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Nonparametric Approach for Non-Gaussian Vector Stationary Processes

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Suppose that $\{\mathbf{z}(t)\}$ is a non-Gaussian vector stationary process with spectral density matrix $f(\lambda)$. In this paper we consider the testing problem $H: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda = c$ against $A: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda \neq c$, where $K\{\cdot\}$ is an appropriate function and c is a given constant. For this problem we propose a test T_n based on $\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda$, where $\hat{f}_n(\lambda)$ is a nonparametric spectral estimator of $f(\lambda)$, and we define an efficacy of T_n under a sequence of nonparametric contiguous alternatives. The efficacy usually depnds on the fourth-order cumulant spectra f_4^Z of $\mathbf{z}(t)$. If it does not depend on f_4^Z, we say that T_n is non-Gaussian robust. We will give sufficient conditions for T_n to be non-Gaussian robust. Since our test setting is very wide we can apply the result to many problems in time series. We discuss interrelation analysis of the components of $\{\mathbf{z}(t)\}$ and eigenvalue analysis of $f(\lambda)$. The essential point of our approach is that we do not assume the parametric form of $f(\lambda)$. Also some numerical studies are given and they confirm the theoretical results. © 1996 Academic Press, Inc.

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1. Introduction

The ordinary nonparametric approach for independently and identically distributed random variables has developed in various directions. For example Hallin, Ingenbleek, and Puri [9] and Hallin and Puri [10] introduced a class of linear serial rank statistics for the problem of testing a given ARMA model against other ARMA models. They derived the asymptotic distributions of the proposed test statistics under the null as well as alternative hypotheses and gave an explicit formulation of the asymptotically most powerful test under a sequence of contiguous ARMA alternatives. Dzhaparidze [6] considered a class $\mathscr{F}$ of goodness-of-fit tests for testing a simple hypothesis about the form of the spectral density. He investigated the asymptotic properties of test $T \in \mathscr{F}$ under a sequence of nonparametric contiguous alternatives.

Suppose that $\{z(t);\ t=0, \pm 1, ...\}$ is a p-dimensional non-Gaussian vector stationary process with spectral density matrix $f(\lambda)$. In this paper we consider the testing problem

$$H\colon \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda = c,$$

against

$$A\colon \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda \neq c, \tag{1.1}$$

where $K\{\cdot\}$ is a holomorphic function defined on $\mathbf{C}^{p^2}$ and c is a given constant. For this problem we propose a test statistic T_n based on $\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda$, where $\hat{f}_n(\lambda)$ is a nonparametric spectral estimator of $f(\lambda)$. In Section 2 we investiage the asymptotic properties of T_n, and introduce an efficacy of T_n, $\mathrm{eff}(T_n)$, which measures a goodness of T_n. Usually $\mathrm{eff}(T_n)$ depends on the fourth-order cumulant spectra of $\{\mathbf{z}(t)\}$. If it does not depend on the fourth-order cumulant spectra, then we say that T_n is non-Gaussian robust. For a Gaussian scalar process, Taniguchi and Kondo [20] and Kondo and Taniguchi [14] proved some superiority of T_n to the existing methods. In this paper we develop the discussion beyond their scope. In Section 3 we give sufficient conditions for T_n to be non-Gaussian robust in typical examples of $K\{\cdot\}$. Our test setting is unexpectedly wide and can be applied to many problems in time series. In Section 4 we discuss interrelation analysis of the components of $\{\mathbf{z}(t)\}$. The measure of linear dependence $F_{X,Y}$ is known to be an important quantity in econometrics and is related to the causality (e.g., Geweke [8]). We can see that our setting (1.1) includes the testing problem

$$H\colon F_{X,Y} = c \qquad \text{against} \qquad A\colon F_{X,Y} \neq c.$$

In Section 5 we deal with a testing problem for the integral of certain function of the eigenvalues of $f(\lambda)$. It is shown that our setting (1.1) also includes this problem. We apply this to the principal components analysis of $\{\mathbf{z}(t)\}$.

The functional of the spectral density matrix $\int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda$ represents so many important indices in time series analysis. Therefore we will be able to find many other unexpected applications of (1.1). Here it may be noted that we will establish the $\sqrt{n}$-consistent asymptotic theory for $\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda$.

Our approach is designed for essentially nonparametric hypotheses and so is different from that of Hallin *et al.* [9, 10]. Also, since Dzhaparidze's class $\mathscr{F}$ of tests is based on $\int_{-x}^{x} \{I_n(\lambda) f(\lambda)^{-1}\}\, d\lambda$, where $I_n(\lambda)$ is the periodogram, his problem does not include our testing problem (1.1).

As for the notations used in this paper, we denote the set of all integers by J, and denote Kronecker's delta by $\delta(t, s)$.

2. Basic Theory

In this section we formulate some basic theorems concerning the nonparametric testing problem. Let $\{\mathbf{z}(t);\ t \in J\}$ be a vector-valued linear process generated as

$$\mathbf{z}(t) = \sum_{s=0}^{\infty} G(s)\, \mathbf{e}(t-s), \qquad t \in J,$$

where the $\mathbf{z}(t)$'s have p components and the $\mathbf{e}(t)$'s are p-vectors such that $E\{\mathbf{e}(t)\} = 0$ and $E\{\mathbf{e}(t)\, \mathbf{e}(s)'\} = \delta(t, s)\, \Omega$, with Ω a $p \times p$ positive definite matrix; the $G(s)$'s are $p \times p$ matrices. We denote the (a, b) component of Ω and $G(s)$ by Ω_{ab} and $G_{ab}(s)$, respectively, and denote the ath component of $\mathbf{z}(t)$ and $\mathbf{e}(t)$ by $z_a(t)$ and $e_a(t)$, respectively. Initially, we make the following assumption.

Assumption 1. (i) For $a, b = 1, ..., p$,

$$\sum_{s=0}^{\infty} (1 + s^2) |G_{ab}(s)| < \infty.$$

(ii) In the unit disc $|z| \leqslant 1$,

$$\det \left\{ \sum_{s=0}^{\infty} G(x)\, z^s \right\} \neq 0.$$

Under this assumption the process $\{\mathbf{z}(t)\}$ is a second-order stationary process with the spectral density matrix

$$f(\lambda) = \frac{1}{2\pi} A(\lambda)\, \Omega A(\lambda)^*, \qquad -\pi < \lambda \leqslant \pi,$$

where $A(\lambda) = \sum_{s=0}^{\infty} G(s)\, e^{i\lambda s}$. The (a, b) component of $f(\lambda)$ and $A(\lambda)$ are denoted by $f_{ab}(\lambda)$ and $A_{ab}(\lambda)$, respectively. For a partial realization $\{\mathbf{z}(1), ..., \mathbf{z}(n)\}$, the periodogram matrix is defined as

$$I_n(\lambda) = \frac{1}{2\pi n} \left\{ \sum_{i=1}^{n} \mathbf{z}(t)\, e^{i\lambda t} \right\} \left\{ \sum_{t=1}^{n} \mathbf{z}(t)\, e^{i\lambda t} \right\}^*.$$

Since Gaussianity is not assumed for the process we need the following assumption.

Assumption 2. (i) $\{\mathbf{e}(t)\}$ is fourth-order stationary. (ii) The joint fourth-order cumulants $c^e_{abcd}(t_1, t_2, t_3)$ of $e_a(t)$, $e_b(t+t_1)$, $e_c(t+t_2)$, $e_d(t+t_3)$ satisfy

$$\sum_{t_1, t_2, t_3 = -\infty}^{\infty} |c^e_{abcd}(t_1, t_2, t_3)| < \infty, \qquad a, b, c, d = 1, ..., p.$$

Then $\{e(t)\}$ has the fourth-order cumulant spectral density

$$f^e_{abcd}(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{2\pi}\right)^3 \sum_{t_1, t_2, t_3 = -\infty}^{\infty} c^e_{abcd}(t_1, t_2, t_3)\, e^{-i(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3)}.$$

Similarly we can define $c^z_{abcd}(t_1, t_2, t_3)$ and $f^z_{abcd}(\lambda_1, \lambda_2, \lambda_3)$, respectively, the fourth-order cumulant and spectral density of the process $\{\mathbf{z}(t)\}$.

