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QED revisited: proving equivalence between path integral and stochastic quantization

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Abstract

We perform the stochastic quantization of scalar QED based on a generalization of the stochastic gauge fixing scheme and its geometric interpretation. It is shown that the stochastic quantization scheme exactly agrees with the usual path integral formulation.

1. Introduction

The stochastic quantization scheme of Parisi and Wu [1] has been applied to QED since many years. Nice agreement with conventional calculations was found in several explicit examples (for reviews see, e.g., [2,3]), a general equivalence proof so far was lacking.

The main idea of "stochastic quantization" is to view Euclidean quantum field theory as the equilibrium limit of a statistical system coupled to a thermal reservoir. This system evolves in a new additional time direction which is called stochastic time until it reaches the equilibrium limit for infinite stochastic time. In the equilibrium limit the stochastic averages become identical to ordinary Euclidean vacuum expectation values.

There are two equivalent formulations of stochastic quantization: in one formulation all fields have an additional dependence on stochastic time. Their stochastic time evolution is determined by a Langevin equation which has a drift term constructed from the gradient of the classical action of the system. The expectation values of observables are obtained by ensemble averages over the Wiener measure.

Equivalently one has a Fokker–Planck equation for the probability distribution characterizing the stochastic evolution of the system. Now expectation values of observables are defined in terms of functional integrals with respect to the stochastic time dependent Fokker–Planck probability distribution. The equilibrium limit of the probability distribution provides the Euclidean path integral density.

One of the most interesting aspects of this new quantization scheme lies in its rather unconventional treatment of gauge field theories, in specific of Yang–Mills theories. We recall that originally it was formulated by Parisi and Wu [1] without the introduction of gauge fixing terms and without the usual Faddeev–Popov ghost fields; later on a modified ap-

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proach named stochastic gauge fixing was given by Zwanziger [4]; further generalizations and a globally valid path integral were advocated in [5–8].

The main difficulty for providing an equivalence proof in the case of QED appears to be a rather non-trivial topological obstruction; all previous attempts failed in the past years to identify the standard—gauge fixed—QED action as a Fokker–Planck equilibrium distribution.

In this Letter we introduce new modifications of the original Parisi–Wu stochastic process of QED, yet keeping expectation values of gauge invariant observables unchanged: the modified stochastic process not only has damped flows along the vertical direction [4] but also is modeled on a specific manifold of gauge and matter fields with associated *flat* connection [5–8]. It is precisely in this case that the standard—gauge fixed—QED path integral density can indeed be identified with the equilibrium limit of the underlying Fokker–Planck probability distribution.

In Section 2 the geometrical setting for QED is introduced and the associated bundle structure of the space of gauge potentials and matter fields is summarized. We introduce adapted coordinates, the corresponding vielbeins and metrics.

The generalized stochastic process for QED is presented in Section 3, Section 4 is devoted to the derivation of the conventional QED path integral density as the equilibrium solution.

2. The geometrical setting of QED

In this section we present the major geometrical structures of QED. We collect in a somewhat formal style all the necessary ingredients which are needed for a compact and transparent formulation of the stochastic quantization scheme of QED.

2.1. Gauge fields

Let $P \to M$ be a principal $U(1)$-bundle over the n-dimensional boundaryless connected, simply connected and compact Euclidean manifold M. The photon fields are regarded as elements of the affine space $\mathcal{A}$ of all connections on P. The gauge group $\mathcal{G}$ is given by $\mathcal{G} = C^\infty(M; U(1))$ with Lie algebra $\operatorname{Lie}\mathcal{G} = C^\infty(M; u(1))$; here $\operatorname{Lie}(U(1)) = u(1) = i\mathbf{R}$. The action of $\mathcal{G}$ on $\mathcal{A}$ is defined by

