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Cohomology of Algebraic Groups

DAVID HERTZIG[1]

Mathematics Department, Purdue University, Lafayette, Indiana

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The purpose of this paper is to expound a technique for the computation of certain cohomology of algebraic groups. A natural approach is to restrict attention to a Borel subgroup B. Since B splits as a semidirect product of B_u (unipotent part) and a maximal torus T, a standard exact-sequence argument leads to the heart of the problem, namely cohomology of B_u with action of the torus T. One now notes that B_u is a product of "one-parameter subgroups" and the procedure is to consider the restriction of a cocycle to these elementary subgroups and derive conditions on it so that it may extend to the whole group. Using the cohomology relation it is possible to find a neat normal form for a cocycle. In the case of the field with two elements a maximal torus is trivial and hence the trivial action of T does not give helpful restrictions on a cocycle; it is in this case that the computation is most tedious.

The special case worked out in detail in this paper is in fact quite general and gives a sufficiently complete exposition of the technique to enable one to make the computations in a particular special case.

Such explicit computation appears necessary in the cohomological approach to the "congruence subgroup problem" and the desirability of a technique was pointed out to the author by H. Bass.

Let G be a subgroup of $GL(n, k)$, the group of $n \times n$ nonsingular matrices with coefficients in the field k of characteristic p. Note G operates by inner automorphism on the space $\mathfrak{M}_n(k)$ of all $n \times n$ matrices over k; let $\mathfrak{g}$ be a subspace of $\mathfrak{M}_n(k)$ stable under this operation. Recall that a 1-cocycle from G to $\mathfrak{g}$ is a map

$$f : G \to \mathfrak{g}$$

satisfying

$$f(gg') = f(g) + f(g')^g$$

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for all $g, g' \in G$. The set of all 1-cocycles from G to $\mathfrak{g}$ is a commutative group $Z^1(G, \mathfrak{g})$. A 1-cocycle $f \in Z^1(G, \mathfrak{g})$ is a 1-coboundary if there exists $A \in \mathfrak{g}$ such that

$$f(g) = A^g - A$$

for all $g \in G$. The set of all 1-coboundaries is a subgroup $B^1(G, \mathfrak{g}) < Z^1(G, \mathfrak{g})$. The quotient group $H^1(G, \mathfrak{g})$ is the 1-cohomology group of G into $\mathfrak{g}$.

If H is a subgroup of $GL(n, k)$ which normalizes G, and also under which $\mathfrak{g}$ is stable, we can consider the action of H in the cohomology. More precisely, we denote $Z_0{}^1(G, \mathfrak{g})$ the subgroup of $Z^1(G, \mathfrak{g})$ consisting of the cocycles f satisfying

$$f(g^h) = f(g)^h$$

for all $h \in H$, $g \in G$. We denote $H_0{}^1(G, \mathfrak{g})$ the image of $Z_0{}^1(G, \mathfrak{g})$ in $H^1(G, \mathfrak{g})$.

In this paper we restrict our considerations to the following situations:

$G =$ triangular unipotent matrices

$H =$ diagonal unimodular matrices

$\mathfrak{g} = \mathfrak{M}_n(k)$

$\mathfrak{g}_0 =$ matrices of trace 0

$G_{\rho\mu} =$ subgroup of G consisting of the elementary matrices $\{I + xe_{\rho\mu}\}_{x \in k}$ $(\rho < \mu)$, where, as usual, $e_{\rho\mu}$ is the matrix whose (ρ, μ)th coordinate is 1 and all others are 0.

If f is a 1-cocycle from G to $\mathfrak{g}$, its restriction $f^{\rho\mu}$ to $G_{\rho\mu}$ is a 1-cocycle from $G_{\rho\mu}$ to $\mathfrak{g}$. The isomorphism $x \to I + xe_{\rho\mu}$ of k with $G_{\rho\mu}$ will be used to identify $G_{\rho\mu}$ with k. Then if $f^{\rho\mu}(x)$ is the matrix $(f_{ij}^{\rho\mu}(x))$, we have n^2 functions

$$f_{ij}^{\rho\mu} : k \to \mathfrak{g}$$

satisfying

$$f_{ij}^{\rho\mu}(x + y) = f_{ij}^{\rho\mu}(x) + f_{ij}^{\rho\mu}(y) + \delta_{i\rho} x f_{\mu j}^{\rho\mu}(y) - \delta_{j\mu} x f_{i\rho}^{\rho\mu}(y) - \delta_{i\rho}\delta_{j\mu} x^2 f_{\mu\rho}^{\rho\mu}(y).$$

Since $f_{ij}^{\rho\mu}(x + y) = f_{ij}^{\rho\mu}(y + x)$, we must have

$$\begin{aligned} &\delta_{i\rho} x f_{\mu j}^{\rho\mu}(y) - \delta_{j\mu} x f_{i\rho}^{\rho\mu}(y) - \delta_{i\rho}\delta_{j\mu} x^2 f_{\mu\rho}^{\rho\mu}(y) \\ &= \delta_{i\rho} y f_{\mu j}^{\rho\mu}(x) - \delta_{j\mu} y f_{i\rho}^{\rho\mu}(x) - \delta_{i\rho}\delta_{j\mu} y^2 f_{\mu\rho}^{\rho\mu}(x). \end{aligned}$$

Therefore for $i = \rho$, $j \neq \mu$ we get

$$x f_{\rho j}^{\rho\mu}(y) = y f_{\mu j}^{\rho\mu}(x)$$

and for $i \neq \rho$, $j = \mu$ we get

$$x f^{\rho\mu}_{i\rho}(y) = y f^{\rho\mu}_{i\rho}(x).$$

Denoting $c^{\rho\mu}_{ij} = f^{\rho\mu}_{ij}(1)$, the above imply

$$f^{\rho\mu}_{i\rho}(x) = c^{\rho\mu}_{i\rho} x \qquad (i \neq \rho),$$

$$f^{\rho\mu}_{\mu j}(x) = c^{\rho\mu}_{\mu j} x \qquad (j \neq \mu).$$

Then taking $i = \rho$, $j = \mu$ we find

$$f^{\rho\mu}_{\mu\mu}(x) - f^{\rho\mu}_{\rho\rho}(x) = (c^{\rho\mu}_{\mu\mu} - c^{\rho\mu}_{\rho\rho})\, x + c^{\rho\mu}_{\mu\rho}(x - x^2).$$