Assumption 3. Let D be an open subset of $\mathbf{C}^{p^2}$. $K: D \to \mathbb{R}$ is holomorphic.

Consider the testing problem

$$H: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda = c,$$

against

$$A: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda \neq c, \tag{2.1}$$

where c is a given constant. This test setting is unexpectedly wide and can be applied to many problems in time series. Several important applications will be given in Sections 3–5. For the problem (2.1) we propose a test based on $\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda$, where $\hat{f}_n(\lambda)$ is a nonparametric spectral estimator of $f(\lambda)$. In this paper we use

$$\hat{f}_n(\lambda) = \int_{-\pi}^{\pi} W_n(\lambda - \mu)\, I_n(\mu)\, d\mu.$$

Here $W_n(\cdot)$ satisfies the following.

Assumption 4. (i) $W(x)$ is bounded, even, nonnegative and such that

$$\int_{-\infty}^{\infty} W(x)\, dx = 1.$$

(ii) For $M = 0(n^{\alpha})(\frac{1}{4} < \alpha < \frac{1}{2})$, the function $W_n(\lambda) = MW(M\lambda)$ can be expanded as

$$W_n(\lambda) = \frac{1}{2\pi} \sum_{l} w\left(\frac{l}{M}\right) \exp(-il\lambda),$$

where $w(x)$ is a continuous, even function with $w(0) = 1$, $|w(x)| \leqslant 1$ and $\int_{-\infty}^{\infty} w(x)^2\, dx < \infty$, and satisfies

$$\lim_{|x| \to 0} \frac{1 - w(x)}{|x|^2} = \kappa_2 < \infty.$$

Under our assumptions it is not difficult to check that the assumptions of Theorems 9 and 10 in Hannan [11, Section V] are satisfied, whence

$$\hat{f}_n(\lambda) - f(\lambda) = O_p\{\sqrt{M/n}\}, \tag{2.2}$$

uniformly in $\lambda \in [-\pi, \pi]$.

We proceed to discuss the asymptotic theory for $\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda$. For this we impose the following conditions on the process $\{\mathbf{z}(t)\}$, as in Hosoya and Taniguchi [12]. We denote the σ-field generated by $\{e(s): s \leqslant t\}$ by $B(t)$.

Assumptions 5. (i) For $a, b = 1, ..., p$ and $m \in J$,

$$\mathrm{Var}[E\{e_a(t)\, e_b(t+m) \mid B(t-\tau)\} - \delta(m, 0)\, \Omega_{ab}] = O(\tau^{-2-\varepsilon}), \qquad \varepsilon > 0,$$

uniformly in t.

(ii) $E\,|E\{e_a(t_1)\, e_b(t_2)\, e_c(t_3)\, e_d(t_4)|\, B(t_1 - \tau)\} - E\{e_a(t_1)\ e_b(t_2)\ e_c(t_3)\ e_d(t_4)\}| = O(\tau^{-1-\eta})$, uniformly in t_1, where $t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$ and $\eta > 0$.

First, we state the following basic result, whose proof we have put in Section 6.

THEOREM 1. *Suppose that Assumptions* 1–5 *hold. Under the null hypothesis H*,

$$S_n = \sqrt{n}\left[\int_{-\pi}^{\pi} K\{\hat{f}_n(\lambda)\}\, d\lambda - c\right]$$

has asymptotically a normal distribution with mean zero and variance $v_1(f) + v_2(f_4^z)$, *where*

$$v_1(f) = 4\pi \int_{-\pi}^{\pi} \operatorname{tr}[f(\lambda)\, K^{(1)}\{f(\lambda)\}]^2\, d\lambda$$

and

$$v_2(f_4^z) = 2\pi \sum_{r,\, t,\, u,\, s=1}^{p} \iint_{-\pi}^{\pi} K_{rt}^{(1)}(\lambda_1)\, K_{us}^{(1)}(\lambda_2)\, f_{rtus}^z(-\lambda_1, \lambda_2, -\lambda_2)\, d\lambda_1\, d\lambda_2.$$

Here $K^{(1)}\{f(\lambda)\}$ *is the first derivative of* $K\{f(\lambda)\}$ *at* $f(\lambda)$ (*see* [15]), *and* $K_{rt}^{(1)}(\lambda)$ *is the* (r, t) *component of* $K^{(1)}\{f(\lambda)\}$.

Here it should be noted that $\sqrt{n}$ consistency holds in Theorem 1 despite (2.2). This is due to the fact that integration of $\hat{f}_n$ recovers $\sqrt{n}$ consistency. For the testing problem (2.1) we are led to estimate the asymptotic variance $v_1(f) + v_2(f_4^z)$ of S_n. In view of Theorem 1 we can propose $v_1(\hat{f}_n)$ as a consistent estimator of $v_1(f)$. Regarding estimation of $v_2(f_4^z)$, Taniguchi [18] and Keenan [13] gave consistent estimators of $v_2(f_4^z)$ when the process concerned is scalar-valued. In what follows we extend Taniguchi's estimator to the case when the process is vector-valued. This extension is not straightforward and requires a modification of the scalar case.

All moments of $\{\mathbf{z}(t)\}$ up to eighth-order are assumed to exist and we set

$$c_{a_1 \cdots a_k}^z(t_1, ..., t_{k-1}) = \operatorname{cum}\{z_{a_1}(0), z_{a_2}(t_1), ..., z_{a_k}(t_{k-1})\},$$

$a_1, ..., a_k = 1, ..., p;\ k = 1, 2, ..., 8.$

Assumption 6. For each $j = 1, 2, ..., k-1$ and any k-tuple $a_1, a_2, ..., a_k$ we have

$$\sum_{t_1, ..., t_{k-1} = -\infty}^{\infty} |t_j c_{a_1 \cdots a_k}^z(t_1, ..., t_{k-1})| < \infty,$$

$k = 2, 3, ..., 8.$

Under Assumption 6 we may define the kth order ($k=2, ..., 8$) cumulant spectral density by

$$f^z_{a_1 \cdots a_k}(\lambda_1, ..., \lambda_{k-1}) = (2\pi)^{-k+1} \sum_{t_1, ..., t_{k-1}=-\infty}^{\infty} c^z_{a_1 \cdots a_k}(t_1, ..., t_{k-1}) \exp\left(-i \sum_{j=1}^{k-1} \lambda_j t_j\right).$$

Define

$$d_{a_j}(\lambda) = \sum_{t=1}^{n} z_{a_j}(t)\, e^{it\lambda}.$$

We impose a further assumption.

Assumption 7. (i) $H(x)$ is a real-valued function, even, of bounded variation, and such that

$$\int_{-\pi}^{\pi} H(x)\, dx = 1,$$

and $H(x)=0$ for $x \notin [-\pi, \pi]$.

(ii) $\{B_n\}$ is a sequence which satisfies $B_n \to 0$, $B_n^2 n \to \infty$ as $n \to \infty$.

Henceforth we set $H_n(x) = B_n^{-1} H(B_n^{-1} x)$. The following proposition gives a consistent estimator of

$$\iint_{-\pi}^{\pi} K(\lambda_1, \lambda_2) f^z_{a_1 a_2 a_3 a_4}(-\lambda_1, \lambda_2, -\lambda_2)\, d\lambda_1\, d\lambda_2, \tag{2.3}$$

where $K(\lambda_1, \lambda_2)$ is continuous on $[-\pi, \pi] \times [-\pi, \pi]$.