$$A \to A^g = A + g^{-1}dg. \tag{2.1}$$

Let us define the subgroup $\mathcal{G}_0 \subset \mathcal{G}$ where $\mathcal{G}_0 = \mathcal{G}/U(1)$ denotes the group of all gauge transformations reduced by the constant ones. Since $\mathcal{G}_0$ acts freely on $\mathcal{A}$ we consider the principal $\mathcal{G}_0$ bundle $\hat{\pi}: \mathcal{A} \to \mathcal{M} = \mathcal{A}/\mathcal{G}$. One can prove that $\mathcal{A} \to \mathcal{M}$ is trivializable, a global section $\hat{\sigma}: \mathcal{M} \to \mathcal{A}$ being given by $\hat{\sigma}([A]) = A^{\omega(A)^{-1}}$. Here $\omega(A) \in \mathcal{G}_0$ is defined by

$$\omega(A) = \exp\bigl[\Delta^{-1} d^*(A - A_0)\bigr], \tag{2.2}$$

where Δ denotes the invertible Laplacian which is acting on the Lie algebra $\operatorname{Lie}\mathcal{G}_0$; $A_0 \in \mathcal{A}$ is a chosen fixed background connection. Note that $\omega(A)$ fulfills

$$\omega\bigl(A^g\bigr) = \omega(A)g \tag{2.3}$$

with $g \in \mathcal{G}_0$.

2.2. Matter fields

In order to discuss scalar matter fields ϕ we chose a representation ρ of $g \in \mathcal{G}_0$ on the vector space $V = \mathbf{C}$; $\rho(g)$ simply denotes multiplication with g. We consider the associated vector bundles $E = P \times_\rho V$ on M. Scalar fields are described by appropriately chosen sections of E. In the following we denote by $\mathcal{F}$ the space of scalar fields.

The action of $\mathcal{G}_0$ on $\Phi^i := (A, \phi) \in \mathcal{A} \times \mathcal{F}$ is given by

$$\begin{aligned} &\Phi^i = (A, \phi) \quad \longrightarrow \\ &\bigl(\Phi^g\bigr)^i = \bigl(A^g, \phi^g\bigr) = \bigl(A + g^{-1}dg, g^{-1}\phi\bigr). \end{aligned} \tag{2.4}$$

We remind that $\mathcal{A} \times \mathcal{F} \xrightarrow{\pi} \mathcal{A} \times_{\mathcal{G}_0} \mathcal{F}$ is trivializable iff $\mathcal{A} \xrightarrow{\hat{\pi}} \mathcal{M}$ is trivializable. Indeed, using the previous construction of $\omega(A)$ we obtain a global section $\sigma: \mathcal{A} \times_{\mathcal{G}_0} \mathcal{F} \to \mathcal{A} \times \mathcal{F}$ by defining

$$\begin{aligned} \sigma\bigl([A, \phi]\bigr) &= \bigl(\hat{\sigma}\bigl([A]\bigr), \phi^{\omega(A)^{-1}}\bigr) \\ &= \bigl(A^{\omega(A)^{-1}}, \phi^{\omega(A)^{-1}}\bigr). \end{aligned} \tag{2.5}$$

The tangent space of the configuration space $\mathcal{A} \times \mathcal{F}$ is given by $T_{(A,\phi)}(\mathcal{A} \times \mathcal{F}) = \Omega^1(M; i\mathbf{R}) \times \mathcal{F}$. Noting that $v_\phi \in \mathcal{F}$ by construction transforms equivariantly we obtain a $\mathcal{G}$-invariant Riemannian metric

$h: T(\mathcal{A}\times\mathcal{F})\times T(\mathcal{A}\times\mathcal{F})\to\mathbf{R}$ by defining

$$h_{(A,\phi)}\big((\tau_A^1, v_\phi^1), (\tau_A^2, v_\phi^2)\big) = \langle\tau_A^1, \tau_A^2\rangle + \langle v_\phi^1, v_\phi^2\rangle. \tag{2.6}$$

Here

$$\langle\alpha,\beta\rangle = \frac{1}{2}\int_M (\bar\alpha\wedge *\beta + \alpha\wedge *\bar\beta) \tag{2.7}$$

and $\alpha,\beta\in T\mathcal{A}$, or $T\mathcal{F}$, respectively; $*$ is the hodge operator on M, $\bar\alpha$ denotes complex conjugation of α.

