Support Sizes of Directed Triple Systems

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In this paper, we determine the spectrum of support sizes of directed triple systems, for all $\lambda$. © 1996 Academic Press, Inc.

1. Introduction

A $\lambda$-fold triple system of order $v$, denoted TS($v$, $\lambda$), is a pair ($V$, $A$) where $V$ is a $v$-set and $A$ is a collection of 3-subsets (called blocks or triples) of $V$, such that each 2-subset of $V$ is contained in exactly $\lambda$ triples. A triple system is called simple if it contains no repeated triples.

A $\lambda$-fold directed triple system of order $v$, denoted DTS($v$, $\lambda$), is a pair ($V$, $B$) where $V$ is again a $v$-set while $B$ is a collection of ordered 3-subsets (called directed or transitive triples) of $V$, such that each ordered pair of distinct elements of $V$ is contained in exactly $\lambda$ directed triples. We note that each directed triple contains 3 ordered 2-subsets and the ordered 2-subsets contained in the directed triple ($a$, $b$, $c$) are ($a$, $b$), ($a$, $c$) and ($b$, $c$). A directed triple system is also called a transitive triple system in the literature.

Let ($V$, $B$) be a DTS($v$, $\lambda$). If we replace each directed triple ($a$, $b$, $c$) $\in$ $B$ by the (unordered) triple {$a$, $b$, $c$}, then the resulting collection of triples contains each 2-subset of $V$ precisely $2\lambda$ times. This produces a TS($v$, $2\lambda$), called the underlying triple system of the DTS($v$, $\lambda$). A directed triple system is called simple if it contains no repeated directed triples, and it is pure if the underlying triple system is simple.

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The following two results for the existence of triple systems and directed triple systems are well known:

**Lemma 1** [6]. There exists a TS\((v, \lambda)\) if and only if
\[
\lambda(v - 1) \equiv 0 \pmod{2} \quad (1.1)
\]

**Lemma 2** [7, 9]. There exists a DTS\((v, \lambda)\) if and only if
\[
\lambda(v - 1) \equiv 0 \pmod{3} \quad (1.2)
\]

It is easy to verify that the number of directed triples contained in a DTS\((v, \lambda)\) is
\[
b = \frac{\lambda v(v - 1)}{3} \quad (1.3)
\]

Let \(B^*\) be the set of all the distinct directed triples contained in a DTS\((v, \lambda)\) and let \(b^* = |B^*|\). \(B^*\) is called the support of the DTS\((v, \lambda)\) and \(b^*\) called the support size of the DTS\((v, \lambda)\).

Designs with repeated blocks have interesting applications in statistics [4] and have been studied extensively. The support size problem for triple systems has been almost completely solved [2].

For directed triple systems, let
\[
SSD(v, \lambda) = \{b^* | \text{there is a DTS}(v, \lambda) \text{ of support size } b^*\} \quad (1.4)
\]

Our purpose in this paper is to determine the set SSD\((v, \lambda)\) for every \(v\) and every \(\lambda\). Let
\[
M_{v, \lambda} = \min \{\lambda, 3(v - 2)\} v(v - 1)/3
\]
\[
DS(v, \lambda) = \begin{cases} 
\{v(v - 1)/3, v(v - 1)/3 + 2, v(v - 1)/3 + 3, \ldots, M_{v, \lambda}\} & \text{if } v \equiv 0, 1 \pmod{3} \\
\{(v(v - 1) + 22)/3, (v(v - 1) + 22)/3 + 1, \ldots, M_{v, \lambda}\} & \text{if } \lambda \equiv 0 \pmod{3} \text{ and } v \equiv 2 \pmod{3}, v \neq 2 \\
\emptyset & \text{if } \lambda \not\equiv 0 \pmod{3} \text{ and } v \equiv 2 \pmod{3}.
\end{cases} \quad (1.5)
\]

The main result of this paper is the following theorem.

**Theorem 1.1.** For all \(v \geq 4\) and all \(\lambda\),
\[
SSD(v, \lambda) = \begin{cases} 
DS(v, \lambda) & \text{if } \lambda \neq 3(v - 2) \\
DS(v, \lambda) \setminus \{M_{v, \lambda} - 1\} & \text{if } \lambda = 3(v - 2).
\end{cases} \quad (1.6)
\]
2. Necessary Conditions

**Lemma 2.1.** If \( v \equiv 0 \) or 1 (mod 3), then \( \text{SSD}(v, \lambda) \subseteq \text{DS}(v, \lambda) \).

**Proof.** Each DTS\((v, \lambda)\) contains \( \lambda(v-1)/3 \) directed triples and each directed triple can repeat at most \( \lambda \) times. Thus we have
\[
\lambda(v-1)/3 \leq b^* \leq \lambda(v-1)/3 = M_{v, \lambda} \tag{2.1}
\]
Obviously \( b^* = v(v-1)/3 + 1 \) is impossible, the conclusion follows.

The main part of this section is to deal with the case when \( v \equiv 2 \) (mod 3) and \( \lambda \equiv 0 \) (mod 3). In this case, there is no DTS\((v, 1)\) exist. Let \((V, B)\) be a DTS\((v, \lambda)\) with \( v \equiv 2 \) (mod 3) and \( \lambda \equiv 0 \) (mod 3). For each \( i = 1, 2, ..., \lambda \) let \( B \) denote the multiset of \( i \)-times repeated directed triples. Let \( |B| = ib_i \) be the number of directed triples contained in \((B)\). Then we have
\[
\sum_{i=1}^\lambda ib_i = \lambda(v-1)/3. \tag{2.2}
\]
Since \( v \equiv 2 \) (mod 3), then we have
\[
\sum_{i=1}^\lambda ib_i = \lambda/3. \tag{2.3}
\]
For each \( x \in V \), let \( B(x) \) denote the multiset of directed triples in \( \bigcup_{i=1}^{\lambda-1} B_i \) containing \( x \) and let \( d(x) = |B(x)| \).

**Lemma 2.2.** If \( x \neq y \), \( d(x) = d(y) = \lambda \), then \( |B(x) \cap B(y)| = 0 \).

**Proof.** If there is a directed triple \( B \) containing both \( x \) and \( y \), then \( B \) must be of the form \((x, y, a)\) or \((x, a, y)\). If \( B = (x, y, a) \). Since \( B \) can repeat at most \( \lambda - 1 \) times and so there exists a directed triple of form \((x, b, y)\). Then there exist \( \lambda \) directed triples containing \((y, a)\) and \( \lambda \) directed triples containing \((b, y)\) and this implies that \( d(y) \geq 2\lambda \) which is impossible. The case \( B = (x, a, y) \) can be proved similarly.