PROPOSITION 1. *Under Assumptions* 1–7, *the following is a consistent estimator of* (2.3):

$$U_n = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1} \sum_{j_2} \sum_{j_3} H_n\left\{\frac{2\pi(j_2+j_3)}{n}\right\} K\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}\right)$$
$$\times \frac{1}{8\pi^3 n} d_{a_1}\left\{\frac{2\pi(-j_1+j_2+j_3)}{n}\right\} d_{a_2}\left(\frac{2\pi j_1}{n}\right) d_{a_3}\left(\frac{-2\pi j_2}{n}\right) d_{a_4}\left(\frac{-2\pi j_3}{n}\right) \tag{2.4.1}$$

$$-\int_{-\pi}^{\pi} [K(\lambda, -\lambda) \hat{f}_{a_1 a_3}(-\lambda) \hat{f}_{a_2 a_4}(\lambda) + K(\lambda, \lambda) \hat{f}_{a_1 a_4}(-\lambda) \hat{f}_{a_3 a_3}(\lambda)\Big]\, d\lambda \tag{2.4.2}$$

$$-\frac{H(0)}{B_n}\left(\frac{2\pi}{n}\right)^2 \sum_{j_1}\sum_{j_2} K\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}\right)$$

$$\times\left(\frac{1}{2\pi n}\right)^2 d_{a_1}\left(\frac{-2\pi j_1}{n}\right) d_{a_2}\left(\frac{2\pi j_1}{n}\right) d_{a_3}\left(\frac{-2\pi j_2}{n}\right) d_{a_4}\left(\frac{2\pi j_2}{n}\right), \quad (2.4.3)$$

where, here and subsequently, sums with respect to $j_1, ...$ *are from* $[-n/2]+1$ *to* $[n/2]$, *and* $\hat{f}_{a_k a_j}(\lambda)$ *is the nonparametric spectral estimator of* $f_{a_k a_j}(\lambda)$ *defined previously.*

The proof is given in Section 6.

Let $\hat{v}_2$ be the consistent estimator of $v_2(f_4^z)$ given in the manner of Proposition 1. Then, Theorem 1, Proposition 1, and Slutsky's theorem together yield the following.

THEOREM 2. *Suppose that Assumptions* 1–7 *hold. Then, under the null hypothesis H*,

$$T_n = S_n/\sqrt{v_1(\hat{f}_n) + \hat{v}_2}$$

has asymptotically the standard normal distribution $\mathcal{N}(0, 1)$. *In particular, if the process is Gaussian,*

$$T_n^G = S_n/\sqrt{v_1(\hat{f}_n)}$$

converges in distribution to $\mathcal{N}(0, 1)$.

From Theorem 2 we can propose the test of H given by the critical region

$$[|T_n| > t_\alpha], \quad (2.5)$$

where t_α is defined by

$$\int_{t_\alpha}^{\infty} (2\pi)^{-1/2} \exp\left(-\frac{x^2}{2}\right) dx = \frac{\alpha}{2}.$$

Next we introduce a measure of goodness of our test. Let $a(\lambda)$ be a $p \times p$ matrix whose entries $a_{kl}(\lambda)$ $(k, l = 1, ..., p)$ are square integrable functions on $[-\pi, \pi]$. We assume that $a(\lambda)$ is positive definite for each $\lambda \in [-\pi, \pi]$. Consider a sequence of alternative spectral density matrices

$$g_n(\lambda) = f(\lambda) + \frac{1}{\sqrt{n}} a(\lambda). \quad (2.6)$$

Let $E_{g_n}(\cdot)$ and $V_f(\cdot)$ denote the expectation under $g_n(\lambda)$ and the variance under $f(\lambda)$, respectively. It is natural to define an efficacy of T_n by

$$\mathrm{eff}(T_n) = \lim_{n \to \infty} \frac{E_{g_n}(S_n)}{\sqrt{V_f(S_n)}}, \tag{2.7}$$

in line with the usual definition for a sequence of "parametric alternatives" (e.g., Randles and Wolfe [16, pp. 147–149]). Then we see that

$$\begin{aligned}\mathrm{eff}(T_n) &= \lim_{n \to \infty} \frac{\sqrt{n} \int_{-\pi}^{\pi} [K\{g_n(\lambda)\} - K\{f(\lambda)\}]\, d\lambda}{\sqrt{v_1(f) + v_2(f_4^z)}} \\ &= \frac{\int_{-\pi}^{\pi} \mathrm{tr}[K^{(1)}\{f(\lambda)\}\, a(\lambda)]\, d\lambda}{\sqrt{v_1(f) + v_2(f_4^z)}}. \end{aligned} \tag{2.8}$$

For another test T_n^* we can define an asymptotic relative efficiency (ARE) of T_n relative to T_n^* by

$$\mathrm{ARE}(T_n, T_n^*) = \left\{ \frac{\mathrm{eff}(T_n)}{\mathrm{eff}(T_n^*)} \right\}^2. \tag{2.9}$$

For a Gaussian scalar process, Taniguchi and Kondo [20] and Kondo and Taniguchi [14] proved some superiority of T_n to the existing methods. In this paper we develop the discussion beyond their scope. In later sections we will discuss non-Gaussian robustness of T_n, interrelation analysis for the components of $\{\mathbf{z}(t)\}$ and eigenvalue analysis of the spectral density matrix.

3. Non-Gaussian Robustness

In the previous section we evaluated the efficacy of T_n. If the process is not Gaussian, it depends on a non-Gaussian quantity $v_2(f_4^z)$. Henceforth we say that T_n is non-Gaussian robust if $v_2(f_4^z) = 0$. This definition means that if T_n is non-Gaussian robust then its efficacy attains the "Gaussian efficacy"

$$\int_{-\pi}^{\pi} \mathrm{tr}[K^{(1)}\{f(\lambda)\}\, a(\lambda)]\, dy / \sqrt{v_1(f)} \tag{3.1}$$

although the process is not Gaussian. We deal with the same process $\{\mathbf{z}(t)\}$ as in Section 2. We impose a further assumption on the innovation process $\{\mathbf{e}(t)\}$.

Assumption 8. (i) $\{\mathbf{e}(t); t \in J\}$ is a family of i.i.d. random vectors with $E\{\mathbf{e}(t)\} \equiv 0$ and the variance matrix Ω.

(ii) The fourth-order cumulants

$$\kappa_{abcd} = \text{cum}\{e_a(t), e_b(t), e_c(t), e_d(t)\},$$

exist for $a, b, c, d = 1, ..., p$, and $t \in J$.

Then we get the following proposition, whose proof we have put in Section 6.

PROPOSITION 2. *Suppose that Assumptions* 1 *and* 8 *hold. If*

$$\int_{-\pi}^{\pi} A^*(\lambda)\, K^{(1)}\{f(\lambda)\}\, A(\lambda)\, d\lambda = 0 \qquad (p \times p\text{-null matrix}), \tag{3.2}$$

then $v_2(f_4^z) = 0$.

In this section we consider the testing problem related to the following three measures of nearness between $p \times p$-spectral density matrices $f(\lambda)$ and $g(\lambda)$.