It then follows that

$$f^{\rho\mu}_{ij}(x + y) = f^{\rho\mu}_{ij}(x) + f^{\rho\mu}_{ij}(y) \qquad (i \neq \rho, j \neq \mu),$$

$$f^{\rho\mu}_{i\mu}(x + y) = f^{\rho\mu}_{i\mu}(x) + f^{\rho\mu}_{i\mu}(y) - c^{\rho\mu}_{i\rho} xy \qquad (i \neq \rho),$$

$$f^{\rho\mu}_{\rho j}(x + y) = f^{\rho\mu}_{\rho j}(x) + f^{\rho\mu}_{\rho j}(y) + c^{\rho\mu}_{\mu j} xy \qquad (j \neq \mu),$$

$$f^{\rho\mu}_{\rho\mu}(x + y) = f^{\rho\mu}_{\rho\mu}(x) + f^{\rho\mu}_{\rho\mu}(y) + (c_{\mu\mu} - c^{\rho\mu}_{\rho\rho})\, xy + c^{\mu}_{\mu\rho}(xy - x^2 y - xy^2),$$

with

$$f^{\rho\mu}_{i\rho}(x) = c^{\rho\mu}_{i\rho} x \qquad (i \neq \rho),$$

$$f^{\rho\mu}_{\mu j}(x) = c^{\rho\mu}_{\mu j}(x) \qquad (j \neq \mu),$$

and

$$f^{\rho\mu}_{\mu\mu}(x) - f^{\rho\mu}_{\rho\rho}(x) = (c^{\rho\mu}_{\mu\mu} - c^{\rho\mu}_{\rho\rho})\, x + c^{\rho\mu}_{\mu\rho}(x - x^2).$$

For the case $n = 2$ we have $G = G_{12}$ and necessarily $\rho = 1$, $\mu = 2$; we simply denote $f^{\rho\mu}_{ij} = f_{ij}$ and $c^{\rho\mu}_{ij} = c_{ij}$ $(1 \leqslant i, j \leqslant 2)$. The relations reduce to

$$f_{11}(x + y) = f_{11}(x) + f_{11}(y) + c_{21} xy,$$

$$f_{12}(x + y) = f_{12}(x) + f_{12}(y) + (c_{22} - c_{11} + c_{21})\, xy - c_{21}(xy^2 + x^2 y),$$

$$f_{21}(x) = c_{21} x,$$

$$f_{22}(x) = f_{11}(x) + (c_{22} - c_{11} + c_{21})\, x - c_{21} x^2.$$

If $p = 2$, setting $x = y$ gives $c_{21} = c_{22} - c_{11} + c_{21} = 0$. If $p = 3$, setting $x = \pm y$ leads to $c_{21} = 0$.

Define

$$\alpha = \begin{cases} \frac{1}{2}(c_{22} - c_{11} + c_{21}) & \text{if} \quad p \neq 2, \\ 0 & \text{if} \quad p = 2; \end{cases}$$

$$\beta = \begin{cases} \frac{1}{6} c_{21} & \text{if} \quad p \neq 2, 3, \\ 0 & \text{if} \quad p = 2, 3. \end{cases}$$

Then we may write

$$f(x) = \begin{pmatrix} \rho_0(x) & \rho_1(x) \\ 0 & \rho_0(x) \end{pmatrix} + \alpha \begin{pmatrix} -x & x^2 \\ 0 & x \end{pmatrix} + \beta \begin{pmatrix} 3x^2 & -2x^3 \\ 12x & -3x^2 \end{pmatrix}$$

where

$$\rho_0(x+y) = \rho_0(x) + \rho_0(y),$$
$$\rho_1(x+y) = \rho_1(x) + \rho_1(y).$$

If

$$h = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

and

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

the condition $f(g^h) = f(g)^h$ becomes

$$f(u^2x) = \begin{pmatrix} f_{11}(x) & u^2 f_{12}(x) \\ u^{-2} f_{21}(x) & f_{22}(x) \end{pmatrix}$$

or

$$\rho_0(u^2x) - \alpha u^2 x + 3\beta u^4 x^2 = \rho_0(x) - \alpha x + 3\beta x^2,$$
$$\rho_1(u^2x) + \alpha u^4 x^2 - 2\beta u^6 x^3 = u^2 \rho_1(x) + \alpha u^2 x^2 - 2\beta u^2 x^3,$$
$$12\beta u^2 x = 12\beta u^{-2} x,$$
$$\rho_0(u^2x) + \alpha u^2 x - 3\beta u^4 x^2 = \rho_0(x) + \alpha x - 3\beta x^2.$$

From the third equation we obtain

$$\beta = 0 \qquad \text{unless} \qquad k = \mathbf{F}_5 .$$

From the first and fourth equations we obtain

$$\alpha = 0 \qquad \text{unless} \qquad k = \mathbf{F}_3 .$$

However, if $k = \mathbf{F}_3$ we may take

$$A = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

so that

$$A^x - A = -\alpha \begin{pmatrix} -x & x^2 \\ 0 & x \end{pmatrix};$$

hence, up to cohomology, there is no loss of generality in assuming $\alpha = 0$.

We also have

$$\rho_0(u^2x) = \rho_0(x),$$
$$\rho_1(u^2x) = u^2\rho_1(x).$$

Suppose $k \neq \mathbf{F}_2, \mathbf{F}_3$; choose $u \in k$, $u \neq 0$, such that $u^2 \neq 1$, i.e., $v = u - u^{-1} \neq 0$. Then for $x \in k$

$$\begin{aligned}\rho_0(x) = \rho_0(v^2x) &= \rho_0(u^2x) - 2\rho_0(x) + \rho_0(u^{-2}x)\\ &= \rho_0(x) - 2\rho_0(x) + \rho_0(x) = 0.\end{aligned}$$

Therefore, $\rho_0(x) = \gamma x$ with $\gamma = 0$ unless $k = \mathbf{F}_2$ or $\mathbf{F}_3$. If $k = \mathbf{F}_2$ we may take

$$A = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

so that

$$A^x - A = \begin{pmatrix} \gamma x & \gamma x \\ 0 & \gamma x \end{pmatrix};$$

then up to cohomology we see there is no loss of generality in assuming $\gamma = 0$ unless $k = \mathbf{F}_3$.

The condition on ρ_1 implies

$$\rho_1(x^2) = \delta x^2$$

so that

$$\begin{aligned}\delta(1 + x)^2 = \rho_1((1 + x)^2) &= \rho_1(1) + 2\rho_1(x) + \rho_1(x^2)\\ &= \delta + 2\rho_1(x) + \delta x^2\end{aligned}$$

and

$$2\rho_1(x) = 2\delta x.$$

Therefore if $p \neq 2$, $\rho_1(x) = \delta x$. On the other hand, if $p = 2$ and k is perfect, every element is a square and again we have $\rho_1(x) = \delta x$. It is easy to construct other examples if k is not perfect.

If $p \neq 2$ let

$$A = \begin{pmatrix} -\frac{1}{2}\delta & 0 \\ 0 & \frac{1}{2}\delta \end{pmatrix}$$

so that

$$A^x - A = \begin{pmatrix} 0 & \delta x \\ 0 & 0 \end{pmatrix};$$

then, up to cohomology, there is no loss of generality in assuming $\delta = 0$ unless $p = 2$.