**Lemma 2.3.** If there exist \( n \) distinct elements \( x_1, x_2, ..., x_n \in V \) such that
\[
|B(x_i) \cap B(x_j)| = 0, \quad \forall i, j = 1, 2, ..., n, \quad i \neq j,
\]
then
\[
\sum_{i=1}^n d(x_i) \leq \sum_{i=1}^{\lambda-1} ib_i - 5\lambda/3. \tag{2.4}
\]
**Proof.** If

$$|B(x_i) \cap B(x_j)| = 0 \quad \forall i, j = 1, 2, ..., n, \quad i \neq j,$$

and

$$\sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{j-1} ib_j - \frac{2j}{3},$$

then there exist exactly $2\lambda/3$ directed triples $x_1, x_2, ..., x_{2\lambda/3} \in \bigcup_{i=1}^{j-1} B$, such that $x_i \notin B(x_j), 1 \leq i < 2\lambda/3, 1 \leq j \leq n$. Then for each $x \in \bigcup_{i=1}^{j-1} B$, we have $d(x) \neq 0 \pmod{\lambda}$, a contradiction.

**Lemma 2.4.** For each $x \in V$, $d(x) \leq \sum_{i=1}^{j-1} ib_j - (5\lambda/3)$.

**Proof.** The conclusion follows from Lemma 2.2, Lemma 2.3 and the fact that $d(x) \equiv 0 \pmod{\lambda}$.

Let $t_n$ be the number of elements $x \in V$ with $d(x) = n\lambda$. Let

$$\sum_{i=1}^{j-1} ib_j = (k + 5/3)\lambda \quad (2.5)$$

**Lemma 2.5.** For $1 \leq i \leq k$,

- If $2i < k + 1$ and $t_i \geq 1$, then
  $$t_{k+1-i} + \sum_{j=1}^{i-1} j/t_{k+1-i+j} \leq 2t_i,$$

- If $2i = k + 1$ and $t_i \geq 1$, then
  $$t_i - 1 + \sum_{j=1}^{i-1} j/t_{k+1-i+j} \leq 2t_i,$$

- If $2i > k + 1$ and $t_i \geq 1$, then
  $$t_{k+1-i} + \sum_{j=1}^{i-1} j/t_{k+1-i+j} - 2i + k + 1 \leq 2t_i.$$

**Proof.** For any $x, y \in V, x \neq y$, we have $|B(x) \cap B(y)| \equiv 0 \pmod{\lambda}$. Thus, if $d(x) + d(y) = (k + 1 + j)\lambda, x \neq y$, then $|B(x) \cap B(y)| \geq j\lambda$. By Lemma 2.3, if $d(x) + d(y) = (k + 1 + j)\lambda, x \neq y$, then $|B(x) \cap B(y)| \geq \lambda$. Let $x \in V, d(x) = i\lambda$, then there are exactly $i\lambda$ directed triples containing $x$. So the number of elements other than $x$ contained in these $i\lambda$ directed triples are $2i\lambda$. The conclusion follows.
Let $k = 4$, the following lemma can be proved in a similar way.

**Lemma 2.6.** Suppose

$$\sum_{i=1}^{k-1} ib_i = 17\lambda/3.$$  

If there exist $s$ pairs of elements $x_1, x_2, ..., x_s, y_1, y_2, ..., y_s \in V$, such that $d(x_i) = d(y_i) = 2\lambda$, $B(x_i) \cap B(y_i) = \emptyset$, $1 \leq i \leq s$, then

$$3t_4 + 2t_3 + 3(s-1) + (t_2 - 2s) + t_1 \leq 8.$$  

If there exist $x, y \in V$ such that $d(x) = \lambda$, $d(y) = 3\lambda$ and $B(x) \cap B(y) = \emptyset$, then

$$2(t_3 - 1) + t_2 + (t_1 - 1) \leq 8.$$  

With the above preparations, now we are in a position to prove the following result.

**Lemma 2.7.**

$$\sum_{i=1}^{k-1} ib_i \geq 20\lambda/3. \quad (2.6)$$  

**Proof.** Since

$$\sum_{i=1}^{k-1} ib_i \equiv 2\lambda/3 \, (\text{mod} \, \lambda).$$  

then we may write

$$\sum_{i=1}^{k-1} ib_i = (k + 5/3)\lambda.$$  

By Lemma 2.4, we have $k \geq 1$ and $t_i = 0$ if $i \geq k + 1$. So we have

$$\sum_{i=1}^{k} i\lambda t_i = (3k + 5)\lambda. \quad (2.7)$$  

By Lemma 2.2 and Lemma 2.3, $k = 1$ is impossible.

If $k = 2$, then $t_1 \leq 2$. Then by (2.7), $t_1 = 1$, $t_2 = 5$. Let $i = 2$, then $t_1 + t_2 - 1 = 5 > 4$, a contradiction to Lemma 2.5 (iii).
If \( k = 3 \), then \( t_1 \leq 3 \) and \( t_i = 0 \) if \( i \geq 4 \). By Lemma 2.5, if \( t_2 \neq 0 \), then \( t_2 + t_3 \leq 4 \); if \( t_3 \neq 0 \), then \( t_1 + t_2 + 2(t_3 - 1) \leq 6 \). It can be checked that the equation
\[
t_1 + 2t_2 + 3t_3 = 14
\]
has no solutions satisfying the above conditions.

If \( k = 4 \), then \( t_1 \leq 4 \) and \( t_i = 0 \) if \( i \geq 5 \). The equation
\[
t_1 + 2t_2 + 3t_3 + 4t_4 = 17
\]
has the following 34 solutions satisfying \( t_1 \leq 4 \):

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<td>(5) 0 1 7 0</td>
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Lemma 2.6(i) eliminates cases (1), (2), (3), (4), (6), (7), (9), (10), (12), (15), (16), (17), (19), (20) and (21); Lemma 2.6(ii) eliminates cases (8), (11) and (13); Lemma 2.5 eliminates all the remaining 16 cases.

This proves the lemma.

**Lemma 2.8.** If \( v \equiv 2 \pmod{3} \) and \( \lambda \equiv 0 \pmod{3} \), then
\[
\sum_{i=1}^{3} b_i \geq \left\{ \frac{v(v-1) + 22}{3} \right\}.
\]

(2.8)
Proof. Let
\[ \sigma(\lambda) = \begin{cases} (\lambda - 2)/2, & \text{if } \lambda \equiv 0 \pmod{6} \\ (\lambda - 1)/2, & \text{if } \lambda \equiv 3 \pmod{6}. \end{cases} \]

For $1 \leq k \leq \sigma(\lambda)$, it is easy to see that
\[ \sum_{j=1}^{k} b_j \geq \sum_{i=1}^{k} b_{\lambda-i}. \]  (2.9)

By Lemma 2.7, there exists a non-negative integer $h$ such that
\[ \sum_{i=1}^{\lambda-1} ib_i = (20 + 3h) \lambda/3. \]  (2.10)

We prove that
\[ \sum_{i=1}^{\lambda-1} b_i \geq 14 + 2h. \]  (2.11)