(I) Likelihood ratio measure. Define

$$D_{LR}(f, g) = \int_{-\pi}^{\pi} \left[\log \frac{\det\{g(\lambda)\}}{\det\{f(\lambda)\}} + \text{tr}\{f(\lambda)\, g(\lambda)^{-1}\} - p \right] d\lambda. \tag{3.3}$$

Altough the process concerned is not Gaussian, we can formally make the Gaussian likelihood ratio GLR. The above measure is an approximation of $n^{-1}E(GLR)$. We refer to it as a likelihood ratio measure between $f(\lambda)$ and $g(\lambda)$. If we are interested in the testing problem

$$H_{\text{LR}}: D_{\text{LR}}\left(f, \frac{1}{2\pi} I_p\right) = c,$$

against

$$A_{\text{LR}}: D_{\text{LR}}\left(f, \frac{1}{2\pi} I_p\right) \neq c, \tag{3.4}$$

where c is a given constant and I_p is the $p \times p$-identity matrix, we can set down

$$D_{\text{LR}}\left(f, \frac{1}{2\pi} I_p\right) = \int_{-\pi}^{\pi} K_{\text{LR}}\{f(\lambda)\}\, d\lambda,$$

where $K_{LR}\{f(\lambda)\} = -\log\det\{f(\lambda)\} + 2\pi \operatorname{tr}\{f(\lambda)\} - p - p\log 2\pi$. Let $T_n^{(LR)}$ be the test statistic for (3.4) which is given in the manner of Theorem 2. By $T_n^{(LR)}$, the nearness of $f(\lambda)$ to the white noise will be examined.

(II) α-entropy measure. For $\alpha \in (0, 1)$, Albrecht [1] introduced the α-entropy measure between $f(\lambda)$ and $g(\lambda)$,

$$D_\alpha(f, g) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [\log\det\{(1-\alpha) I_p + \alpha f(\lambda) g(\lambda)^{-1}\} - \alpha \log\det\{f(\lambda) g(\lambda)^{-1}\}] \, d\lambda,$$

which measures a nearness of $f(\lambda)$ to $g(\lambda)$. We set down

$$D_\alpha\left(f, \frac{1}{2\pi} I_p\right) = \int_{-\pi}^{\pi} K_\alpha\{f(\lambda)\} \, d\lambda,$$

where

$$K_\alpha\{f(\lambda)\} = (1/4\pi)[\log\det\{(1-\alpha) I_p + 2\pi\alpha f(\lambda)\} - \alpha\log\det\{2\pi f(\lambda)\}].$$

Consider the testing problem

$$H_\alpha: \int_{-\pi}^{\pi} K_\alpha\{f(\lambda)\} \, d\lambda = c$$

against

$$A_\alpha: \int_{-\pi}^{\pi} K_\alpha\{f(\lambda)\} \, d\lambda \neq c. \tag{3.5}$$

Similarly we can make the test statistic $T_n^{(\alpha)}$ for (3.5) in the manner of Theorem 2.

(III) Log-square distance. Define

$$D_{LOG} = \int_{-\pi}^{\pi} [\log\det\{f(\lambda)\} - \log\det\{g(\lambda)\}]^2 \, d\lambda.$$

We set down

$$D_{LOG}(f, I_p) = \int_{-\pi}^{\pi} K_{LOG}\{f(\lambda)\} \, d\lambda,$$

where $K_{LOG}\{f(\lambda)\} = [\log\det\{f(\lambda)\}]^2$. For the testing problem

$$H_{LOG}: \int_{-\pi}^{\pi} K_{LOG}\{f(\lambda)\} \, d\lambda = c,$$

against

$$A_{\mathrm{LOG}}\colon \int_{-\pi}^{\pi} K_{\mathrm{LOG}}\{f(\lambda)\}\, d\lambda \neq c, \tag{3.6}$$

we can construct the test statistic $T_n^{(\mathrm{LOG})}$ in the manner of Theorem 2.

The above three measures $D_{\mathrm{LR}}(\cdot)$, $D_\alpha(\cdot)$, and $D_{\mathrm{LOG}}(\cdot)$ enable us to test how $f(\lambda)$ is distant from the white noise process nonparametrically. The following theorem is concerned with the non-Gaussian robustness of $T_n^{(\mathrm{LR})}$, $T_n^{(\alpha)}$, and $T_n^{(\mathrm{LOG})}$.

THEOREM 3. *Suppose that Assumptions* 1–8 *hold.*

(i) *For the testing problem* (3.4), $T_n^{(\mathrm{LR})}$ *is non-Gaussian robust if*

$$\sum_{j=0}^{\infty} G(j)'\, G(j) = \Omega^{-1}. \tag{3.7}$$

(ii) *For the testing problem* (3.5), $T_n^{(\alpha)}$ *is non-Gaussian robust if*

$$\int_{-\pi}^{\pi} \{(1-\alpha)\, A(\lambda)^{-1}\, A^*(\lambda)^{-1}\, \Omega^{-1} + \alpha I_p\}^{-1}\, d\lambda = 2\pi I_p. \tag{3.8}$$

(iii) *For the testing problem* (3.6), $T_n^{(\mathrm{LOG})}$ *is non-Gaussian robust if the spectral density matrix is expressed as*

$$f(\lambda) = \exp\left\{\sum_{j\neq 0} A_j \cos(j\lambda)\right\}, \tag{3.9}$$

where the A_j*'s are* $p \times p$*-matrices and* $\exp\{\cdot\}$ *is the matrix exponential* (*for the definition, see Bellman* [2, p. 169]).

Now we give a numerical example related to Theorem 3.

EXAMPLE 1. Let $\{\mathbf{z}(t); t \in J\}$ be a scalar process with spectral density $f(\lambda) = \exp\{a \cos \lambda\}$ and $Ez(t) = 0$. For this process we consider the testing problem (3.6), i.e.,

$$\begin{aligned} H_{\mathrm{LOG}}&\colon \int_{-\pi}^{\pi} \{\log f(\lambda)\}^2\, d\lambda = a^2\pi \\ A_{\mathrm{LOG}}&\colon \int_{-\pi}^{\pi} \{\log f(\lambda)\}^2\, d\lambda \neq a^2\pi. \end{aligned} \tag{3.10}$$

Then the test statistic is

$$T_n^{(\mathrm{LOG})} = \frac{\sqrt{n}\,[\int_{-\pi}^{\pi} \{\log \hat{f}_n(\lambda)\}^2\, d\lambda - a^2\pi]}{4\sqrt{\pi}\,[\int_{-\pi}^{\pi} \{\log \hat{f}_n(\lambda)\}^2\, d\lambda]^{1/2}}.$$

TABLE I

a		Case (1)	Case (2)
0.5	Mean of $T_n^{(\mathrm{LOG})}$	-0.028	0.040
	Variance of $T_n^{(\mathrm{LOG})}$	0.972	1.078
	Frequency of $T_n^{(\mathrm{LOG})} < -1.64$	0.050	0.058
	Frequency of $T_n^{(\mathrm{LOG})} > 1.64$	0.040	0.056
0.6	Mean of $T_n^{(\mathrm{LOG})}$	-0.036	0.016
	Variance of $T_n^{(\mathrm{LOG})}$	1.066	1.015
	Frequency of $T_n^{(\mathrm{LOG})} < -1.64$	0.056	0.062
	Frequency of $T_n^{(\mathrm{LOG})} > 1.64$	0.060	0.042

In view of Theorem 3, $T_n^{(\mathrm{LOG})}$ is evidently non-Gaussian robust. For this fact we deal with the following two cases. We can express $\mathbf{z}(t)$ as

$$\mathbf{z}(t) = \sum_{s=0}^{\infty} g(s)\, e(t-s), \qquad t \in J.$$

Case (1). $e(t)$; $t \in J$, are i.i.d. as $\mathcal{N}(0, 1)$ (Gaussian case).
Case (2). $e(t)$; $t \in J$, are i.i.d. with probability density

$$p(x) = \exp(-x+1)$$

(non-Gaussian case).

For Cases (1) and (2) we generated $z(1), ..., z(1024)$, respectively. Then we calculated $T_n^{(\mathrm{LOG})}$ with $M = 100$ for Cases (1) and (2) and iterated this procedure 1000 times. We get Table I. The table agrees with our theoretical results approximately.