If $p = 2$ in the case of cohomology into $\mathfrak{g}$ we can take

$$A = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$A^x - A = \begin{pmatrix} 0 & \delta x \\ 0 & 0 \end{pmatrix},$$

and then in this case too we may assume $\delta = 0$.

We now have

$$f(x) = \begin{pmatrix} \gamma x + 3\beta x^2 & \rho_1(x) - 2\beta x^3 \\ 2\beta x & \gamma x - 3\beta x^2 \end{pmatrix}$$

with $\gamma = 0$ unless $k = \mathbf{F}_3$, $\beta = 0$ unless $k = \mathbf{F}_5$, $\rho_1 \equiv 0$ unless $p = 2$. If $p = 2$ and k is perfect, $\rho_1(x) = \delta x$ with $\delta = 0$ for cohomology into $\mathfrak{g}$. Moreover, if $f(x)$ is to be of trace 0, $\gamma = 0$. Thus

PROPOSITION 1. *For $n = 2$, $H_0^1(G, \mathfrak{g}) = 0$ provided k is perfect if $p = 2$, except for $k = \mathbf{F}_3$ or $\mathbf{F}_5$ in which cases $H_0^1(G, \mathfrak{g}) \cong k$.*

Also if $p \neq 2$, $H_0^1(G, \mathfrak{g}_0) = 0$, except for $k = \mathbf{F}_5$. In the cases $k = \mathbf{F}_5$ or $p = 2$ and k perfect, $H_0^1(G, \mathfrak{g}_0) \cong k$.

We now assume $n \geqslant 3$ and $k \neq \mathbf{F}_2$. Let

$$h = \begin{pmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{pmatrix}$$

with $u_1 u_2 \cdots u_n = 1$. The condition

$$f(g^h) = f(g)^h$$

implies

$$f_{ij}^{\rho\mu}(u_\rho u_\mu^{-1} x) = u_i u_j^{-1} f_{ij}^{\rho\mu}(x).$$

First consider the case $n = 3$, and let $\lambda \neq \rho, \mu$ so that $\{\lambda, \rho, \mu\} = \{1, 2, 3\}$. Suppose $i = \rho$; then

$$f_{\rho j}^{\rho\mu}(u_\rho u_\mu^{-1} x) = u_\rho u_j^{-1} f_{\rho j}^{\rho}(x).$$

Taking $u_\rho = y$, $u_\lambda = y^{-1}$, $u_\mu = 1$, $x = 1$ we get

$$f_{\rho j}^{\rho\mu}(y) = c_{\rho j}^{\rho\mu} u_j^{-1} y$$

$$= \begin{cases} c_{\rho\rho}^{\rho\mu} & (j = \rho), \\ c_{\rho\mu}^{\rho\mu} y & (j = \mu), \\ c_{\rho\lambda}^{\rho\mu} y^2 & (j = \lambda). \end{cases}$$

Then $c_{\rho\lambda}^{\rho\mu} u_\rho{}^2 u_\mu^{-2} x^2 = c_{\rho\lambda}^{\rho\mu} u_\rho u_\lambda^{-1} x^2$. Since $u_\rho u_\lambda u_\mu = 1$, taking $u_\mu = t \neq 0, 1$, this reduces to

$$c_{\rho\lambda}^{\rho\mu}(1 - t^3) = 0.$$

Thus if $k \neq \mathbf{F}_4$, $c_{\rho\lambda}^{\rho\mu} = 0$.

Suppose $i = \mu$; then

$$f_{\mu j}^{\rho\mu}(u_\rho u_\mu^{-1} x) = u_\mu u_j^{-1} f_{\mu j}^{\rho\mu}(x)$$

so that, for $j \neq \mu$,

$$c_{\mu j}^{\rho\mu} u_\rho u_\mu^{-1} x = u_\mu u_j^{-1} c_{\mu j}^{\rho\mu} x, \qquad c_{\mu j}^{\rho\mu}(u_\rho u_j - u_\mu{}^2) = 0.$$

Therefore

$$f_{\mu\rho}^{\rho\mu} = 0 \quad \text{if} \quad k \neq \mathbf{F}_3,$$

$$f_{\mu\lambda}^{\rho\mu} = 0 \quad \text{if} \quad k \neq \mathbf{F}_4.$$

Also,

$$f_{\mu\mu}^{\rho\mu}(u_\rho u_\mu^{-1} x) = f_{\mu\mu}^{\rho\mu}(x)$$

so that

$$f_{\mu\mu}^{\rho\mu}(x) = \begin{cases} c_{\mu\mu}^{\rho\mu} & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Suppose $i = \lambda$; then

$$f_{\lambda j}^{\rho\mu}(u_\rho u_\mu^{-1} x) = u_\lambda u_j^{-1} f_{\lambda j}^{\rho\mu}(x).$$

Taking $u_\rho = 1$, $u_\mu = y^{-1}$, $u_\lambda = y$, $x = 1$, we get

$$f_{\lambda j}^{\rho\mu}(y) = c_{\lambda j}^{\rho\mu} y u_j^{-1}$$

$$= \begin{cases} c_{\lambda\rho}^{\rho\mu} y & (j = \rho), \\ c_{\lambda\mu}^{\rho\mu} y^2 & (j = \mu), \\ c_{\lambda\lambda}^{\rho\mu} & (j = \lambda). \end{cases}$$

Then $c_{\lambda\rho}^{\rho\mu} u_\rho u_\mu^{-1} x = u_\lambda u_\rho^{-1} c_{\lambda\rho}^{\rho\mu} x$. Since $u_\rho u_\lambda u_\mu = 1$, taking $u_\rho = t \neq 0, 1$, this reduces to

$$c_{\lambda\rho}^{\rho\mu}(1 - t^3) = 0.$$

Thus if $k \neq \mathbf{F}_4$, then $c_{\lambda\rho}^{\rho\mu} = 0$.

Also, $c_{\lambda\mu}^{\rho\mu} u_\rho^{\,2} u_\mu^{-2} x^2 = u_\lambda u_\mu^{-1} c_{\lambda\mu}^{\rho\mu} x^2$ and again, if $k \neq \mathbf{F}_4$, $c_{\lambda\mu}^{\rho\mu} = 0$.