2.3. Adapted coordinates

Let the globally defined gauge fixing surface Σ in $\mathcal{A}\times\mathcal{F}$ be defined by

$$\begin{aligned}\Sigma &= \operatorname{Im}\sigma \\ &= \big\{(B,\psi)\in\mathcal{A}\times\mathcal{F} \mid \\ &\qquad (B,\psi) = \big(A^{\omega(A)^{-1}}, \phi^{\omega(A)^{-1}}\big)\big\},\end{aligned} \tag{2.8}$$

where ω is given by (2.2). Note that B and ψ are invariant under the action of $\mathcal{G}_0$ which trivially follows from (2.4) and from (2.3); B satisfies the "gauge fixing condition"

$$d^*(B - A_0) = 0. \tag{2.9}$$

We define the adapted coordinates $\Psi^\mu = \{B, \psi, g\}$ via the bundle maps $\chi:\Sigma\times\mathcal{G}_0\to\mathcal{A}\times\mathcal{F}$ and $\chi^{-1}:\mathcal{A}\times\mathcal{F}\to\Sigma\times\mathcal{G}_0$, where

$$\chi(B,\psi,g) = \big(B^g, \psi^g\big) \tag{2.10}$$

and

$$\begin{aligned}\chi^{-1}(A,\phi) &= \big(\sigma\big([A,\phi]\big), \omega(A)\big) \\ &= \big(A^{\omega(A)^{-1}}, \phi^{\omega(A)^{-1}}, \omega(A)\big).\end{aligned} \tag{2.11}$$

The differentials $T\chi$ and $T\chi^{-1}$ are calculated straightforwardly (compare also with [7])

$$\begin{aligned}&T\chi(\zeta_B, v_\psi, Y_g) \\ &\quad = \big(\zeta_B + d\theta_g(Y_g), g^{-1}\big(v_\psi - \theta_g(Y_g)\Phi\big)\big),\end{aligned} \tag{2.12}$$

as well as

$$\begin{aligned}&T\chi^{-1}(\tau_A, v_\phi) \\ &\quad = \big(\mathbf{P}\tau_A, \omega(A)\big(v_\phi + \big(\Delta^{-1}d^*\tau_A\big)\phi\big), \\ &\qquad \omega(A)\Delta^{-1}d^*\tau_A\big).\end{aligned} \tag{2.13}$$

Here $(\zeta_B, v_\psi)\in T_B\Sigma$, $Y_g\in T_g\mathcal{G}$ and $(\tau_A, v_\phi)\in T_{(A,\phi)}(\mathcal{A}\times\mathcal{F})$. The Maurer–Cartan form on $\mathcal{G}_0$ is denoted by θ, $\mathbf{P}$ is the transversal projector $\mathbf{P} = \mathbf{1} - d\Delta^{-1}d^*$. From the differentials $T\chi$ and $T\chi^{-1}$ we read off the vielbeins $e^i{}_\mu = \delta\Phi^i/\delta\Psi^\mu$ and their inverses $E^\mu{}_i = \delta\Psi^\mu/\delta\Phi^i$ corresponding to the change of variables $\Psi^\mu = \{B,\psi,g\}\leftrightarrow\Phi^i = \{A,\phi\}$.

Above we defined a Riemannian structure on $\mathcal{A}\times\mathcal{F}$ in terms of the $\mathcal{G}_0$ invariant metric h. In adapted coordinates this metric is given now as the pullback $G = \chi^* h$; explicitly we obtain

$$\begin{aligned}&G_{(B,\psi,g)}\big((\zeta_B^1, v_\psi^1, Y_g^1), (\zeta_B^2, v_\psi^2, Y_g^2)\big) \\ &\quad = \langle\zeta_B^1, \mathbf{P}\zeta_B^2\rangle + \langle\theta_g(Y_g^1), (\Delta + |\psi|^2)\theta_g(Y_g^2)\rangle \\ &\qquad + \langle v_\psi^1, v_\psi^2\rangle - \langle\theta_g(Y_g^1)\psi, v_\psi^2\rangle - \langle v_\psi^1, \theta_g(Y_g^2)\psi\rangle.\end{aligned} \tag{2.14}$$

Here $(\zeta_B^1, v_\psi^1), (\zeta_B^2, v_\psi^2)\in T_{(B,\psi)}\Sigma$ and $Y_g^1, Y_g^2\in T_g\mathcal{G}$. In matrix notation we have $G = e^\dagger e$ and $G^{-1} = EE^\dagger$. The determinant of G is given by $\det G = \Delta$.