4. Interrelation Analysis of the Components of the Process

Our test can be applied to testing for interrelation of the components of $\{\mathbf{z}(t)\}$. Suppose that the process $\{\mathbf{z}(t); t \in J\}$ has the form

$$\mathbf{z}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{pmatrix}$$

with $\mathbf{x}(t)$, q vector-valued, and $\mathbf{y}(t)$, r vector-valued; $q + r = p$ and has the spectral density matrix

$$f(\lambda) = \begin{bmatrix} f_{xx}(\lambda) & f_{xy}(\lambda) \\ f_{yx}(\lambda) & f_{yy}(\lambda) \end{bmatrix}. \tag{4.1}$$

Denote by $H\{\cdot\}$ the linear closed manifold generated by $\{\cdot\}$ and denote by $\text{proj}[\mathbf{x}(t) \mid H\{\cdot\}]$ the projection of $\mathbf{x}(t)$ on $H\{\cdot\}$. We consider the residual process

$$\mathbf{u}_1(t) = \mathbf{x}(t) - \text{proj}[\mathbf{x}(t) \mid H\{\mathbf{x}(t-1), \mathbf{x}(t-2), ...\}],$$
$$\mathbf{v}_1(t) = \mathbf{y}(t) - \text{proj}[\mathbf{y}(t) \mid H\{\mathbf{y}(t-1), \mathbf{y}(t-2), ...\}],$$
$$\mathbf{u}_2(t) = \mathbf{x}(t) - \text{proj}[\mathbf{x}(t) \mid H\{\mathbf{x}(t-1), \mathbf{x}(t-2), ...; \mathbf{y}(t-1), \mathbf{y}(t-2), ...\}],$$
$$\mathbf{v}_2(t) = \mathbf{y}(t) - \text{proj}[\mathbf{y}(t) \mid H\{\mathbf{x}(t-1), \mathbf{x}(t-2), ...; \mathbf{y}(t-1), \mathbf{y}(t-2), ...\}],$$

and

$$\mathbf{u}_3(t) = \mathbf{x}(t) - \text{proj}[\mathbf{x}(t) \mid H\{\mathbf{x}(t-1), \mathbf{x}(t-2), ...; \mathbf{y}(t), \mathbf{y}(t-1), ...\}].$$

The measure of linear feedback from $Y = \{\mathbf{y}(t)\}$ to $X = \{\mathbf{x}(t)\}$ is defined by

$$F_{Y \to X} = \log[\det\{\text{Var}(\mathbf{u}_1(t))\}/\det\{\text{Var}(\mathbf{u}_2(t))\}].$$

Symmetrically,

$$F_{X \to Y} = \log[\det\{\text{Var}(\mathbf{v}_1(t))\}/\det\{\text{Var}(\mathbf{v}_2(t))\}].$$

The measure of instanteous linear feedback

$$F_{X \cdot Y} = \log[\det\{\text{Var}(\mathbf{u}_2(t))\}/\det\{\text{Var}(\mathbf{u}_3(t))\}]$$

has motivation similar to that of the above two measures. The following

$$F_{X,\, Y} = \log[\det\{\text{Var}(\mathbf{u}_1(t))\} \det\{\text{Var}(\mathbf{v}_1(t))\}/\det \Omega]$$

is called the measure of linear dependence. Then it is known that

$$F_{X,\, Y} = F_{Y \to X} + F_{X \to Y} + F_{X \cdot Y} \tag{4.2}$$

and

$$F_{X,\, Y} = \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda, \tag{4.3}$$

where

$$K\{f(\lambda)\} = -\frac{1}{2\pi} \log[\det\{I_q - f_{xy}(\lambda) f_{yy}^{-1}(\lambda) f_{yx}(\lambda) f_{xx}^{-1}(\lambda)\}]$$

(see [8]). Since $F_{Y \to X}$, $F_{X \to Y}$, and $F_{X \cdot Y}$ are important econometric measures which represent "strength of causality," the testing problem

$$H\colon F_{X,\, Y} = c,$$

against

$$A\colon F_{X,\,Y} \neq c \tag{4.4}$$

is important. This is exactly an example of our testing problem. Hence we can test (4.4) by using T_n in Theorem 2.

EXAMPLE 2. Let $\mathbf{z}(t) = \{x(t), y(t)\}'$ be a two-dimensional linear process generated by

$$\mathbf{z}(t) = \mathbf{e}(t) + \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \mathbf{e}(t-1), \qquad t \in J, \tag{4.5}$$

where $\mathbf{e}(t)$'s are i.i.d. $\mathcal{N}(0, I_2)$. We denote the spectral density matrix of $\mathbf{z}(t)$ by

$$f(\lambda) = \begin{pmatrix} f_{xx}(\lambda) & f_{xy}(\lambda) \\ f_{yx}(\lambda) & f_{yy}(\lambda) \end{pmatrix}.$$

Consider the testing problem

$$H\colon F_{X,\,Y} = \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda = c,$$

against,

$$A\colon F_{X,\,Y} \neq c, \tag{4.6}$$

where $K\{f(\lambda)\} = -(1/2\pi) \log\{1 - f_{xy}(\lambda) f_{yy}^{-1}(\lambda) f_{yx}(\lambda) f_{xx}^{-1}(\lambda)\}$. For this problem we generated $\mathbf{z}(1), ..., \mathbf{z}(1024)$ given by (4.5) for $\varepsilon = 1.15, 1.20, 1.25$. Then we calculated T_n in Theorem 2 and iterated this procedure 500 times. The results are given by Table II. We can see that they agree with the theoretical results.

Next, we turn to an investigation of another interrelation analysis. Let $\mathbf{z}(t) = \{\mathbf{u}(t)', \mathbf{v}(t)', \mathbf{w}(t)')'$ be a p-dimensional process satisfying all the

TABLE II

ε	Mean of T_n	Variance of T_n	Frequency of $T_n < -1.64$	Frequency of $T_n > 1.64$
1.15	0.077	1.073	0.058	0.058
1.20	0.067	1.006	0.042	0.056
1.25	0.076	1.000	0.050	0.052

assumptions in Section 2, where $\mathbf{u}(t)$, $\mathbf{v}(t)$, and $\mathbf{w}(t)$ are q, r, and s component processes, respectively. Correspondingly we have the partition

$$f(\lambda)=\begin{bmatrix} f_{uu}(\lambda) & f_{uv}(\lambda) & f_{uw}(\lambda) \\ f_{vu}(\lambda) & f_{vv}(\lambda) & f_{vw}(\lambda) \\ f_{wu}(\lambda) & f_{wv}(\lambda) & f_{ww}(\lambda) \end{bmatrix},$$

and the spectral representation

$$\mathbf{z}(t)=\int_{-\pi}^{\pi} e^{-it\lambda}\, d\xi(\lambda),$$

with $d\xi(\lambda)=(d\xi_u(\lambda)', d\xi_v(\lambda)', d\xi_w(\lambda)')'$. Hannan [11] considered a test for association for $\mathbf{u}=\{\mathbf{u}(t)\}$ with $\mathbf{v}=\{\mathbf{v}(t)\}$ (at frequency λ) after allowing for any effects of $\mathbf{w}=\{\mathbf{w}(t)\}$. The hypothesis is given by

$$H_\lambda : f_{uv}(\lambda)-f_{uw}(\lambda) f_{ww}(\lambda)^{-1} f_{wv}(\lambda)=0, \tag{4.7}$$