Combining this with the conditions already obtained for $f^{\rho\mu}$ to be a cocycle we conclude

$$f_{\rho\mu}^{\rho\mu}(x) = c_{\rho\mu}^{\rho\mu} x,$$

$$f_{\rho\lambda}^{\rho\mu}(x) = c_{\rho\lambda}^{\rho\mu} x^2 \qquad (c_{\rho\lambda}^{\rho\mu} = 0 \quad \text{if} \quad k \neq \mathbf{F}_4),$$

$$f_{\lambda\mu}^{\rho\mu}(x) = c_{\lambda\mu}^{\rho\mu} x^2 \qquad (c_{\lambda\mu}^{\rho\mu} = 0 \quad \text{if} \quad k \neq \mathbf{F}_4),$$

$$f_{ij}^{\rho\mu}(x) = 0 \qquad \text{otherwise.}$$

Let

$$g_1 = I + xe_{23}, \qquad g_2 = I + ye_{12}, \qquad g_3 = I + xye_{13},$$

so that

$$g_2 g_1 = g_1 g_2 g_3 .$$

Therefore

$$f(g_2) + f(g_1)^{g_2} = f(g_1) + f(g_2)^{g_1} + f(g_3)^{g_1 g_2}.$$

It is easily verified that this condition gives

$$f_{\rho\mu}^{\rho\mu}(x) = c_{\rho\mu}^{\rho\mu} x,$$

$$f_{13}^{12}(x) = c_{13}^{12} x^2 \qquad (c_{13}^{12} = 0 \quad \text{if} \quad k \neq \mathbf{F}_4),$$

$$f_{13}^{23}(x) = c_{13}^{23} x^2 \qquad (c_{13}^{23} = 0 \quad \text{if} \quad k \neq \mathbf{F}_4),$$

$$f_{ij}^{\rho\mu}(x) = 0 \qquad \text{otherwise,}$$

along with

$$c_{13}^{13} = c_{12}^{12} + c_{23}^{23} .$$

Taking

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{12}^{12} & 0 \\ 0 & 0 & -c_{12}^{12} \end{pmatrix}$$

and setting $\bar{f}(g) = A^g - A$, we get

$$\bar{f}^{12}_{12}(x) = c^{12}_{12}x, \qquad \bar{f}^{13}_{13}(x) = -c^{12}_{12}x, \qquad \bar{f}^{23}_{23}(x) = -2c^{12}_{12}x$$

and $\bar{f}^{\rho\mu}_{ij} = 0$ otherwise. Hence, up to cohomology, there is no loss of generality in assuming $c^{12}_{12} = 0$.

If $p \neq 3$, let $a = \frac{1}{3} c^{13}_{13}$ and

$$A = \begin{pmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix},$$

whereas if $p = 3$, let $a = c^{13}_{13}$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Then if $\bar{f}(g) = A^g - A$, we get

$$\bar{f}^{13}_{13}(x) = c^{13}_{13}x, \qquad \bar{f}^{23}_{23}(x) = c^{23}_{23}x, \qquad \bar{f}^{\rho\mu}_{ij} = 0 \qquad \text{otherwise.}$$

We now have

$$f: \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \alpha x^2 + \beta y^2 + \gamma z \\ 0 & 0 & \gamma x \\ 0 & 0 & 0 \end{pmatrix},$$

with $\alpha = 0$, $\beta = 0$ unless $k = \mathbf{F}_4$, and $\gamma = 0$ unless $p = 3$, and we are in the case of $\mathfrak{g}_0$.

PROPOSITION 2. *For $n = 3$, and $k \neq \mathbf{F}_2$, if $p \neq 3$ and $k \neq \mathbf{F}_4$,*

$$H_0^1(G, \mathfrak{g}) = H_0^1(G, \mathfrak{g}_0) = 0.$$

If $p = 3$,

$$H_0^1(G, \mathfrak{g}) = 0, \qquad H_0^1(G, \mathfrak{g}_0) \cong k.$$

If $k = \mathbf{F}_4$,

$$H_0^1(G, \mathfrak{g}) = H_0^1(G, \mathfrak{g}_0) \cong k^{(2)}.$$

Now assume $n \geq 4$. Taking $u_\rho = u_\mu = 1$, we have

$$(u_i - u_j) f^{\rho\mu}_{ij}(x) = 0.$$

It follows easily that, except in the case $n = 4$ and $k = \mathbf{F}_3$, we must have

$$f^{\rho\mu}_{ij} = 0 \qquad \text{for} \qquad i \neq \rho, \mu \qquad \text{or} \quad j \neq \rho, \mu.$$

We also easily deduce $f^{\rho\mu}_{\rho\mu}(x) = c^{\rho\mu}_{\rho\mu}x$, as well as $c^{\rho\mu}_{\mu\rho} = 0$ if $k \neq \mathbf{F}_3$. The conditions

$$f^{\rho\mu}_{\mu\mu}(x+y) = f^{\rho\mu}_{\mu\mu}(x) + f^{\rho\mu}_{\mu\mu}(y) - c^{\rho\mu}_{\mu\rho}xy,$$

$$f^{\rho\mu}_{\rho\rho}(x+y) = f^{\rho\mu}_{\rho\rho}(x) + f^{\rho\mu}_{\rho\rho}(y) + c^{\rho\mu}_{\rho\rho}xy,$$

imply

$$c^{\rho\mu}_{\mu\mu} = -c^{\rho\mu}_{\rho\rho} = c^{\rho\mu}_{\mu\rho}.$$

The relation

$$f^{\rho\mu}_{\rho\mu}(x+y) = f^{\rho\mu}_{\rho\mu}(x) + f^{\rho\mu}_{\rho\mu}(y) + (c^{\rho\mu}_{\mu\mu} - c^{\rho\mu}_{\rho\rho})\,xy + c^{\rho\mu}_{\mu\rho}(xy - x^2y - xy^2)$$

then gives

$$f^{\mu\rho}_{\mu\rho} = 0, \qquad f^{\rho\mu}_{\rho\rho} = 0, \qquad f^{\rho\mu}_{\mu\mu} = 0.$$

Since for $i \neq \rho, \mu$ we have

$$f^{\rho\mu}_{ii}(x+y) = f^{\rho\mu}_{ii}(x) + f^{\rho\mu}_{ii}(y)$$

we also have $f^{\rho\mu}_{ii} = 0$ in the case $n = 4$ and $k = \mathbf{F}_3$.

Finally then, we must have

$$f^{\rho\mu}(x) = c^{\rho\mu}_{\rho\mu}xe_{\rho\mu},$$

except in the case $n = 4$ and $k = \mathbf{F}_3$, in which case we can conclude

$$f^{\rho\mu}(x) = x\{c^{\rho\mu}_{\rho\mu}e_{\rho\mu} + c^{\rho\mu}_{\lambda\sigma}e_{\lambda\sigma} + c^{\rho\mu}_{\sigma\lambda}e_{\sigma\lambda}\}$$

where $\{\rho, \mu, \sigma, \lambda\} = \{1, 2, 3, 4\}$.

Suppose $\rho < \lambda < \mu$ and let

$$g_1 = I + xe_{\lambda\mu}, \qquad g_2 = I + ye_{\rho\lambda}, \qquad g_3 = I + xye_{\rho\mu}$$

so that

$$g_2g_1 = g_1g_2g_3.$$

The cocycle condition

$$f(g_2) + f(g_1)^{g_2} = f(g_1) + f(g_2)^{g_1} + f(g_3)^{g_1g_2}$$

then gives

$$c^{\rho\mu}_{\rho\mu} = c^{\rho\lambda}_{\rho\lambda} + c^{\lambda\mu}_{\lambda}$$

and, in the case $n = 4$ and $k = \mathbf{F}_3$,

$$f^{\rho\mu}_{\lambda\sigma} = 0,$$

except for f^{23}_{14}.