Case 1. If $\lambda \equiv 0 \pmod{6}$: By (2.9), we may write
\[ \sum_{i=1}^{\sigma(\lambda)} b_i = \sum_{i=\sigma(\lambda)+2}^{\lambda-1} b_i + t \]

for some $t \geq 0$. Then
\[
\sum_{i=1}^{\lambda-1} ib_i = \sigma(\lambda) \left\{ \sum_{i=\sigma(\lambda)+2}^{\lambda-1} b_i + t \right\} - \sum_{i=1}^{\sigma(\lambda)-1} (\sigma(\lambda) - i) b_i + \sum_{i=\sigma(\lambda)+1}^{\lambda-1} ib_i = \frac{b_{\lambda/2}}{2} + \lambda \sum_{i=\sigma(\lambda)+2}^{\lambda-1} b_i + \sum_{i=1}^{\sigma(\lambda)-1} (\sigma(\lambda) - i)(b_{\lambda-i} - b_i) + (\lambda - 2) t/2. \]

It follows from (2.9) that
\[ \alpha = \sum_{i=\sigma(\lambda)+2}^{\lambda-1} \left\{ i - \sigma(\lambda) \right\} b_i - \sum_{i=1}^{\sigma(\lambda)-1} (\sigma(\lambda) - i) b_i \leq 0. \]

Thus we have
\[ \alpha = \sum_{i=1}^{\sigma(\lambda) - 1} (\sigma(\lambda) - i)(b_{\lambda-i} - b_i) \leq 0. \]
Then by (2.10), we have
\[ b_{\lambda /2} = \frac{(20 + 3h) \lambda /3 - \lambda \sum_{i=\sigma(\lambda) + 1}^{\lambda - 1} b_i - \lambda - (\lambda - 2) t/2}{\lambda/2} \]
\[ = \frac{(20 + 3h) \lambda /3 + t - \lambda}{\lambda/2} \geq 40/3 + 2h. \]

Since \( b_{\lambda /2} \) is an integer, then
\[ b_{\lambda /2} \geq 14 + 2h. \]

**Case 2.** If \( \lambda \equiv 3 \mod 6 \): By (2.9), we may write
\[ \sum_{i=1}^{\sigma(\lambda)} b_i = \sum_{i=\sigma(\lambda) + 1}^{\lambda - 1} b_i + t \]
for some \( t \geq 0 \). Then, by (2.10),
\[ \sigma(\lambda) \left\{ \sum_{i=1}^{\sigma(\lambda) - 1} b_i + t \right\} - \sum_{i=1}^{\sigma(\lambda) - 1} (\sigma(\lambda) - i) b_i + \sum_{i=\sigma(\lambda) + 1}^{\lambda - 1} i b_i \]
\[ = \lambda \sum_{i=1}^{\sigma(\lambda) + 1} b_i + \sum_{i=\sigma(\lambda) + 2}^{\lambda - 1} (t - 1 - \sigma(\lambda)) b_i + \sigma(\lambda) t - \sum_{i=1}^{\sigma(\lambda) - 1} (\sigma(\lambda) - i) b_i \]
\[ = (20 + 3h) \lambda /3. \]

It follows from (2.9) that
\[ \sum_{i=1}^{\lambda - 1 \vee \sigma(\lambda) + 1} b_i = \frac{(20 + 3h) \lambda /3 - \sigma(\lambda) t - \lambda}{\lambda} \]
and
\[ \sum_{i=1}^{\lambda - 1 \vee \sigma(\lambda) + 1} b_i = 2 \sum_{i=\sigma(\lambda) + 1}^{\lambda - 1} b_i + t = (40 + 6h)/3 + (t - 2\lambda) \lambda. \]

Since \( \lambda \leq 0, t \geq 0 \), then
\[ \sum_{i=1}^{\lambda - 1 \vee \sigma(\lambda) + 1} b_i \geq 14 + 2h. \]

It follows from (2.10) and
\[ \sum_{i=1}^{\lambda} i b_i = r(v - 1) \lambda /3 \]
that

\[ b_j = (v(v - 1) - 20 - 3h)/3. \]

Then, by (2.6), we have

\[ \sum_{i=1}^{j} b_i \geq 14 + 2h + (v(v - 1) - 20 - 3h)/3 \geq (v(v - 1) + 22)/3. \]

This completes the proof.

Combining Lemma 2.1 and Lemma 2.8, we have proved the following theorem:

**Theorem 2.1.**

\[ \text{SSD}(v, \lambda) \subseteq \text{DS}(v, \lambda). \]

3. **Recursive Constructions**

In this section, we give our main recursive constructions for directed triple systems with various support sizes. Our constructions depend heavily on the embeddings of simple triple systems and embeddings of simple directed triple systems.

Let \((V, A)\) and \((U, B)\) be a TS\((v, \lambda)\) and a TS\((u, \lambda)\), respectively. If \(V \subseteq U\) and \(A\) is a subcollection of \(B\), then \((V, A)\) is called a subsystem of \((U, B)\), or \((V, A)\) is said to be embedded in \((U, B)\). The embedding problem for simple triple systems was completely solved in [10]:

**Theorem 3.1.** The necessary and sufficient conditions for the embedding of a simple TS\((v, \lambda)\) in a simple TS\((u, \lambda)\) are

\[ \lambda(u - 1) \equiv \lambda(v - 1) \equiv 0 \pmod{2} \]

\[ \lambda(u - 1) \equiv \lambda(v - 1) \equiv 0 \pmod{6} \quad (3.1) \]

\[ u \geq 2v + 1. \]

The following theorem can be proved in a similar way [1]:

**Theorem 3.2.** The necessary and sufficient conditions for the embedding of a simple DTS\((v, \lambda)\) in a simple DTS\((u, \lambda)\) are

\[ \lambda(u - 1) \equiv \lambda(v - 1) \equiv 0 \pmod{3} \]

\[ u \geq 2v + 1. \]
The concept of embedding can be generalized in the following way.

Let \( u, v \) and \( \lambda \) be given positive integers, \( \mu \) be a given non-negative integer, \( 0 \leq \mu \leq \lambda \). A DTS\((u, v; \lambda, \mu)\) is a triple \((U, V, B)\) where \( U \) is a \( u \)-set, \( V \) is a \( v \)-subset of \( U \), \( B \) is a collection of directed triples of \( U \) such that:

(i) each ordered 2-subset of \( V \) is contained in exactly \( \mu \) directed triples of \( B \); (ii) each ordered 2-subset of \( U \), not both elements from \( V \), is contained in exactly \( \lambda \) directed triples of \( B \). A DTS\((u, v; \lambda, \mu)\) is called simple if it contains no repeated directed triples.

It can be easily checked that each DTS\((u, v; \lambda, \mu)\) contains \( \lambda(u-1)/3 - (\lambda-\mu)(v-1)/3 \) directed triples. Let

\[
SSD(u, v; \lambda, \mu) = \{ b^* \mid \text{there is a DTS}(u, v; \lambda, \mu) with } b^* \text{ distinct directed triples} \}.
\]

For the application of incomplete directed triple systems in determining the spectrum of support sizes for directed triple systems, we have the following lemma.

**Lemma 3.1.** If \((V, A)\) is a DTS\((v, \mu)\) with support size \( x \), \((U, V, B)\) is DTS\((u, v; \lambda, \mu)\) with support size \( y \), and \( A \cap B = \emptyset \), then \((U, A \cup B)\) is a DTS\((u, \lambda, \mu)\) with support size \( x + y \).