which means that $d\xi_u(\lambda)-f_{uw}(\lambda) f_{ww}(\lambda)^{-1}\, d\xi_w(\lambda)$ is incoherent with $d\xi_v(\lambda)-f_{vw}(\lambda) f_{ww}(\lambda)^{-1}\, d\xi_w(\lambda)$ and all of the apparent association between $\mathbf{u}$ and $\mathbf{v}$ is truly due only to their common association with $\mathbf{w}$. For a given λ, Hannan [11] developed the testing theory for (4.7) based on the asymptotic normality of the finite Fourier transformations of $\{\mathbf{z}(t)\}$ in a neighborhood of λ. In view of our testing problem we can consider a test for association for $\mathbf{u}$ with $\mathbf{v}$ at “all the frequency $\lambda\in[-\pi,\pi]$” after allowing for any effects of $\mathbf{w}$. The hypothesis is written as

$$H: f_{uv}(\lambda)-f_{uw}(\lambda) f_{ww}(\lambda)^{-1} f_{wv}(\lambda)=0 \qquad \text{for all} \quad \lambda\in[-\pi,\pi],$$

which is equivalent to

$$H: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda=0, \tag{4.8}$$

where

$$K\{f(\lambda)\}=\mathrm{tr}[\{f_{uv}(\lambda)-f_{uw}(\lambda) f_{ww}(\lambda)^{-1} f_{wv}(\lambda)\} \\ \times \{f_{vu}(\lambda)-f_{vw}(\lambda) f_{ww}(\lambda)^{-1} f_{wu}(\lambda)\}].$$

Therefore we can apply all the results in Section 2 to this testing problem.

5. Eigenvalue Analysis of the Spectral Density Matrix

It is well known that the eigenvalues play a fundamental role in multivariate problems. In this section we investigate the principal components analysis of vector time series, which is related to the eigenvalues of the spectral density matrix. First, we state the following lemma which is due to Magnus and Neudecker [15].

LEMMA 1. *Let* $\mu_1, ..., \mu_p$ *be the eigenvalues of a matrix* $M_0 \in \mathbf{C}^{p^2}$ *and assume that* μ_i *is simple (i.e.,* $\mu_1 > \mu_2 > \cdots > \mu_p$*). Then a scalar function* $\mu_{(i)}$ *exists, defined in a neighborhood* $\mathcal{N}(M_0) \subset \mathbf{C}^{p^2}$ *of* M_0*, such that* $\mu_{(i)}(M_0) = \mu_i$ *and* $\mu_{(i)}(M)$ *is a simple eigenvalue of* M *for every* $M \in \mathcal{N}(M_0)$*. Moreover,* $\mu_{(i)}$ *is* ∞ *time differentiable on* $\mathcal{N}(M_0)$*, and the differential is*

$$d\mu_{(i)} = \mathrm{tr}\left[\left\{\prod_{\substack{j=1\\ j\neq 1}}^{p} \left(\frac{\mu_j I_p - M_0}{\mu_j - \mu_i}\right)\right\} dM\right]. \tag{5.1}$$

Next we turn to discuss the principal components analysis of the vector linear process $\mathbf{z}(t)$ defined in Section 2. Suppose that the spectral density matrix $f(\lambda)$ has the simple eigenvalues $\mu_1(\lambda) > \mu_2(\lambda) > \cdots > \mu_p(\lambda)$. Then the variance of the error series by the q ($q < p$) principal components is given by

$$\int_{-\pi}^{\pi} \left\{ \sum_{j=q+1}^{p} \mu_j(\lambda) \right\} d\lambda. \tag{5.2}$$

If we are interested in the degree of the above measure we can set down the testing problem

$$H: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda = c,$$

against

$$A: \int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda \neq c, \tag{5.3}$$

where $K\{f(\lambda)\} = \sum_{j=q+1}^{p} \mu_j(\lambda)$. In view of Lemma 1 the derivative of $K\{\cdot\}$ is

$$K^{(1)}\{f(\lambda)\} = \sum_{i=q+1}^{p} \left[\prod_{\substack{j=1\\ j\neq 1}}^{p} \left\{\frac{\mu_j(\lambda)\, I_p - f(\lambda)}{\mu_j(\lambda) - \mu_i(\lambda)}\right\}\right].$$

Thus we may construct the test T_n of Theorem 2 for (5.3), whence we can test (5.3) by using T_n.

In Sections 3, 4, and 5 we gave several applications of our testing problem. Since the functional of the spectral density matrix $\int_{-\pi}^{\pi} K\{f(\lambda)\}\, d\lambda$ represents so many important indices in time series analysis, we will be able to find many other unexpected applications of our testing problem.

At this end of this section we summarize the merits of our approach:

(1) Our approach is designed for essentially nonparametric hypotheses. So we do not need any parametric assumptions on the spectral density matrix.

(2) Our test statistic is based on the nonparametric spectral estimator. So we do not need iterative methods to calculate it.

(3) Since our approach is based on the integral of the spectral density matrix $f(\lambda)$ we can develop the $\sqrt{n}$-consistent asymptotic theory (although the nonparametric spectral estimator $\hat{f}_n(\lambda)$ is not a $\sqrt{n}$-consistent estimator of $f(\lambda)$).

(4) We do not assume the Gaussianity of the process $\{\mathbf{z}(t)\}$.

6. Some Proofs

Proof of Theorem 1. Let $\|a\|$ be the norm of $a=(a_1, ..., a_{p^2})' \in \mathbf{C}^{p^2}$ defined by

$$\|a\| = \left\{ \sum_{j=1}^{p^2} |a_j|^2 \right\}^{1/2}.$$

Since the function K is holomorphic in D it has the Taylor series expansion in an open neighborhood $U \subset D$ (e.g., Brillinger [5]; Stewart and Tall [17]). Let

$$\hat{H}(\lambda) = K\{\hat{f}_n(\lambda)\} - K\{f(\lambda)\} - \mathrm{tr}[K^{(1)}\{f(\lambda)\}\{\hat{f}_n(\lambda) - f(\lambda)\}]. \quad (6.1)$$

Using Cauchy's estimate for the derivatives of K (see Bhattacharya and Rao [3, p. 68]) we can see that there exists $\delta > 0$ such that

$$\hat{H}(\lambda) = O\{\|\hat{f}_n(\lambda) - f(\lambda)\|^2\}, \quad (6.2)$$

in the open ball $B(\delta) = \{\|\hat{f}_n(\lambda) - f(\lambda)\| \leqslant \delta\}$.

For every $\varepsilon > 0$ there exists $c > 0$ such that

$$P\left\{|\hat{H}(\lambda)| \geqslant c\frac{M}{n}\right\} \leqslant P\left\{|\hat{H}(\lambda)| \geqslant c\frac{M}{n} \cap B(\delta)\right\} + P\{\|\hat{f}_n(\lambda) - f(\lambda)\| > \delta\} < \varepsilon,$$

because of (2.2). Hence,

$$\hat{H}(\lambda) = O_p(M/n), \quad (6.3)$$

uniformly in λ. Under the null hypothesis, S_n is written as

$$S_n = \sqrt{n} \int_{-\pi}^{\pi} [K\{\hat{f}_n(\lambda)\} - K\{f(\lambda)\}]\, d\lambda.$$

It follows from (6.3) that

$$\begin{aligned} S_n &= \sqrt{n} \int_{-\pi}^{\pi} \mathrm{tr}[K^{(1)}\{f(\lambda)\}\{\hat{f}_n(\lambda) - f(\lambda)\}]\, d\lambda + o_p(1) \\ &= L_n + o_p(1) \qquad \text{(say)}. \end{aligned} \tag{6.4}$$