The first condition may be reformulated as

$$c_{\rho\mu}^{\rho\mu} = c_{1\mu}^{1\mu} - c_{1\rho}^{1\rho}.$$

Define α and β as follows:

$$\alpha = \begin{cases} 0 & \text{if} \quad p \mid n, \\ -\dfrac{1}{n} \sum\limits_i c_{1i}^{1i} & \text{if} \quad p \nmid n; \end{cases}$$

$$\beta = \begin{cases} -\sum\limits_i c_{1i}^{1i} & \text{if} \quad p \mid n, \\ 0 & \text{if} \quad p \nmid n. \end{cases}$$

Define $A = (a_{ij})$ as follows:

$$a_{ij} = \delta_{ij}\{\alpha + c_{1i}^{1i} + \delta_{in}\beta\}.$$

Then if $\bar{f}(g) = A^g - A$, we have

$$\bar{f}^{\rho\mu}(x) = c_{\rho\mu}^{\rho\mu} x e_{\rho\mu} + \delta_{\mu n} \beta x c_{\rho n}^{\rho n} e_{\rho n}$$

so that, up to cohomology, we may assume

$$f^{\rho\mu}(x) = \delta_{\mu n} \gamma x e_{\rho n}$$

with $\gamma = 0$ if $p \nmid n$, except the case $n = 4$, $k = \mathbf{F}_3$, $(\rho, \mu) = (2, 3)$, where we have

$$f^{23}(x) = \delta x e_{14}.$$

If $p \mid n$, let $A = \gamma e_{nn}$ and $\bar{f}(g) = A^g - A$. Then

$$\bar{f}^{\rho\mu}(x) = \delta_{\mu n} \gamma x e_{\rho n}.$$

PROPOSITION 3. *For $n = 4$ and $k \neq \mathbf{F}_2$, if $k \neq \mathbf{F}_3$,*

$$H_0^1(G, \mathfrak{g}) = 0.$$

If $p \neq 2$, $H_0^1(G, \mathfrak{g}_0) = 0$ and if $p = 2$, $H_0^1(G, \mathfrak{g}_0) \cong k$. If $k = \mathbf{F}_3$,

$$H_0^1(G, \mathfrak{g}) = H_0^1(G, \mathfrak{g}_0) \cong k.$$

PROPOSITION 4. *For $n \geqslant 5$ and $k \neq \mathbf{F}_2$,*

$$H_0^1(G, \mathfrak{g}) = 0.$$

If $p \nmid n$, $H_0^1(G, \mathfrak{g}_0) = 0$ *and if* $p \mid n$,

$$H_0^1(G, \mathfrak{g}_0) \cong k.$$

In the case $k = \mathbf{F}_2$ the torus H is trivial and so compatibility with the action of H gives no conditions on the cocycles. However, every function $f : \mathbf{F}_2 \to \mathbf{F}_2$ satisfying $f(0) = 0$ is simply of the form $f(x) = cx$ ($c = 0$ or 1), so that (in the earlier notation)

$$f_{ij}^{\rho\mu}(x) = c_{ij}^{\rho\mu} x$$

or

$$f^{\rho\mu}(x) = xc^{\rho\mu}$$

for a constant matrix $c^{\rho\mu}$.

Thus $f^{\rho\mu}(x + y) = f^{\rho\mu}(x) + f^{\rho\mu}(y)$ and the cocycle condition then implies $c^{\rho\mu}$ must commute with $e_{\rho\mu}$. Hence

$$c_{i\rho}^{\rho\mu} = \delta_{i\rho} c_{\mu\mu}^{\rho\mu}, \qquad c_{\mu j}^{\rho\mu} = \delta_{j\mu} c_{\rho\rho}^{\rho\mu}.$$

Now let $\lambda < \mu$, $\rho < \sigma$, $\sigma \neq \lambda$, $\rho \neq \mu$ and

$$g_1 = I + e_{\lambda\mu}, \qquad g_2 = I + e_{\rho\sigma}.$$

Then $g_1 g_2 = g_2 g_1$ so that

$$f(g_1) + f(g_2)^{g_1} = f(g_2) + f(g_1)^{g_2}$$

or

$$f(g_1)^{g_2} + f(g_1) = f(g_2)^{g_1} + f(g_2),$$

which may be written

$$[e_{\rho\sigma}, c^{\lambda\mu}] + c_{\sigma\rho}^{\lambda\mu} e_{\rho\sigma} = [e_{\lambda\mu}, c^{\rho\sigma}] + c_{\mu\lambda}^{\rho\sigma} e_{\lambda\mu}.$$

Hence

$$\sum_{j \neq \sigma} c_{\sigma j}^{\lambda\mu} e_{\rho j} + \sum_{i \neq \rho} c_{i\rho}^{\lambda\mu} e_{i\sigma} + (c_{\sigma\sigma}^{\lambda\mu} + c_{\rho\rho}^{\lambda\mu} + c_{\sigma\rho}^{\lambda\mu})\, e_{\rho\sigma}$$

$$= \sum_{j \neq \mu} c_{\mu j}^{\rho\sigma} c_{\lambda j} + \sum_{i \neq \lambda} c_{i\lambda}^{\rho\sigma} e_{i\mu} + (c_{\mu\mu}^{\rho\sigma} + c_{\lambda\lambda}^{\rho\sigma} + c_{\mu\lambda}^{\rho\sigma})\, e_{\lambda\mu}.$$

If $\sigma = \mu$ this reduces to

$$\sum_{i \neq \rho} c_{i\rho}^{\lambda\mu} e_{i\mu} + (c_{\mu\mu}^{\lambda\mu} + c_{\rho\rho}^{\lambda\mu})\, e_{\rho\mu} = \sum_{i \neq \lambda} c_{i\lambda}^{\rho\mu} e_{i\mu} + (c_{\mu\mu}^{\rho\mu} + c_{\lambda\lambda}^{\rho\mu})\, e_{\lambda\mu}.$$

If $\rho = \lambda$ this gives no information; assume then $\rho \neq \lambda$. Since this is symmetric in ρ and λ, we might as well assume $\rho < \lambda < \mu$. Then

$$c^{\rho\mu}_{\rho\lambda} = c^{\lambda\mu}_{\mu\mu} + c^{\lambda\mu}_{\rho\rho},$$

$$c^{\lambda\mu}_{\lambda\rho} = c^{\rho\mu}_{\mu\mu} + c^{\rho\mu}_{\lambda\lambda},$$

$$c^{\lambda\mu}_{i\rho} = c^{\rho\mu}_{i\lambda} \qquad \text{for} \qquad i \neq \lambda, \rho.$$