**Lemma 3.2.** If \( \lambda \geq 1 \), \( 0 \leq t \leq \min\{\lambda-1, (3v-1)v\} \), \( t \neq 1 \), then

\[
v(v+1) + (t-1)(v+1) \in SSD(2v+1, v; \lambda, 0).
\]

**Proof.** Let \( V = \{\infty_1, \infty_2, \ldots, \infty_v\}, U = V \cup Z_{v+1} \). Let

\[
F_j = \{(a, b) \mid a, b \in Z_{v+1}, b - a = j \pmod{v+1} \} = Z_{v+1} \setminus \{0\}
\]

For any permutation \( \pi \) on the set \([1, 2, \ldots, v]\), let

\[
\pi(A) = \bigcup_{j=1}^v \{(a, \infty_j, b) \mid (a, b) \in F_{\sigma(j)}\}
\]

\[
\pi(B) = \bigcup_{j=1}^v \{(\infty_j, a, b) \mid (a, b, \infty_j) \in F_{\sigma(j)}\}.
\]

For \( 0 \leq i \leq v-1 \), let \( \sigma_i = \sigma' \) where \( \sigma \) is the following permutation on the set \([1, 2, \ldots, v]\)

\[
\sigma(j) = \begin{cases} j + 1, & \text{if } 1 \leq j \leq v - 1 \\ 1, & \text{if } j = v \end{cases}
\]
Then, for \( i = 0, 1, 2, \ldots, v - 1 \), \((U, V, \sigma_i(A))\) are \( v \) pairwise disjoint \( DTS(2v + 1, v; 1, 0) \) and \((U, V, \sigma_j(B))\) are \( v \) pairwise disjoint simple \( DTS(2v + 1, v; 2, 0) \) and

\[
\sigma_i(A) \cap \sigma_j(B) = \emptyset, \quad 0 \leq i, j \leq v - 1.
\]

For \( 2 \leq k \leq v \), let \( \tau_k \) be the following permutation on \( \{1, 2, \ldots, v\} \):

\[
\tau_k(j) = \begin{cases} 
  j + 1, & \text{if } 1 \leq j < k - 1, \\
  1, & \text{if } j = k, \\
  j, & \text{if } k + 1 \leq j \leq v.
\end{cases}
\]

Then \( \tau_v = \sigma \) and for \( 2 \leq k \leq v \), we have

\[
|\tau_k(A) \cup \sigma(A)| = (v + k)(v + 1),
\]

\[
|\tau_k(A) \cap \sigma(A)| = \begin{cases} 
  0, & \text{if } 2 \leq i \leq v - 1, \ i \neq v + 1 - k \\
  v + 1, & \text{if } i = v + 1 - k \\
  v + 1, & \text{if } 0 \leq i \leq v - 1.
\end{cases}
\]

Write \( t = (2m + n) v + k \), \( 0 \leq m \leq v \), \( 0 \leq n \leq v - 1 \), \( 0 \leq k \leq v - 1 \).

(i) If \( k = 0 \), let \( B \) be the multiset containing a copy for each \( \sigma_i(B) \), \( 0 \leq i \leq m - 1 \), a copy for each \( \sigma_i(A) \), \( 1 \leq i \leq n \), and \( \lambda - (2m + n) \) copies of \( \sigma_0(A) \).

(ii) If \( 1 \leq k \leq v - 1 \), we may suppose \( n \leq v - 2 \). If \( k + n < v \), let \( B \) be the multiset containing a copy for each \( \sigma_i(B) \), \( 0 \leq i \leq m - 1 \), a copy for each \( \sigma_i(A) \), \( 2 \leq i \leq n + 1 \), and \( \lambda - (2m + n + 1) \) copies of \( \sigma_0(A) \). If \( k + n \geq v \), let \( B \) be the multiset containing a copy for each \( \sigma_i(B) \), \( 0 \leq i \leq m - 1 \), a copy for \( \tau_{k+1}(A) \) and for each \( \sigma_i(A) \), \( 2 \leq i \leq n + 1 \), and \( \lambda - (2m + n + 1) \) copies of \( \sigma_0(A) \).

Then in each case, \((U, V, B)\) is \( DTS(2v + 1, v; 1, 0) \) with support size \((v + 1)(t + 1)\). This completes the proof.

**Lemma 3.3.** If \( \lambda \geq 1 \), \( 0 \leq t \leq \min\{(\lambda - 1)v, (3v - 1)v\} \), \( t \neq 1 \), then

\[
(v + 1)(v + 4) + t(v + 4) \in SSD(2v + 4, v; \lambda, 0).
\]

**Proof.** Let \( V = \{\infty_1, \infty_2, \ldots, \infty_v\} \), \( U = V \cup Z_{v+4} \). Let

\[
D = \{(i, i + v + 2, i + v + 1) | i \in Z_{v+4}\},
\]

\[
F_j = \{(a, b) | a, b \in Z_{v+4}, b - a \equiv j (\mod v + 4)\},
\]

\[
j \in Z_{v+4} \setminus \{0, v + 1, v + 2, v + 3\}.
\]
For any permutation $\pi$ on $\{1, 2, ..., v\}$, let

$$\pi(A) = \left\{ \bigcup_{j=1}^{v} \{ (a, \pi_j, b) \mid (a, b) \in F_{\pi(j)} \} \right\} \cup D,$$

$$\pi(B) = \left\{ \bigcup_{j=1}^{v} \{ (\pi_j, a, b), (a, b, \pi_j) \} \mid (a, b) \in F_{\pi(j)} \right\} \cup 2D.$$

Then $(U, V, \pi(A))$ is a $DTS(2v + 4, v; 1, 0)$ and $(U, V, \pi(B))$ is a $DTS(2v + 4, v; 2, 0)$ with support size $(2v + 1)(v + 4)$.

Now define $\sigma_i(0 \leq i \leq v - 1)$ and $\tau_k(2 \leq k \leq v - 1)$ as in Lemma 3.2, the conclusion then follows in a similar way, we omit the details.

