( \alpha_i \geq 0 \), \( \sum_{i=1}^{n} \alpha_i = 1 \) and \( \sum_{i=1}^{n} \alpha_i F(u_0, u_i) \cap (-\infty, 0) \neq \emptyset \), which is a contradiction with the assumption (ii).

Finally, from the assumption (i), we know the condition (iii) of Theorem 2.4 is also satisfied. Therefore there exists \( \bar{x} \in K \) such that \( \langle t, u - \bar{x} \rangle \geq 0 \), for all \( t \in S(\bar{x}) \), \( u \in A \). \( \Box \)

Next we will verify a quasi-variational inequality.

**Theorem 3.6.** Let \( K \) be a \( \sigma \)-compact convex subset of a finite-dimensional normed space \( X \) and let \( C : K \to 2^X \) be an upper hemi-continuous set-valued mapping with nonempty convex closed values. We consider a function \( \varphi : K \times K \to \mathbb{R} \) satisfying

(i) \( \varphi \) is lsc in \( x \) on \( K \);

(ii) \( \forall x \in K, \ y \to \varphi(x,y) \) is concave;

(iii) \( \sup_{y \in K} \varphi(y,y) = 0 \);

(iv) there is a nonempty compact subset \( A \) of \( K \) such that

\[
A \cap \left\{ x \in K : \sup_{y \in C(x)} \varphi(x,y) \leq 0 \right\} = \emptyset,
\]

and for every \( x \in K \setminus A \), there exists \( u \in A \cap C(x) \) satisfying

\[
\varphi(x,u) > \sup_{y \in K} \varphi(y,y); \quad \text{and}
\]

(v) the subset \( \{ x \in K : \sup_{y \in C(x)} \varphi(x,y) \leq 0 \} \) is closed.

Then there exists a point \( \bar{x} \in K \) satisfying

(I) \( \bar{x} \in C(\bar{x}) \), and

(II) \( \sup_{y \in C(\bar{x})} \varphi(\bar{x}, y) \leq 0 \).

**Proof.** We shall argue by reduction to the absurd. Denote \( \alpha(x) = \sup_{y \in C(x)} \varphi(x,y) \). If the conclusion is false for all \( x \in K \), we would have either \( \alpha(x) > 0 \) or \( x \notin C(x) \). The later implies that there exists \( p \in X^* \) such that \( \langle p, x \rangle - \sigma(C(x), p) > 0 \). We set

(a) \( V_0 := \{ x \in K : \alpha(x) > 0 \} \), and
(b) \( V_n(p) := \{ x \in K_n : \langle p, x \rangle - \sigma(C(x), p) > 0 \} \).

Where \( K = \bigcup_{n=1}^{\infty} K_n \) and each \( K_n \) is a compact subset of \( X \).
Obviously, for each $K_n$, we have
$$K_n \subset V_0 \cup \left( \bigcup_{p \in X^*} V_n(p) \right).$$

The assumption on the set-valued map $C$ and the condition (i) imply that the sets $V_0$ and $V_n(p)$ are open. Since each $K_n$ is compact, there exist finite $p_{n1}, p_{n2}, \ldots, p_{nm_n}$ such that
$$K_n \subset V_0 \cup \left( \bigcup_{i=1}^{m_n} V_n(p_{ni}) \right).$$

Therefore
$$K = \bigcup_{n=1}^{\infty} K_n = V_0 \cup \bigcup_{n=1}^{\infty} \left( \bigcup_{i=1}^{m_n} V_n(p_{ni}) \right).$$

By the unity partition theorem, there exists a continuous partition of unity
$$\{g_0, g_{11}, g_{12}, \ldots, g_{1m_1}; g_{21}, g_{22}, \ldots, g_{2m_2}; \ldots; g_{n1}, g_{n2}, \ldots, g_{nm_n}; \ldots\},$$
which satisfies

1. $g_{ni}$ is continuous on $K$. There exists $V_{ni}(p_{ni})$, for each $x \in K$, such that $x \in V_{ni}(p_{ni})$ and only finite $g_{ni}$ are nonzero on it;
2. $0 \leq g_{ni}(x) \leq 1$, $g_0(x) + \sum_{n=1}^{\infty} \sum_{i=1}^{m_n} g_{ni}(x) = 1$, for all $x \in K$; and
3. for each $g_{ni}$, there exists $V_{ni}(p_{ni})$ such that $\text{supp}(g_{ni}) \subset V_{ni}(p_{ni})$, or $\text{supp}(g_{0}) \subset V_0$.