Similarly, considering the case $\lambda = \rho$, $\sigma < \mu$ (and replacing σ by λ), we have for $\rho < \lambda < \mu$

$$c^{\rho\lambda}_{\mu\lambda} = c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\rho\rho},$$

$$c^{\rho\mu}_{\lambda\mu} = c^{\rho\lambda}_{\mu\mu} + c^{\rho\lambda}_{\rho\rho},$$

$$c^{\rho\mu}_{\lambda j} = c^{\rho\lambda}_{\mu j} \qquad \text{for} \qquad j \neq \lambda, \mu.$$

Finally, in the case $\sigma \neq \mu$ and $\rho \neq \lambda$ we obtain

$$c^{\lambda\mu}_{\sigma j} = c^{\rho\sigma}_{\mu j} = 0 \qquad \text{for} \qquad j \neq \sigma, \mu,$$

$$c^{\lambda\mu}_{i\rho} = c^{\rho\sigma}_{i\lambda} = 0 \qquad \text{for} \qquad i \neq \rho, \lambda,$$

$$c^{\lambda\mu}_{\sigma\mu} = c^{\rho\sigma}_{\rho\lambda}, \qquad c^{\lambda\mu}_{\lambda\rho} = c^{\rho\sigma}_{\mu\sigma}, \qquad c^{\lambda\mu}_{\rho\rho} = c^{\lambda\mu}_{\sigma\sigma}, \qquad c^{\rho\sigma}_{\lambda\lambda} = c^{\rho\sigma}_{\mu\mu}.$$

Now suppose $\rho < \lambda < \mu$ and take

$$g_1 = I + e_{\lambda\mu}, \qquad g_2 = I + e_{\rho\lambda}, \qquad g_3 = I + e_{\rho\mu}.$$

Then $g_2 g_1 = g_1 g_2 g_3$ so that

$$f(g_2) + f(g_1)^{g_2} = f(g_1) + f(g_2)^{g_1} + f(g_3)^{g_1 g_2}$$

or

$$f(g_3)^{g_1 g_2} = f(g_1) + f(g_1)^{g_1} + f(g_2) + f(g_2)^{g_2}.$$

We have

$$\begin{aligned} f(g_1) + f(g_1)^{g_2} &= [e_{\rho\lambda}, c^{\lambda\mu}] + c^{\lambda\mu}_{\lambda\rho} e_{\rho\lambda} \\ &= \sum_{i \neq \rho} c^{\lambda\mu}_{i\rho} e_{i\lambda} + \sum_{j \neq \lambda} c^{\lambda\mu}_{\lambda j} e_{\rho j} + (c^{\lambda\mu}_{\rho\rho} + c^{\lambda\mu}_{\lambda\lambda} + c^{\lambda\mu}_{\lambda\rho})\, e_{\rho\lambda} \\ &= c^{\lambda\mu}_{\lambda\rho} e_{\lambda\lambda} + \sum_{j \neq \lambda} c^{\lambda\mu}_{\lambda j} e_{\rho j} + (c^{\lambda\mu}_{\rho\rho} + c^{\lambda\mu}_{\lambda\lambda} + c^{\lambda\mu}_{\lambda\rho})\, e_{\rho\lambda} \end{aligned}$$

and

$$f(g_2) + f(g_2)^{g_1} = [e_{\lambda\mu}, c^{\rho\lambda}] + c^{\rho\lambda}_{\mu\lambda} e_{\lambda\mu}$$

$$= \sum_{i\neq\lambda} c^{\rho\lambda}_{i\lambda} e_{i\mu} + \sum_{j\neq\mu} c^{\rho\lambda}_{\mu j} e_{\lambda j} + (c^{\rho\lambda}_{\lambda\lambda} + c^{\rho\lambda}_{\mu\mu} + c^{\lambda}_{\mu\lambda})\, e_{\lambda\mu}$$

$$= \sum_{i\neq\lambda} c^{\rho\lambda}_{i\lambda} e_{i\mu} + c^{\rho\lambda}_{\mu\lambda} e_{\lambda\lambda} + (c^{\rho\lambda}_{\lambda\lambda} + c^{\rho\lambda}_{\mu\mu} + c^{\rho\lambda}_{\mu\lambda})\, e_{\lambda\mu} .$$

We have

$$f(g_3)^{g_2} = c^{\rho\mu} + [e_{\rho\lambda}, c^{\rho\mu}] + c^{\rho\mu}_{\lambda\rho} e_{\rho\lambda}$$

$$= c^{\rho\mu} + \sum c^{\rho\mu}_{i\rho} e_{i\lambda} + \sum c^{\rho\mu}_{\lambda j} e_{\rho j}$$

$$= c^{\rho\mu} + (c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda})\, e_{\rho\lambda} + c^{\rho\mu}_{\lambda\mu} e_{\rho\mu} .$$

Now

$$(c^{\rho\mu})^{g_1} = c^{\rho\mu} + [e_{\lambda\mu}, c^{\rho\mu}] + c^{\rho\mu}_{\mu\lambda} e_{\lambda\mu} = c^{\rho\mu} + \sum c^{\rho\mu}_{i\lambda} e_{i\mu} + \sum c^{\mu\mu}_{\mu j} e_{\lambda j}$$

$$= c^{\rho\mu} + c^{\rho\mu}_{\mu\lambda} e_{\rho\mu} + (c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\mu\mu})\, e_{\lambda\mu} ,$$

$$(e_{\rho\lambda})^{g_1} = e_{\rho\lambda} + e_{\rho\mu} , \qquad (e_{\rho\mu})^{g_1} = e_{\rho\mu}$$

and so

$$f(g_3)^{g_1 g_2} = c^{\rho\mu} + (c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda})\, e_{\rho\lambda} + (c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\mu\mu})\, e_{\lambda\mu}$$

$$+ (c^{\rho\mu}_{\rho\lambda} + c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\lambda\mu})\, e_{\rho\mu} .$$