**Lemma 3.4.** Let $\delta = 1, 2$ or $3$, $\lambda \geq \delta(v + 1)$, and $1 \leq t \leq \min\{\lambda - \delta(v + 1), 3(v - 2)\}$, then

$$tv(v + 1) + 2\delta v(2v + 1)(v + 1)/3 \in SSD(2v + 1, v; \lambda, \delta(v + 1)).$$

**Proof.** Let $V = \{\infty_1, \infty_2, ..., \infty_s\}$, $U = V \cup Z_{\tau}$. For $0 \leq i \leq v - 1$, let $\sigma_i(A)$ and $\sigma_i(B)$ be defined as in Lemma 3.2. Let

$$A_1 = \{ (a, b, c), (c, b, a) \mid \{ a, b, c \} \subset Z_{\tau} \} \cup \{ (\infty_i, a, \infty_j) \},$$

$$(\infty_j, a, \infty_i) \mid a \in Z_{\tau}, \infty_i, \infty_j \in V \cup \sigma_{\tau - 2}(A) \cup \sigma_{\tau - 1}(A);$$

$$A_2 = \{ (y, x, z), (z, x, y) \mid (x, y, z), (z, y, x) \in A_1 \};$$

$$A_3 = \{ (x, z, y), (y, z, x) \mid (x, y, z), (z, y, x) \in A_1 \}.$$  

Then, for $i = 1, 2, 3$, $(U, V, A_i)$ are three pairwise disjoint simple

$DTS(2v + 1, v; 2, 0)$ and

$$A_i \cap \sigma_j(A) = \emptyset, \quad A_i \cap \sigma_j(B) = \emptyset, \quad 1 \leq i \leq 3, \quad 0 \leq j \leq v - 3.$$

Let $\lambda - \delta(v + 1) = \lambda_0$, for $1 \leq t \leq \min\{\lambda - \delta(v + 1), 3(v - 2)\}$, from $\sigma_i(A)$ and $\sigma_j(B)$, $0 \leq i \leq v - 3$, it can be easily form a $DTS(2v + 1, v; \lambda_0, 0)$ with support size $tv(v + 1)$. Now let

$$D = \bigcup_{i=0}^{\delta} A_i,$$

then $(U, V, D)$ is a $DTS(2v + 1, v; \lambda, \delta(v + 1))$ with support size $tv(v + 1) + 2\delta(v + 1)(v + 1)/3$.

The following lemma can also be proved in a similar way.
Lemma 3.5. Let $\delta = 1, 2$ or $3$, $\lambda \geq \delta(v + 4)$ and $1 \leq t \leq \min\{\lambda - \delta(v + 4), 3(v - 2)\}$, then
\[ t(v + 1)(v + 4) + \delta(2v + 4)(2v + 3)(v + 4)/3 \in SSD(2v + 4, v; \lambda, \delta(v + 4)). \]

4. Determining the Spectrum of Support Sizes for $DTS(v, \lambda)$s of Small Order

Lemma 4.1. For any $\lambda \geq 2$ and $v \equiv 0$ or 1 (mod 3),
\[ DS(v, 2) \subseteq SSD(v, \lambda). \]

Proof. It is proved in [5, 8] that if $v \equiv 0$ or 1 (mod 3), then for each $k \in \{0, 1, \ldots, v(v - 1)/3 - 2, v(v - 1)/3\}$, there exists a pair of $DTS(v, 1)$, say $(V, A)$ and $(V, B)$, such that $|A \cap B| = k$. Take a copy of $A$ and $\lambda - 1$ copies of $B$, this gives a $DTS(v, \lambda)$ with support size $2v(v - 1)/3 - k$ and the conclusion follows.

Corollary. For all $v \equiv 0$ or 1 (mod 3), $SSD(v, 2) = DS(v, 2)$.

Lemma 4.2. $SSD(4, \lambda) = DS(4, \lambda)$ for all $\lambda \geq 1$.

Proof. Let $V = Z_4$ and
\begin{align*}
B_1 &= \{012, 103, 231, 320\}, \\
B_2 &= \{012, 103, 230, 321\}, \\
B_3 &= \{012, 130, 203, 321\}, \\
B_4 &= \{023, 132, 210, 301\}, \\
B_5 &= \{023, 132, 201, 310\}, \\
B_6 &= \{031, 120, 213, 302\}, \\
B_7 &= \{032, 123, 210, 301\}, \\
B_8 &= \{031, 123, 213, 302\}, \\
B_9 &= \{013, 102, 231, 320\}, \\
B_{10} &= \{021, 130, 203, 312\}, \\
B_11 &= \{032, 123, 201, 310\}, \\
B_{12} &= \{023, 130, 201, 312\},
\end{align*}
where we denote the directed triple $(a, b, c)$ by $abc$. 


Then each \((V, B_i)\) is a \(DTS(4, 1)\), \(1 \leq i \leq 12\). Let \(b_{1, 1}, \ldots, b_{1, 12}\) be the support size of the \(DTS(4, \lambda_i)\), \((V, B)\), where \(B\) is the collection formed from \(B_{11}, B_{12}, \ldots, B_{12}\). Then it can be checked that
\[
\begin{align*}
 b_{1, 1} &= 4, & b_{1, 2} &= 6, & b_{1, 3} &= 7, & b_{1, 4} &= 8; \\
 b_{1, 5, 8} &= 9, & b_{1, 6, 4} &= 10, & b_{1, 3, 4, 6} &= 11, & b_{1, 4, 6, 2} &= 12; \\
 b_{1, 5, 7, 8} &= 13, & b_{1, 2, 4, 6} &= 14, & b_{1, 3, 4, 6} &= 15, & b_{2, 5, 6, 7} &= 16; \\
 b_{1, 4, 6, 10, 11} &= 17, & b_{1, 2, 4, 6, 10, 11} &= 18, & b_{1, 5, 6, 7, 8} &= 19; \\
 b_{1, 4, 6, 8, 10, 11} &= 21, & b_{1, 2, 4, 6, 8, 10, 11} &= 22, & b_{1, 5, 6, 7, 8, 9} &= 23.
\end{align*}
\]

This completes the proof.

**Lemma 4.3.** \(SSD(5, \lambda) = DS(5, \lambda)\) for all \(\lambda \equiv 0 \pmod{3}\).

**Proof.** There is a \(DTS(5, \lambda)\) if and only if \(\lambda \equiv 0 \pmod{3}\). Let \(V = \mathbb{Z}_4\), \(U = \{\infty\} \cup \mathbb{Z}_4\) and let
\[
\begin{align*}
 B_1 &= \{0 \times 1, 1 \times 2, 2 \times 3, 3 \times 0, 0 \times 2, 1 \times 3, \\
 &\hspace{1cm} 2 \times 0, 3 \times 1, 0 \times 3, 1 \times 0, 2 \times 1, 3 \times 2\}, \\
 B_2 &= \{\infty 01, \infty 12, \infty 23, \infty 30, \infty 02, \infty 13, \infty 10, \infty 21, \infty 32, \infty 03, \infty 20, \infty 31\}, \\
 B_3 &= \{01 \times 12, 23 \times 30, 02 \times 13, 13 \times 010, \infty 21, \infty 32, \infty 03, \infty 20, \infty 31\};
\end{align*}
\]

Let \(\lambda = 3\lambda_0\), then \((U, V, \lambda_0, B_1)\) is a \(DTS(5, 4; 3\lambda_0, \lambda_0)\) with support size 12; for \(\lambda_0 \geq 2\), \((U, V, (\lambda_0 - 1) \times B_1 \cup B_2)\) is a \(DTS(5, 4; 3\lambda_0, \lambda_0)\) with support size 24; for \(\lambda_0 \geq 3\), \((U, V, (\lambda_0 - 2) \times B_1 \cup B_2 \cup B_3)\) is a \(DTS(5, 4; 3\lambda_0, \lambda_0)\) with support size 36. Since we can form a \(DTS(4, 2\lambda_0)\) with support size \(x \in SSD(4, 2\lambda_0)\), then by Lemma 3.1 and Lemma 4.2, it can be checked that \(DS(5, \lambda) \setminus \{14, 15, 17\} \subset SSD(5, \lambda)\) if \(\lambda = 3\lambda_0 \geq 3\), and \(DS(5, \lambda) \setminus \{14, 15, 17, 29\} \subset SSD(5, \lambda)\) if \(\lambda = 3\lambda_0 \geq 6\).