Now we define the function $\psi : K \times K \to \mathbb{R}$ by
$$\psi(x, y) = g_0(x)\varphi(x, y) + \sum_{n=1}^{\infty} \sum_{i=1}^{m_n} g_{ni}(x)\langle p_{ni}, x - y \rangle.$$

For each $x \in K$, it follows from (1) that only finite $g_{ni}(x)\langle p_{ni}, x - y \rangle$ are nonzero in some neighborhood of $x$. From assumptions (i) and (ii), we know that $\psi(x, y)$ is lower semicontinuous in $x$ and concave in $y$.

Next we show that $\psi(x, y)$ satisfying the condition (iii) of Lemma 3.1. Indeed, since $A \cap \{x \in K: \alpha(x) \leq 0\} = \emptyset$, we have $A \subset V_0$. So
$$K \setminus A = (K \setminus V_0) \cup (V_0 \setminus A).$$

If $x \in K \setminus A$, then $x \in K \setminus V_0$ or $x \in V_0 \setminus A$. If $x \in K \setminus V_0$, then $g_0(x) = 0$ and there exist $g_{ni}$ and $V_{ni}(p_{ni})$ such that $g_{ni}(x) > 0$, $x \in V_{ni}(p_{ni})$. By assumption (iv), there exists $u \in A \cap C(x)$ such that $\varphi(x, u) > \sup_{y \in K} \varphi(y, y)$. So
$$\langle p_{ni}, x - u \rangle = \langle p_{ni}, x \rangle - \langle p_{ni}, u \rangle > \langle p_{ni}, x \rangle - \sigma(C(x), p_{ni}) > 0.$$

Hence,
$$\psi(x, u) = \sum_{g_{ni}(x) \neq 0} g_{ni}(x)\langle p_{ni}, x - u \rangle > 0 = \sup_{y \in K} \psi(y, y).$$
If \( x \in V_0 \setminus A \), we also have

\[
\psi(x, u) = g_0(x)\varphi(x, u) + \sum_{g_{ni}(x) \neq 0} g_{ni}(x)(p_{ni}, x - u) > 0 = \sup_{y \in K} \psi(y, y).
\]

By Lemma 3.1, there exists \( \bar{x} \in K \) such that

\[
\sup_{y \in K} \psi(\bar{x}, y) \leq 0.
\]

We shall contradict this inequality by proving that there exists \( \bar{y} \in K \) such that \( \psi(\bar{x}, \bar{y}) > 0 \).

We take

(4) any \( \bar{y} \in C(\bar{x}) \) if \( \alpha(\bar{x}) \leq 0 \), or

(5) \( \bar{y} \in \hat{C}(\bar{x}) \) satisfying \( \varphi(\bar{x}, \bar{y}) > \frac{1}{2} \alpha(\bar{x}) \) if \( \alpha(\bar{x}) > 0 \).

Since \( g_0, g_{11}, g_{12}, \ldots, g_{1m_1}; g_{21}, g_{22}, \ldots, g_{2m_2}; \ldots \) is a partition of unity, we have \( g_{ni}(\bar{x}) > 0 \) for at least one index \( i = 1, 2, \ldots \). So \( \psi(\bar{x}, \bar{y}) > 0 \) follows from the assertion:

(6) \( g_0(\bar{x}) > 0 \) implies that \( \varphi(\bar{x}, \bar{y}) > 0 \), or

(7) \( g_{ni}(\bar{x}) > 0 \) implies that \( \langle p_{ni}, \bar{x} - \bar{y} \rangle > 0 \).

Indeed, if \( g_0(\bar{x}) > 0 \), then \( \bar{x} \in V_0 \) consequently, \( \varphi(\bar{x}, \bar{y}) > \frac{1}{2} \alpha(\bar{x}) \). If \( g_{ni}(\bar{x}) > 0 \), then \( \bar{x} \in V(p_{ni}) \) and consequently, \( \langle p_{ni}, \bar{x} \rangle > \sigma(C(\bar{x}), p_{ni}) \geq \langle p_{ni}, \bar{y} \rangle \) (since \( \bar{y} \in C(\bar{x}) \)). Thus, \( \langle p_{ni}, \bar{x} - \bar{y} \rangle > 0 \).

4. Non-cooperative equilibrium in n-person games

Now we consider the decision rules of \( n \) plays that are determined by loss functions. In [3], the set of multi-strategies \( E \) is a compact set. In this section, we discuss the situations in which the set \( E \) is not compact and give a result of the existence for non-cooperative equilibrium in \( n \)-person games in which the set of multi-strategies is a \( \sigma \)-compact convex subset.

Let \( E^i \subset R, \hat{E}^i = \prod_{j \neq i}^n E^i, \hat{x}^i = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in \hat{E}^i \). The set of multi-strategies \( x := (x^i, \hat{x}^i) \) may be written as the set \( E := E^i \times \hat{E}^i \). In a \( n \)-person game, the behavior of the \( i \)th player is defined by a loss function. The related conceptions and discussions can be find in [2–4].