Therefore,

$$\sum_{i\neq\lambda,\rho} c^{\rho\lambda}_{i\lambda} e_{i\mu} + \sum_{j\neq\lambda,\mu} c^{\lambda\mu}_{\lambda j} e_{\rho j} + (c^{\rho\lambda}_{\mu\lambda} + c^{\lambda\mu}_{\lambda\rho})\, e_{\lambda\lambda} + (c^{\lambda\mu}_{\rho\rho} + c^{\lambda\mu}_{\lambda\lambda} + c^{\lambda\mu}_{\lambda\rho})\, e_{\rho\lambda}$$

$$+ (c^{\rho\lambda}_{\rho\lambda} + c^{\lambda\mu}_{\lambda\mu})\, e_{\rho\mu} + (c^{\rho\lambda}_{\lambda\lambda} + c^{\rho\lambda}_{\mu\mu} + c^{\rho\lambda}_{\mu\lambda})\, e_{\lambda\mu}$$

$$= c^{\rho\mu} + (c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda})\, e_{\rho\lambda} + (c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\mu\mu})\, e_{\lambda\mu} + (c^{\rho\mu}_{\rho\lambda} + c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\lambda\mu})\, e_{\rho\mu}$$

$$= \sum_{j\neq\lambda,\mu} c^{\rho\mu}_{\rho j} e_{\rho j} + \sum_{i\neq\rho,\lambda} c^{\rho\mu}_{i\mu} e_{i\mu} + \sum_{i\neq\rho,\mu,\lambda} c^{\rho\mu}_{ii} e_{ii} + c^{\rho\mu}_{\lambda\lambda} e_{\lambda\lambda}$$

$$+ (c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\rho\lambda})\, e_{\rho\lambda} + (c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\mu\mu} + c^{\rho\mu}_{\lambda\mu})\, e_{\lambda\mu}$$

$$+ (c^{\rho\mu}_{\rho\lambda} + c^{\rho\mu}_{\rho\rho} + c^{\rho\mu}_{\lambda\lambda} + c^{\rho\mu}_{\lambda\mu} + c^{\rho\mu}_{\rho\mu})\, e_{\rho\mu} + \sum_{\substack{i\neq\rho,\mu\\ j\neq\rho,\mu\\ i\neq j}} c^{\rho\mu}_{ij} e_{ij} .$$

Hence we must have

$$c_{i\lambda}^{\rho\lambda} = c_{i\mu}^{\rho\mu} \qquad \text{for} \qquad i \neq \rho, \lambda,$$

$$c_{\lambda j}^{\lambda\mu} = c_{\rho j}^{\rho\mu} \qquad \text{for} \qquad j \neq \lambda, \mu,$$

$$c_{ii}^{\rho\mu} = 0 \qquad \text{for} \qquad i \neq \rho, \lambda, \mu,$$

$$c_{\lambda\lambda}^{\rho\mu} = c_{\mu\lambda}^{\rho\lambda} + c_{\lambda\rho}^{\lambda\mu},$$

$$c_{\rho\rho}^{\lambda\mu} + c_{\lambda\lambda}^{\lambda\mu} + c_{\lambda\rho}^{\lambda\mu} = c_{\rho\rho}^{\rho\mu} + c_{\lambda\lambda}^{\rho\mu} + c_{\rho\lambda}^{\rho\mu},$$

$$c_{\lambda\lambda}^{\rho\lambda} + c_{\mu\mu}^{\rho\lambda} + c_{\mu\lambda}^{\rho\lambda} = c_{\lambda\lambda}^{\rho\mu} + c_{\mu\mu}^{\rho\mu} + c_{\lambda\mu}^{\rho\mu},$$

$$c_{\rho\lambda}^{\rho\lambda} + c_{\lambda\mu}^{\lambda\mu} = c_{\rho\mu}^{\rho\mu} + c_{\rho\lambda}^{\rho\mu} + c_{\rho\rho}^{\rho\mu} + c_{\lambda\lambda}^{\rho\mu} + c_{\lambda\mu}^{\rho\mu},$$

$$c_{ij}^{\rho\mu} = 0 \qquad \text{for} \qquad i \neq \rho, \mu, \quad j \neq \rho, \mu, \quad i \neq j.$$

From the first two equations we obtain

$$c_{\mu\lambda}^{\rho\lambda} = c_{\mu\mu}^{\rho\mu}, \qquad c_{\lambda\rho}^{\lambda\mu} = c_{\rho\rho}^{\rho\mu},$$

so that $c_{\lambda\lambda}^{\rho\mu} = c_{\mu\lambda}^{\rho\lambda} + c_{\lambda\rho}^{\lambda\mu} = c_{\mu\mu}^{\rho\mu} + c_{\rho\rho}^{\rho\mu} = 0$, and we have $c_{ii}^{\rho\mu} = 0$ for $i \neq \rho, \mu$. The other relations reduce to

$$c_{\lambda\lambda}^{\lambda\mu} + c_{\rho\rho}^{\lambda\mu} = c_{\rho\lambda}^{\rho\mu}, \qquad c_{\lambda\lambda}^{\rho\lambda} + c_{\mu\mu}^{\rho\lambda} = c_{\lambda\mu}^{\rho\mu},$$

$$c_{\rho\lambda}^{\rho\lambda} + c_{\mu\lambda}^{\lambda\mu} = c_{\rho\mu}^{\rho\mu} + c_{\rho\lambda}^{\rho\mu} + c_{\rho\rho}^{\rho\mu} + c_{\lambda\mu}^{\rho\mu}.$$

We may summarize all the information as follows:

For $\rho < \mu$

$$c_{i\rho}^{\rho\mu} = 0 \qquad (i \neq \rho),$$

$$c_{\mu j}^{\rho\mu} = 0 \qquad (j \neq \mu),$$

$$c_{\rho\rho}^{\rho\mu} = c_{\mu\mu}^{\rho\mu}.$$

For $\rho < \lambda < \mu$,

$$c_{i\rho}^{\lambda\mu} = 0 \qquad (i \neq \rho, \lambda),$$

$$c_{\mu j}^{\rho\lambda} = 0 \qquad (j \neq \lambda, \mu),$$

$$c_{ij}^{\rho\mu} = 0 \qquad (i \neq \rho \text{ and } j \neq \mu),$$

$$c_{i\mu}^{\rho\mu} = c_{i\lambda}^{\rho\lambda} \qquad (i \neq \rho, \lambda),$$

$$c_{\rho j}^{\rho\mu} = c_{\lambda j}^{\lambda\rho} \qquad (j \neq \lambda, \mu),$$

$$c_{\rho\lambda}^{\rho\mu} = c_{\lambda\lambda}^{\lambda\mu} + c_{\rho\rho}^{\lambda\mu},$$

$$c_{\lambda\mu}^{\rho\mu} = c_{\rho\rho}^{\rho\lambda} + c_{\mu\mu}^{\lambda\mu},$$

$$c_{\rho\lambda}^{\rho\lambda} + c_{\lambda\mu}^{\lambda\mu} + c_{\rho\mu}^{\rho\mu} = c_{\lambda\lambda}^{\lambda\mu} + c_{\rho\rho}^{\lambda\mu} + c_{\rho\rho}^{\rho\lambda} + c_{\mu\mu}^{\rho\lambda}.$$

For $\rho < \mu$, $\lambda < \sigma$, $\{\rho, \mu, \lambda, \sigma\}$ distinct,

$$c_{\sigma\mu}^{\rho\mu} = c_{\lambda\rho}^{\lambda\sigma}.$$

Finally,

$$c_{ij}^{\rho,\rho+1} = \delta_{1i}\delta_{nj}c_{1n}^{\rho,\rho+1} \qquad (i \neq \rho, j \neq \rho+1, i \neq j),$$

$$c_{ii}^{\rho,\rho+1} = c_{jj}^{\rho,\rho+1} \qquad (i \neq \rho, \rho+1, j \neq \rho, \rho+1).$$