Now let
\[
\begin{align*}
 B_4 &= \{123, 10 \times 03, 301, 2 \times 1, 3 \times 2, 102, 20 \times 0, 203, 1 \times 3, \\
 &\hspace{1cm} 123, 10 \times 03, 301, 2 \times 1, 3 \times 2, 201, \infty 02, 302, 3 \times 1\}, \\
 B_5 &= \{032, 01 \times 12, 210, 3 \times 0, 2 \times 3, 013, 31 \times 0, 312, 02 \times 0, \\
 &\hspace{1cm} 032, 01 \times 12, 210, 3 \times 0, \infty 23, 310, \infty 13, 213, 2 \times 0\}.
\end{align*}
\]
Then \((U, \lambda_0 B_4)\) is a \(DTS(5, \lambda)\) with support size 14, \((U, \lambda_0 \cdot B_5)\) is a \(DTS(5, \lambda)\) with support size 15. In \(B_5\), replace the directed triples 032, \(\infty \infty 0\) and 310 by 023, 321, \(\infty \infty \infty\) and 01, this gives a \(DTS(5, \lambda)\) with support size 17. For \(\lambda = 3\lambda_0 \geq 6\), \((U, (\lambda_0 - 1) \cdot B_4 \cup B_5)\) is a \(DTS(5, \lambda)\) with support size 29.

This completes the proof.

**Lemma 4.4.** \(SSD(6, \lambda) = DS(6, \lambda)\) for all \(\lambda \geq 1\).

**Proof.** By Lemma 4.1, we may suppose \(\lambda \geq 3\). Let \(V = Z_6\), the following 12 pairwise disjoint \(DTS(6, 1) \cdot (V, S_i), 1 \leq i \leq 12\) can be found [3]:

- \(S_1 = \{102, 203, 304, 405, 501, 214, 325, 431, 542, 153\}\);
- \(S_2 = \{213, 314, 415, 510, 012, 420, 532, 043, 254, 305\}\);
- \(S_3 = \{051, 152, 253, 354, 450, 204, 310, 421, 032, 143\}\);
- \(S_4 = \{435, 530, 031, 132, 234, 042, 154, 205, 410, 521\}\);
- \(S_5 = \{540, 041, 142, 243, 345, 052, 103, 215, 320, 531\}\);
- \(S_6 = \{324, 425, 520, 021, 123, 430, 541, 053, 104, 315\}\);

and for \(1 \leq i \leq 6\),

\[S_{i+6} = \{(c, b, a) \mid (a, b, c) \in S_i\}\].

Let

- \(A_1 = \{120, 023, 304, 405, 501, 214, 325, 431, 542, 153\}\);
- \(A_2 = \{012, 203, 304, 405, 510, 214, 325, 431, 542, 153\}\);
- \(A_3 = \{012, 203, 304, 405, 510, 214, 325, 431, 542, 513\}\);
- \(A_4 = \{012, 230, 034, 405, 510, 214, 325, 431, 542, 153\}\);
- \(A_5 = \{012, 230, 034, 405, 150, 214, 325, 431, 542, 513\}\);
- \(A_6 = \{021, 230, 034, 405, 150, 124, 325, 431, 542, 531\}\);
- \(A_7 = \{021, 230, 034, 405, 150, 124, 325, 431, 542, 513\}\);

For \(b^* = 10t\), \(2 \leq t \leq \min\{\lambda, 12\}\), let \(B_0 = (\lambda - t + 1) \cdot S_1 \cup \{\cup_{i=2}^{12} S_i\}\), then \((V, B_0)\) is a \(DTS(6, \lambda)\) with support size \(b^* = 10t\).

For \(b^* = 10t + r\), \(1 \leq r \leq 9\) by Lemma 4.1, we may assume \(2 \leq t \leq \min\{\lambda - 1, 11\}\). Let \(B_1 = (\lambda - t) \cdot A_1 \cup S_1 \cup \{\cup_{i=2}^{12} S_i\}\). Since \(|A_1 \cap S_1| = 8\), \(|A_1 \cap S_{11}| = |A_1 \cap S_{12}| = 1\), then \((V, B_1)\) is a \(DTS(6, \lambda)\) with support size \(b^* = 10t + 1\). Let
$\{\lambda - t\} A_1 \cup S_{11} \cup \left\{ \bigcup_{i=2}^{t-2} S_i \right\}$, if $t \neq 11$

$B_9 = \left\{ \begin{array}{ll}
(\lambda - 12) A_1 \cup A_2 \cup \left\{ \bigcup_{i=2}^{12} S_i \right\} & \text{if } t = 11 < \lambda - 1
\end{array} \right.$

$B_2 = (\lambda - t) A_2 \cup \left\{ \bigcup_{i=1}^{t} S_i \right\}$,

$B_6 = (\lambda - t) A_2 \cup \left\{ \bigcup_{i=2}^{t+1} S_i \right\}$,

$B_3 = (\lambda - t) A_6 \cup S_6 \cup S_{12} \cup A_8$.

where

$B_4 = \left\{ \begin{array}{ll}
(\lambda - t) A_6 \cup S_6 \cup S_{12} \cup \left\{ \bigcup_{i=1}^{t-2} S_i \right\}, & \text{if } t \leq 7
\end{array} \right.$

$B_7 = \left\{ \begin{array}{ll}
(\lambda - t) A_6 \cup S_6 \cup S_{12} \cup \left\{ \bigcup_{i=2}^{t-2} S_i \right\}, & \text{if } 8 \leq t \leq 9
\end{array} \right.$

$B_3 = \left\{ \begin{array}{ll}
(\lambda - t) A_6 \cup S_6 \cup S_{12} \cup \left\{ \bigcup_{i=1}^{t-1} S_i \right\}, & \text{if } 10 \leq t \leq 11.
\end{array} \right.$

$B_7 = \left\{ \begin{array}{ll}
(\lambda - t) A_3 \cup S_3 \cup S_{12} \cup D_1, & \text{if } t \geq 4
\end{array} \right.$

$B_7 = \left\{ \begin{array}{ll}
(\lambda - t) A_7 \cup D_2, & \text{if } 2 \leq t \leq 3
\end{array} \right.$

where

$D_1 = \left\{ \begin{array}{ll}
\bigcup_{i=2}^{t} S_i, & \text{if } t \leq 9
\end{array} \right.$

$D_1 = \left\{ \begin{array}{ll}
\bigcup_{i=2}^{t} S_i, & \text{if } 10 \leq t \leq 11
\end{array} \right.$

and

$D_2 = \left\{ \begin{array}{ll}
(\lambda - t) A_7 \cup S_1 \cup S_2, & \text{if } t = 2
\end{array} \right.$