Putting

$$J_n = \sqrt{n} \int_{-\pi}^{\pi} \mathrm{tr}[K^{(1)}\{f(\lambda)\}\{I_n(\lambda) - f(\lambda)\}]\, d\lambda,$$

we next show that

$$|L_n - J_n| = o_p(1). \tag{6.5}$$

From the definition of $\hat{f}_n$, L_n is written as

$$\begin{aligned} L_n &= \sqrt{n} \int_{-\pi}^{\pi} \mathrm{tr}\left[K^{(1)}\{f(\lambda)\} \int_{-\pi}^{\pi} \{I_n(\mu) - f(\mu)\}\; W_n(\lambda - \mu)\, d\mu \right] d\lambda \\ &\quad + \sqrt{n} \int_{-\pi}^{\pi} \mathrm{tr}\left[K^{(1)}\{f(\lambda)\} \left\{ \int_{-\pi}^{\pi} f(\mu)\, W_n(\lambda - \mu)\, d\mu - f(\lambda) \right\} \right] d\lambda \\ &= L_n^{(1)} + L_n^{(2)} \qquad \text{(say)}. \end{aligned}$$

By changing variable $M(\lambda - \mu) \to \eta$ we obtain

$$L_n^{(1)} = \sqrt{n} \int_{-\pi}^{\pi} \int_{M(-\pi-\mu)}^{M(\pi-\mu)} \mathrm{tr}\left[K^{(1)} \left\{ f\left(\mu + \frac{\eta}{M} \right) \right\} W(\eta)\, d\eta \{I_n(\mu) - f(\mu)\} \right] d\mu.$$

From Assumption 4(i), we can write

$$|J_n - L_n^{(1)}| = \left| \sqrt{n} \int_{-\pi}^{\pi} \mathrm{tr}[A_M(\mu)\{I_n(\mu) - f(\mu)\}]\, d\mu \right|,$$

where

$$A_M(\mu) = \int_{M(-\pi-\mu)}^{M(\pi-\mu)} K^{(1)} \left\{ f\left(\mu + \frac{\eta}{M} \right) \right\} W(\eta)\, d\eta - K^{(1)}\{f(\mu)\} \int_{-\infty}^{\infty} W(\eta)\, d\eta.$$

In view of Lemma A.3.3 in Hosoya and Taniguchi [12],

$$E\,|J_n - L_n^{(1)}|^2 = 4\pi \int_{-\pi}^{\pi} \mathrm{tr}[f(\mu)\,A_M(\mu)\,f(\mu)\,A_M(\mu)]\,d\mu$$
$$+2\pi \sum_{r,t,u,v=1}^{p} \iint_{-\pi}^{\pi} A_M^{rt}(\mu_1)\,A_M^{uv}(\mu_2)$$
$$\times f_{rtuv}^{z}(-\mu_1, \mu_2, -\mu_2)\,d\mu_1\,d\mu_2 + o(1), \qquad (6.6)$$

where $A_M^{rt}(\mu)$ is the (r, t) component of $A_M(\mu)$. By the dominated convergence theorem it is shown that, for every $\varepsilon > 0$,

$$\lim_{M\to\infty} \|A_M(\mu)\| = 0 \qquad \text{for} \quad \mu \in B_\varepsilon,$$

where $B_\varepsilon = [-\pi+\varepsilon, \pi-\varepsilon]$. Hence, for every $\varepsilon > 0$,

$$I(B_\varepsilon) = \int_{B_\varepsilon} \mathrm{tr}[f(\mu)\,A_M(\mu)\,f(\mu)\,A_M(\mu)]\,d\mu \to 0,$$
$$I^{rtuv}(B_\varepsilon \times B_\varepsilon) = \iint_{B_\varepsilon\times B_\varepsilon} A_M^{rt}(\mu_1)\,A_M^{uv}(\mu_2)\,f_{rtuv}^{z}(-\mu_1, \mu_2, -\mu_2)\,d\mu_1\,d\mu_2 \to 0, \qquad (6.7)$$

$r, t, u, v = 1, ..., p$, as $M \to \infty$. While all the components of $A_M(\mu)$, $f(\mu)$, and $f_{rtuv}^{z}(\ ,\ ,\ )$ are bounded on $B = [-\pi, \pi]$, so there exists $d > 0$ such that

$$|I(B - B_\varepsilon)| \leqslant d\varepsilon, \qquad |I^{rtuv}(B\times B - B_\varepsilon \times B_\varepsilon)| \leqslant d\varepsilon \qquad \text{for} \quad r, t, u, v = 1, ..., p. \qquad (6.8)$$

Since ε is chosen arbitrarily, (6.6), (6.7), and (6.8) imply $|J_n - L_n^{(1)}| = o_P(1)$. Thus the proof of (6.5) is complete if we show $L_n^{(2)} = o(1)$. The bias evaluation method (e.g., Hannan [11, p. 283]) yields

$$\int_{-\pi}^{\pi} f(\mu)\,W_n(\lambda-\mu)\,d\mu - f(\lambda) = O(M^{-2}),$$

uniformly in $\lambda \in B$, which implies

$$L_n^{(2)} = O(\sqrt{n}/M^2) \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus we have proved that the asymptotic distribution of S_n is equivalent to that of J_n. Applying Lemma A.3.3 of Hosoya and Taniguchi [12] to J_n, the proof is completed.

Proof of Proposition 1. First, we evaluate the expectation of the term (2.4.1):

$$E\{(2.4.1)\} = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1}\sum_{j_2}\sum_{j_3} H_n\left\{\frac{2\pi(j_2+j_3)}{n}\right\} K\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}\right)\frac{1}{8\pi^3 n}$$

$$\times\left[\operatorname{cum}\left[d_{a_1}\left\{\frac{2\pi(-j_1+j_2+j_3)}{n}\right\}, d_{a_2}\left(\frac{2\pi j_1}{n}\right),\right.\right.$$

$$\left.\times\, d_{a_3}\left(\frac{-2\pi j_2}{n}\right), d_{a_4}\left(\frac{-2\pi j_3}{n}\right)\right] \tag{6.9.1}$$

$$+\operatorname{cum}\left[d_{a_1}\left\{\frac{2\pi(-j_1+j_2+j_3)}{n}\right\}, d_{a_2}\left(\frac{2\pi j_1}{n}\right)\right]$$

$$\times\operatorname{cum}\left[d_{a_3}\left(\frac{-2\pi j_2}{n}\right), d_{a_4}\left(\frac{-2\pi j_3}{n}\right)\right] \tag{6.9.2}$$

$$+\operatorname{cum}\left[d_{a_1}\left\{\frac{2\pi(-j_1+j_2+j_3)}{n}\right\}, d_{a_3}\left(\frac{-2\pi j_2}{n}\right)\right]$$

$$\times\operatorname{cum}\left[d_{a_2}\left(\frac{2\pi j_1}{n}\right), d_{a_4}\left(\frac{-2\pi j_3}{n}\right)\right] \tag{6.9.3}$$

$$+\operatorname{cum}\left[d_{a_1}\left\{\frac{2\pi(-j_1+j_2+j_3)}{n}\right\}, d_{a_4}\left(\frac{-2\pi j_3}{n}\right)\right]$$

$$\left.\times\operatorname{cum}\left[d_{a_2}\left(\frac{2\pi j_1}{n}\right), d_{a_3}\left(\frac{-2\pi j_2}{n}\right)\right]\right]. \tag{6.9.4}$$