Define

$$\alpha_\rho = \begin{cases} c_{11}^{\rho,\rho+1} & \text{for} \quad \rho > 1, \\ c_{nn}^{12} & \text{for} \quad \rho = 1; \end{cases}$$

$$\beta_\rho = \begin{cases} c_{1n}^{\rho,\rho+1} & \text{for} \quad \rho < n, \\ 0 & \text{for} \quad \rho = n; \end{cases}$$

$$\gamma_{ij} = \begin{cases} c_{1j}^{1i} & \text{for} \quad i > 1, \\ 0 & \text{for} \quad i = 1; \end{cases}$$

$$\epsilon_{ij} = \begin{cases} c_{in}^{jn} & \text{for} \quad j < n, \\ 0 & \text{for} \quad j = n. \end{cases}$$

Then we may deduce

$$\epsilon_{i\rho} = \begin{cases} \gamma_{i\rho} + (\delta_{i2} + \delta_{\rho,n-1})\,\alpha_\rho & \text{for} \quad i > \rho, \\ \gamma_{i\rho} & \text{for} \quad 1 < i < \rho < n. \end{cases}$$

Also,

$$c_{ij}^{\rho\mu} = \begin{cases} \delta_{\mu,\rho+1}(\delta_{i1}\delta_{jn}\beta_\rho + \delta_{ij}\alpha_\rho) & \text{for} \quad i \neq \rho, \quad j \neq \mu, \\ \gamma_{\mu j} + \delta_{j\rho}\delta_{\mu,\rho+1}(1 + \delta_{\rho1})\,\alpha_\rho & \text{for} \quad i \neq \rho, \quad j = \mu, \\ \epsilon_{i\rho} + \delta_{i\mu}\delta_{\mu,\rho+1}(1 + \delta_{\mu n})\,\alpha_\rho & \text{for} \quad i \neq \rho, \quad j = \mu, \\ \gamma_{\mu\rho} + \gamma_{\mu\mu} + \gamma_{\mu1} + \gamma_{\rho\rho} + \gamma_{\rho1} + \delta_{\rho2}\alpha_1 & \text{for} \quad i = \rho, \quad j = \mu. \end{cases}$$

It now follows that

$$\begin{aligned} c^{\rho\mu} = \sum_{i \neq \rho} (\gamma_{i\rho} + \delta_{i1}\epsilon_{1\rho})\, e_{i\mu} + \sum_{j \neq \mu} \gamma_{\mu j} e_{\rho j} \\ + (\gamma_{\mu\rho} + \gamma_{\mu\mu} + \gamma_{\mu1} + \gamma_{\rho\rho} + \gamma_{\rho1})\, e_{\rho\mu} \\ + \delta_{\mu,\rho+1}\{\alpha_\rho I + (1 + \delta_{\rho1})(1 + \delta_{\mu n})\,\beta_\rho e_{1n}\} \\ + \delta_{\mu2}\alpha_1 e_{11} + (\delta_{\rho1} + \delta_{\rho2})\,\alpha_1 e_{2\mu}. \end{aligned}$$

Define A as follows:

$$a_{ij} = \begin{cases} \gamma_{ij} + \delta_{i1}\epsilon_{1j} + \delta_{i2}\delta_{j1}\alpha_1 & \text{for} \quad i \neq j, \\ \gamma_{ii} + \gamma_{i1} + \delta_{i2}\alpha_1 + a + \delta_{i1}b & \text{for} \quad i = j. \end{cases}$$

Then $\mathrm{Tr}(A) = \Sigma(\gamma_{ii} + \gamma_{i1}) + \alpha_1 + na + b$ so that, if n is odd, we may select $b = 0$ and a such that $\mathrm{Tr}(A) = 0$. However, if n is even we may only force one of b and $\mathrm{Tr}(A)$ to be 0.

We now have

$$c^{\rho\mu} = A^{I+e_{\rho\mu}} + A + \delta_{\rho 1} b e_{1\mu} + \delta_{\mu,\rho+1}\{\alpha_\rho I + (1 + \delta_{\rho 1})(1 + \delta_{\mu n})\beta_\rho e_{1n}\}.$$

Hence, up to cohomology, we may assume with no loss of generality,

$$f: \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ 0 & 0 & 1 & \cdots & x_{3n} \\ \vdots & \cdots & & \cdots & \vdots \\ 0 & 0 & & 0 & 1 \end{pmatrix} \to \left(\sum_{\rho=1}^{n-1} \alpha_\rho x_{\rho,\rho+1}\right) I + \left(\sum_{\rho=2}^{n-2} \beta_\rho x_{\rho,\rho+1}\right) e_{in}$$

$$+ b \sum_{j=2} \sum_{k=1} \left(\prod_{i<k} x_{i,i+1}\right) x_{kj} e_{1j},$$

with $b = 0$ if n is odd or if we are in the case of cohomology into $\mathfrak{g}$. Moreover, it is easy to verify that for distinct choices of $\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_{n-2}, b$, we get non-cohomologous cocycles. If we insist $f(x)$ be of trace 0, if n is odd we simply have

$$f(x) = \left(\sum_{\rho=2}^{n-2} \beta_\rho x_{\rho,\rho+1}\right) e_{1n}.$$

PROPOSITION 5. *For* $k = \mathbf{F}_2$,

$$H_0^1(G, \mathfrak{g}) \cong k^{(2n-4)}.$$

If $2 \nmid n$,

$$H_0^1(G, \mathfrak{g}_0) \cong k^{(n-3)}$$

and if $2 \mid n$

$$H_0^1(G, \mathfrak{g}_0) \cong k^{(2n-3)}.$$

We summarize Propositions 1-5.

THEOREM. *Assuming k perfect for the case* $n = 2$, $p = 2$, *we have*

$$H_0^1(G, \mathfrak{g}) = 0,$$

and if $p \nmid n$

$$H_0^1(G, \mathfrak{g}_0) = 0,$$

whereas if $p \mid n$

$$H_0^1(G, \mathfrak{g}_0) \cong k$$

with the following table of exceptions.

n	k	$d = \dim_k H_0^1(G, \mathfrak{g})$	$d_0 = \dim_k H_0^1(G, \mathfrak{g}_0)$
2	$\mathbf{F}_3$	1	0
2	$\mathbf{F}_5$	1	1
3	$\mathbf{F}_4$	2	2
4	$\mathbf{F}_3$	1	1
odd	$\mathbf{F}_2$	$2n - 4$	$n - 3$
even	$\mathbf{F}_2$	$2n - 4$	$2n - 3$

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