**Definition 4.1.** [3] The \( i \)th player’s loss function \( f^i \) is defined by \( f^i : E \rightarrow R \), which evaluates the loss of the \( i \)th player inflicted by each multi-strategy \( x \in E \).

The associated decision rules are defined by

\[
C^i : \hat{E}^i \rightarrow 2^{E^i}, \quad C^i(\hat{x}^i) = \left\{ x^i \in E^i : f^i(x^i, \hat{x}^i) = \inf_{y^i \in E^i} f^i(y^i, \hat{x}^i) \right\}.
\]

Now we defined \( C : E \rightarrow 2^E, C(x) = \prod_{i=1}^n C^i(x^i) \).
Definition 4.2. [3] A non-cooperative equilibrium (or Nash equilibrium) is a fixed point of the set-valued map $C$ on $E$.

As in [3], we consider the function $\varphi : E \times E \to R$ defined by

$$\varphi(x, y) = \sum_{i=1}^{n} (f^i(x^i, \hat{x}^i) - f^i(y^i, \hat{x}^i)).$$

Lemma 4.3. [3, p. 181] The following assertions are equivalent:

(i) $\bar{x} \in E$ is a non-cooperative equilibrium;
(ii) $\forall y \in E$, $\varphi(\bar{x}, y) \leq 0$.

Now we can verify the existence of a non-cooperative equilibrium which generalize the result of Theorem 12.2 in [3, p. 181].

Theorem 4.4. Suppose that, $\forall i \in N := \{1, 2, \ldots, n\}$,

(i) the sets $E^i$ are convex and $\sigma$-compact;
(ii) the function $f^i$ are continuous and the function $y^i \to f^i(y^i, \hat{x}^i)$ are convex; and
(iii) there is a nonempty compact subset $A^i$ of $E^i$ such that for every $y^i \in E^i \setminus A^i$, there exists $u^i \in A^i$ satisfying

$$f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i) > 0.$$ 

Then there exists a non-cooperative equilibrium.

Proof. We introduce the set $E$ and the function $\varphi$ by

(i) $E := E^1 \times \hat{E}^i = \prod_{i=1}^{n} E^i$;
(ii) $\varphi(x, y) = \sum_{i=1}^{n} (f^i(x^i, \hat{x}^i) - f^i(y^i, \hat{x}^i)).$

Since each $E^i$ is a $\sigma$-compact convex subset, so is $E$. The function $\varphi(x, y)$ is continuous on $E \times E$ and the function $y \to \varphi(x, y)$ is concave. Now we shall verify that there exists a nonempty compact $A$ of $E$ such that for every $x \in E \setminus A$, there exists $u \in A$ satisfying

$$\varphi(x, u) > \sup_{x \in E} \varphi(x, x) = 0.$$

We set $A = \prod_{i=1}^{n} A^i$. It follows from the assumption (iii) that $A$ is compact. For every $x \in E \setminus A$, we set $x = (x^1, x^2, \ldots, x^n)$.

(1) If $\forall i$, $x^i \notin A^i$, then $x^i \in E^i \setminus A^i$, thus $x \in \prod_{i=1}^{n} (E^i \setminus A^i)$. From the assumption (iii), there exists $u^i \in A^i$ satisfying

$$f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i) > 0$$.
thus, we set \( u = (u^1, u^2, \ldots, u^n) \), \( u \in \prod_{i=1}^n A^i = A \), and

\[
\varphi(x, u) = \sum_{i=1}^n (f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i)) > 0.
\]

(2) If there exists \( x^i \in A^i \), but not all \( x^i \in A^i \) (if all \( x^i \in A^i \), then \( x \notin E \setminus A \) which is a contradiction), then, we set \( u = (u^1, u^2, \ldots, u^n) \) where if \( x^i \in A^i \) then take \( u^i = x^i \), otherwise take \( u^i \in A^i \) satisfying

\[
f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i) > 0.
\]

Thus \( u \in \prod_{i=1}^n A^i \), and we have

\[
\varphi(x, u) = \sum_{i=1}^n (f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i)) = \sum_{i=1, u^i \neq x^i}^n (f^i(x^i, \hat{x}^i) - f^i(u^i, \hat{x}^i)) > 0 = \sup_{x \in E} \varphi(x, x).
\]

From Lemma 3.1, we obtain that there exists \( \bar{x} \in E \) such that \( \varphi(\bar{x}, y) \leq \sup_{x \in E} \varphi(x, x) = 0 \), for every \( y \in E \). It follows from Lemma 4.3 that \( \bar{x} \) is a non-cooperative equilibrium.

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References