$D_2 = \left\{ \begin{array}{ll}
(\lambda - t) A_7 \cup S_1 \cup S_2 \cup S_3, & \text{if } t = 4
\end{array} \right.$

$D_2 = \left\{ \begin{array}{ll}
(\lambda - t) A_7 \cup \left\{ \bigcup_{i=1}^{t} S_i \right\}, & \text{if } t \leq 8
\end{array} \right.$

$B_4 = \left\{ \begin{array}{ll}
(\lambda - t) A_6 \cup S_{10} \cup \left\{ \bigcup_{i=1}^{8} S_i \right\}, & \text{if } t = 9
\end{array} \right.$

$B_4 = \left\{ \begin{array}{ll}
(\lambda - t) A_6 \cup S_{12} \cup \left\{ \bigcup_{i=2}^{t} S_i \right\}, & \text{if } 10 \leq t \leq 11
\end{array} \right.$
Then it can be checked that for each $1 \leq i \leq 9$, $V(B_i)$ is a $DTS(6, \lambda)$ with support size $b^* = 10t + i$. This completes the proof.

**Lemma 4.5.** If $v \equiv 0$ or $1 \pmod{3}$, then $SSD(7, \lambda) = DS(7, \lambda)$ for all $\lambda \geq 1$.

**Proof.** Let $V_1 = \{a, b, c\}$, $V_2 = Z_4$, $U = V_1 \cup V_2$. Let

**A**$_1 = \{a01, a23, b02, b13, c03, c21, 10a, 32a, 20b, 31b, 30c, 12c\},$

**A**$_2 = \{0a1, 2a3, 0b2, 1b3, 0c3, 2c1, 1a0, 3a2, 2b0, 3b1, 3c0, 1c2\},$

**A**$_3 = \{01a, 23a, 02b, 13b, 03c, 21c, a10, a32, b20, b31, c30, c12\}.$

Then $(U, V_1, A_i)$, $i = 1, 2, 3$ are tree pairwise disjoint $DTS(7, 3; 10)$. Let

**A**$_4 = \{0a2, 1a2, 0a3, 1a3, 1b0, 3b0, 2b1, 3b2, 0c1, 2c0, 3c1, 2c3, a0b, a0c, a1b, a1c, b1c, b2a, b3a, c0b, c2a, c2b, c3a\}$

**A**$_5 = \{a02, a13, a03, a12, b10, b32, b30, b21, 0c1, a23, c02, c20, c31, 0ab, 0ac, 1ab, 1ac, abc, 2ba, 3ba, 3bc, 0cb, 2ca, 2cb, 3ca\}$

**A**$_6 = \{02a, 13a, 03a, 12a, 10b, 30b, 21b, 23b, 01c, 20c, 31c, 23c, a0b, a0c, ab1, ac1, ba2, ba3, ba3, c0b, ca2, cb2, ca3\}$
For $i = 1, 2, 3$, let
\[ B_{6+i} = \{(z, y, x) \mid (x, y, z) \in B_4\} \]
Then, $(U, V_i, A_i)$, $4 \leq i \leq 9$ are 6 pairwise disjoint $DTS(7, 4; 2, 1)$.

Let
\[ A_0 = \{01a, a12, 2a3, 3a0, 0b2, 2b0, 3b1, 0c3, 1c0, 2c1, 3c2\} \]
Then $(U, V_i, A_0)$ is a $DTS(7, 3; 1, 0)$ with $|A_0 \cap A_2| = 7$, $|A_0 \cap A_3| = 1$, $|A_0 \cap A_i| = 3$, $|A_0 \cap A_j| = 0$ for $i = 1, 4, 5, 8, 9$.

By Lemma 4.1, we have
\[ DS(7, 2) = \{14, 16, 17, ..., 28\} \subseteq SS(7, \lambda) \]
for all $\lambda \geq 2$.

For $\lambda \geq 3$, let
\[ D_0 = \{01a, a12, 2a3, 3a0, 0b2, 1b3, 2b0, 3b1, 0c3, 1c0, 2c1, c32\} \]
\[ B_1 = (\lambda - 1) A_0 \cup A_3, \quad B_2 = (\lambda - 1) A_0 \cup A_1, \]
\[ B_3 = (\lambda - 2) A_0 \cup A_2 \cup A_3, \quad B_4 = (\lambda - 2) A_0 \cup A_1 \cup A_2, \]
\[ B_5 = (\lambda - 2) D_0 \cup A_1 \cup A_2, \quad B_6 = (\lambda - 2) A_0 \cup A_1 \cup A_3, \]
\[ B_7 = (\lambda - 2) A_1 \cup A_2 \cup A_3. \]
Then, for $1 \leq i \leq 7$, $(U, V_i, B_i)$ is a $DTS(7, 3; \lambda, 0)$ with support sizes 23, 24, 28, 29, 30, 35, and 36, respectively. It is easy to see that
\[ \{2, 4, 6\} \subseteq SSD(3, \lambda) \quad \text{for} \quad \lambda \geq 3. \]

Thus, by Lemma 2, we have $\{29, 30, ..., 42\} \subseteq SSD(7, \lambda)$ for all $\lambda \geq 3$.

Let $S_0 = A_4$ and $S_i$ be obtained from $A_4$ by replacing the directed triples $0a2$ and $a0b$ by $a02$ and $0ab$, then $(U, V_2, S_i)$ is a $DTS(7, 4; 2)$ with $|S_0 \cap A_4| = 22$ and $|S_i \cap A_5| = 2$. Generally, we may form in this way a set $S_i$ from $A_4$ such that $|S_0 \cap A_4| = 24 - 2i$ and $|S_i \cap A_5| = 2i$ for $0 \leq i \leq 11$.

For $\lambda \geq 4$, let $\lambda = 2\lambda_0 + r$, $\lambda_0 \geq 2$, $0 \leq r \leq 3$. Let
\[ D_1 = A_4 \cup \{abc, cha\}, \]
\[ D_2 = A_4 \cup \{bac, cab\}, \]
\[ D_3 = A_4 \cup \{acb, bea\}. \]
\[
B = \begin{cases}
  (\lambda_0 - t + 1) S_t \cup \left( \bigcup_{j=5}^{t+3} A_j \right) & \text{if } r = 0, \\
  (\lambda_0 - t + 1) S_t \cup \left( \bigcup_{j=5}^{t+3} A_j \right) \cup \left( \bigcup_{k=1}^{r} D_k \right) & \text{if } 1 \leq r \leq 3.
\end{cases}
\]

Then \((U, V_2, B)\) is a DTS\((7, 4; \lambda, \lambda_0)\) with support sizes \(b^* = 24t + 14r - 2i, 0 \leq i \leq 11\). Obviously this DTS\((7, 4; \lambda, \lambda_0)\) is disjoint with any directed triple system on \(V_2\). Since SSD\((4, \lambda) = DS(4, \lambda_0)\), by Lemma 2.3, we have in fact proved that SSD\((7, \lambda) = DS(7, \lambda) \setminus \{43\}\).