It is known that

$$\operatorname{cum}\{d_{a_1}(\lambda_1), ..., d_{a_k}(\lambda_k)\}$$
$$= (2\pi)^{k-1}\,\Delta_n(\lambda_1 + \cdots + \lambda_k)\, f^z_{a_1\cdots a_k}(-\lambda_2, ..., -\lambda_k) + O(1), \tag{6.10}$$

where $\Delta_n = \sum_{t=1}^n e^{i\lambda t}$ and $O(1)$ is uniform in $\lambda_1, ..., \lambda_k$ (e.g., [5]). Using (6.10), (6.9.1) is written as

$$\left(\frac{2\pi}{n}\right)^3 \sum_{j_1}\sum_{j_2}\sum_{j_3}\left[H_n\left\{\frac{2\pi(j_2+j_3)}{n}\right\} K\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}\right)\right.$$

$$\left.\times f^z_{a_1a_2a_3a_4}\left(-\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi j_3}{n}\right) + O(n^{-1})\right]$$

$$= \left(\frac{2\pi}{n}\right)^2 \sum_{j_1}\sum_{j_2}{}^{*}\left(\frac{2\pi}{B_n n}\right)\sum_{j}{}^{**}\left[H\left(\frac{2\pi j}{B_n n}\right) K\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}\right)\right.$$

$$\left.\times f^z_{a_1a_2a_3a_4}\left(-\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi(j-j_2)}{n}\right) + O(n^{-1})\right], \tag{6.11}$$

where $\sum_{j_2}^{*}$ is the sum for j_2, satisfying

$$-\left[\frac{n}{2}\right]+1 \leqslant j_2 \leqslant \left[\frac{n}{2}\right] \qquad \text{and} \qquad -\left[\frac{n}{2}\right] \leqslant j-j_2 \leqslant \left[\frac{n}{2}\right],$$

and $\sum_{j}^{**}$ is the sum for j satisfying

$$-\left[\frac{B_n n}{2}\right]+1 \leqslant j \leqslant \left[\frac{B_n n}{2}\right].$$

We can see that (6.11) is equal to

$$\iint_{-\pi}^{\pi} K(\lambda_1, \lambda_2) f^{z}_{a_1a_2a_3a_4}(-\lambda_1, \lambda_2, -\lambda_2)\, d\lambda_1\, d\lambda_2 + o(1).$$

Similarly,

$$(6.9.2) = \frac{H(0)}{B_n} \iint_{-\pi}^{\pi} K(\lambda_1, \lambda_2) f_{a_1a_2}(-\lambda_1) f_{a_3a_4}(-\lambda_2)\, d\lambda_1\, d\lambda_2 + o(1),$$

$$\{(6.9.3)+(6.9.4)\} = \int_{-\pi}^{\pi} [K(\lambda, -\lambda) f_{a_1a_3}(-\lambda) f_{a_2a_4}(\lambda) + K(\lambda, \lambda) \times f_{a_1a_4}(-\lambda) f_{a_2a_3}(\lambda)]\, d\lambda + o(1). \tag{6.12}$$

In view of Theorem 1, (2.4.2) is a consistent estimator of (6.12). Also it is easily shown that

$$\lim_{n\to\infty} [E\{(2.4.3)\} + (6.9.2)] = 0,$$

which implies that U_n is an asymptotically unbiased estimator of (2.3). From (6.10), it is not difficult to show that

$$\lim_{n\to\infty} \mathrm{Var}\{(2.4.1)\} = 0, \qquad \lim_{n\to\infty} \mathrm{Var}\{(2.4.3)\} = 0,$$

which completes the proof. In our vector-valued case, the term (2.4.2) differs from the corresponding term of the scalar-valued case.

Proof of Proposition 2. Since the process $\{\mathbf{z}(t)\}$ is a linear process satisfying Assumptions 1 and 8, by using fundamental properties of the cumulant (e.g., [5]), we have

$$f^{z}_{rtus}(-\lambda_1, \lambda_2, -\lambda_2) = \left(\frac{1}{2\pi}\right)^3 \sum_{a,b,c,d=1}^{p} \kappa_{abcd} A_{ra}(-\lambda_1)\, A_{tb}(\lambda_1)\, A_{uc}(-\lambda_2)\, A_{sd}(\lambda_2).$$

From the definition of $v_2(f_4^z)$ we obtain

$$v_2(f_4^z) = \left(\frac{1}{2\pi}\right)^2 \sum_{a,b,c,d=1}^{p} \kappa_{abcd} \left\{ \sum_{r,t=1}^{p} \int_{-\pi}^{\pi} K_{rt}^{(1)}(\lambda_1) A_{ra}(-\lambda_1) A_{tb}(\lambda_1) \, d\lambda_1 \right\}$$
$$\times \left\{ \sum_{u,s=1}^{p} \int_{-\pi}^{\pi} K_{us}^{(1)}(\lambda_2) A_{uc}(-\lambda_2) A_{sd}(\lambda_2) \, d\lambda_2 \right\}. \quad (6.13)$$

From (3.2) it follows that

$$\sum_{r,t=1}^{p} \int_{-\pi}^{\pi} A_{ra}(-\lambda_1) K_{rt}^{(1)}(\lambda_1) A_{tb}(\lambda_1) \, d\lambda_1 = 0,$$

for all $a, b = 1, ..., p$, which implies that $(6.13) = 0$.

Proof of Theorem 3. (i) From Magnus and Neudecker [15] it is easy to see that

$$K_{\mathrm{LR}}^{(1)}\{f(\lambda)\} = 2\pi I_p - f(\lambda)^{-1}. \quad (6.14)$$

Since $f(\lambda) = (1/2\pi) A(\lambda) \Omega A(\lambda)^*$ with $A(\lambda) = \sum_{j=0}^{\infty} G(j) e^{i\lambda j}$, we have

$$\int_{-\pi}^{\pi} A^*(\lambda) K_{\mathrm{LR}}^{(1)}\{f(\lambda)\} A(\lambda) \, d\lambda = \int_{-\pi}^{\pi} A^*(\lambda)[2\pi I_p - f(\lambda)^{-1}] A(\lambda) \, d\lambda$$
$$= \int_{-\pi}^{\pi} 2\pi [A^*(\lambda) A(\lambda) - \Omega^{-1}] \, d\lambda. \quad (6.15)$$

By (3.7), we can show that $(6.15) = 0$. Hence, the assertion follows from Proposition 2.

(ii) Similarly, we obtain

$$K_{\alpha}^{(1)}\{f(\lambda)\} = \frac{\alpha}{4\pi} \left[\left\{ (1-\alpha) \frac{I_p}{2\pi} + \alpha f(\lambda) \right\}^{-1} - f(\lambda)^{-1} \right], \quad (6.16)$$

which shows that

$$\int_{-\pi}^{\pi} A^*(\lambda) K_{\alpha}^{(1)}\{f(\lambda)\} A(\lambda) \, d\lambda$$
$$= \frac{\alpha}{4\pi} \int_{-\pi}^{\pi} \left[\left\{ (1-\alpha) \frac{1}{2\pi} A(\lambda)^{-1} A^*(\lambda)^{-1} + \alpha \frac{\Omega}{2\pi} \right\}^{-1} - 2\pi \Omega^{-1} \right] d\lambda. \quad (6.17)$$

Evidently $(6.17) = 0$ if (3.8) is satisfied, whence the assertion follows from Proposition 2.

(iii) It is shown that

$$K_{\text{LOG}}^{(1)}\{f(\lambda)\} = 2 \log \det\{f(\lambda)\} f(\lambda)^{-1}. \tag{6.18}$$

Then

$$\int_{-\pi}^{\pi} A^*(\lambda) K_{\text{LOG}}^{(1)}\{f(\lambda)\} A(\lambda) \, d\lambda = 4\pi \int_{-\pi}^{\pi} \log \det\{f(\lambda)\} \Omega^{-1} \, d\lambda. \tag{6.19}$$

For a $p \times p$-matrix A, it is known that

$$\det\{\exp A\} = e^{\operatorname{tr} A}, \tag{6.20}$$

which implies that (6.19) is equal to

$$4\pi \int_{-\pi}^{\pi} \operatorname{tr} \left\{ \sum_{j \neq 0} A_j \cos(j\lambda) \right\} \Omega^{-1} \, d\lambda = 0.$$

Thus $T_n^{(\text{LOG})}$ is non-Gaussian robust.

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