To complete the proof, let

\[
B = (\lambda - 2) (A_0 \cup \{abc, cba\} \cup A_5 \cup \{013, 120, 231, 302\}).
\]

This gives a DTS\((7, \lambda)\) with support size \(b^* = 43\).

**Lemma 4.6.** For any \(\lambda \equiv 0 \pmod{3}\), SSD\((8, \lambda) = DS(8, \lambda)\).

**Proof.** Let \(V = Z_6, U = Z_6 \cup \{a, b\}\). Let \(B\) be the following collection of directed triples:

\[
\begin{align*}
015, 04a, 14a, 45a, 235, 2a4, 3a4, a54; \\
014, 05a, 1a5, 23a, 245, 35a, 45a, 5a1; \\
302, 310, 421, 513, a20, 1b2, 40b, 52b, a3b, b43, b50, ba1.
\end{align*}
\]

\[
A_1 = \{ (i, a, i + 1), (i, a, i + 2), (i, a, i + 4), (i, b, i + 5) \mid i \in Z_6 \}
\cup \{ (i, a, i + 3), (a, i, b), (i + 3, b, i), (b, i + 3, a) \mid i = 0, 1, 2 \},
\]

\[
A_2 = \{ (i + 1, a, i), (i + 2, a, i), (i + 4, b, i), (i + 5, b, i) \mid i \in Z_6 \}
\cup \{ (i + 3, a, i), (b, i, a), (i, b, i + 3), (a, i + 3, b) \mid i = 0, 1, 2 \},
\]

\[
A_3 = \{ (a, i, i + 1), (i, i + 2, a), (b, i, i + 4), (i, i + 5, b) \mid i \in Z_6 \}
\cup \{ (a, i, i + 3), (a, i + 2), (i, b, i + 3, i), (i + 3, b, a) \mid i = 0, 1, 2 \},
\]

\[
A_4 = \{ (i, i + 1, a), (a, i, i + 2), (i, i + 4, b), (b, i, i + 5) \mid i \in Z_6 \}
\cup \{ (i, i + 3, a), (a, b, i), (i + 3, i, b), (b, a, i + 3) \mid i = 0, 1, 2 \},
\]
\[ A_3 = \{(i + 1, i, a), (a, i + 2, i), (i + 4, i, b), (b, i + 5, i) \mid i \in \mathbb{Z}_6\} \]
\[ \cup \{(i + 3, i, a), (b, a, i), (i, i + 3, b), (a, b, i + 3) \mid i = 0, 1, 2\} , \]

\[ A_6 = \{(a, i + 1, i), (i + 2, i, a), (b, i + 4, i), (i + 5, i, b) \mid i \in \mathbb{Z}_6\} \]
\[ \cup \{(a, i + 3, i), (i, b, a), (b, i, i + 3), (i + 3, a, b) \mid i = 0, 1, 3\} . \]

Then \((U, V, A_i)\), \(1 \leq i \leq 6\) are 6 pairwise disjoint simple \(DTS(8, 6; 3, 1)\)s. Let \(b^*\) be the support size of a \(DTS(8, \lambda)\) with \(\lambda \equiv 0 \pmod{3}\), then \(b^* \geq 26\) by Theorem 2. Let \(U = V \cup Z_s\). Let

\[ A_0 = \{(i, i + 1, i + 2), (i, a, i + 3), (i, a, i + 3), (i, a, i + 3), (i, a, i + 3), (i, c, i + 1), (i, c, i + 2), (i, c, i + 2) \mid i \in \mathbb{Z}_6\} \]

Then \((U, V, A_0)\) is a \(DTS(8, 3; 3, 0)\) with support size 25. In \(A_0\), replace \(j\) pairs of triples of the form \((i, c, i + 1)\) and \((i - 2, c, i)\) by \((c, i, i + 1)\) and \((i - 2, c, i)\) by \((i, a, i + 3)\) and \((i - 3, a, i)\) by \((b, b, i + 4)\) and \((i - 4, b, i)\) by \((b, b, i + 4)\) and \((i - 4, b, i)\), \(0 \leq j, k, l \leq 5\). This gives a \(DTS(8, 3; 3, 0)\) with support size \(25 + j + 2(k + l)\). Taking \(\lambda/3\) copies of this \(DTS(8, 3; 3, 0)\) gives a \(DTS(8, \lambda, 0)\) with the same support size. Since \([2, 4, 6] \subseteq SSD(3, 3)\), then we have \([26, 27, \ldots, 56] \subseteq SSD(8, \lambda)\). As a consequence, we have proved that \(SSD(8, 3) = DS(8, 3)\).

For \(2 \leq t \leq \min(\lambda/3, 6)\), let

\[ B = \left(\frac{\lambda}{3} - t + 1\right) A_1 \cup \left\{ \bigcup_{i = 2}^{t} A_i \right\} , \]

then \((U, V, B)\) is a \(DTS(8, 6; \lambda, \lambda/3)\) with support size \(36t\). Let

\[ A_7 = \{(i, a, i + 1), (i, a, i + 4), (i, b, i + 2), (i, b, i + 5) \mid i \in \mathbb{Z}_6\} \]
\[ \{(i, a, i + 3), (a, i, b), (i + 3, b, i), (b, i + 3, a) \mid i = 0, 1, 2\} . \]

Then \((U, V, A_7)\) is a simple \(DTS(8, 6; 3, 1)\) and \(|A_7 \cup A_1| = 24\).

For \(\lambda \equiv 0 \pmod{3}\), \(\lambda \geq 6\), let

\[ B = (\lambda - 1) A_1 \cup A_7 , \]

then \((U, V, B)\) is a \(DTS(8, 6; \lambda, \lambda/3)\) with support size 48.
Notice that for $\lambda \geq 4$, $SSD(6, \lambda) \supset DS(6, 4) = \{10, 12, 13, ..., 40\}$. Since taking $\lambda$ copies of $A_1$ gives a $DTS(8, 6; \lambda, \lambda/3)$ with support size 36, it is a routine matter to check that for any $\lambda \equiv 0 \pmod{3}$, we have

$$SSD(8, \lambda) = DS(8, \lambda).$$

5. PROOF OF THE MAIN THEOREM

Now we are in a position to prove our main theorem.

**Theorem 5.1.** For $u \geq 4$, if $\lambda u (u - 1) \equiv 0 \pmod{3}$, then $SSD(u, \lambda) = DS(u, \lambda)$.

**Proof.** The theorem has already been proved for $\lambda = 2$ or for $u \leq 8$ and all $\lambda$. So we may suppose $\lambda \geq 3$ and $u \geq 9$. We prove the theorem by induction on $u$. Write

$$u = \begin{cases} 2v + 1, & \text{if } u \equiv 1 \pmod{2} \\ 2v + 4, & \text{if } u \equiv 0 \pmod{2}. \end{cases}$$

For $\lambda < 3(v - 2)$, the conclusion follows from Lemmas 3.1–3.5.

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