3. Parisi–Wu stochastic quantization

For scalar QED we have

$$S_{\mathrm{inv}} = \langle D_A\varphi, D_A\varphi\rangle + m^2\langle\varphi,\varphi\rangle + \frac{1}{2}\langle F, F\rangle, \tag{3.1}$$

where $D_A\varphi = (d - A)\varphi$ and $F = dA$. The Parisi–Wu Langevin equations are given by

$$d\mathcal{A} = -\frac{\delta S_{\mathrm{inv}}}{\delta\mathcal{A}}ds + dU, \tag{3.2}$$

$$d\varphi = -\frac{\delta S_{\mathrm{inv}}}{\delta\bar\varphi}ds + dV, \tag{3.3}$$

where the Wiener increments fulfill

$$dU\,dU = 2\,ds, \qquad dV\,d\bar V = 2\,ds. \tag{3.4}$$

This can be summarized by

$$d\Phi = -\frac{\delta S_{\mathrm{inv}}}{\delta\Phi}ds + d\xi, \quad d\xi\,d\xi^\dagger = 2\cdot\mathbf{1}\,ds. \tag{3.5}$$

Using Ito calculus [9,10] we transform the Parisi–Wu Langevin equations into adapted coordinates

$$d\Psi = \left[-G^{-1}\frac{\delta S_{\mathrm{inv}}}{\delta\Psi} + \frac{\delta G^{-1}}{\delta\Psi}\right]ds + d\zeta, \tag{3.6}$$

where

$$d\zeta\, d\zeta^{\dagger} = 2G^{-1}\, ds. \tag{3.7}$$

The use of adapted coordinates allows to disentangle the complicated dynamics of gauge independent and gauge dependent degrees of freedom; it will be of great value later on. For completeness we note the Fokker–Planck operator in the original variables

$$L[\Phi] = \frac{\delta}{\delta\Phi}\left[\frac{\delta S_{\text{inv}}}{\delta\Phi} + \frac{\delta}{\delta\Phi}\right], \tag{3.8}$$

as well as in the adapted coordinates

$$L[\Psi] = \frac{\delta}{\delta\Psi}G^{-1}\left[\frac{\delta S_{\text{inv}}}{\delta\Psi} + \frac{\delta}{\delta\Psi}\right]. \tag{3.9}$$

We remark that in the case of the Parisi–Wu processes diffusion along the vertical direction takes place and no equilibrium distribution is approached. Thus a Fokker–Planck formulation of the Parisi–Wu stochastic quantization scheme is impossible: the gauge invariance of the action S_{inv} is leading to divergencies along the vertical directions when trying to normalize the Fokker–Planck density.

4. Generalized stochastic quantization

4.1. Geometric obstruction

Our equivalence proof relies on specific allowed modifications of the metric on the field space, which governs the stochastic process. These modifications correspondingly are implying changes of the associated Fokker–Planck operator. We are going to show that this can be achieved in such a way that the resulting Fokker–Planck operator has a positive kernel and is annihilated on its *right* by the standard gauge fixed QED path integral density. In order for this to be the case, however, a certain integrability condition for the drift term of the considered stochastic process has to be fulfilled. Surprisingly similar as in the pure Yang–Mills theory also in the Abelian QED case there appears a violation of this condition; it is only after a nontrivial modification of the underlying stochastic processes (see next subsection) that this obstruction can be overcome.

Proceeding step by step we first note (see Zwanziger [4]) that a damping force along the gauge orbit has to be introduced in order to maintain the probabilistic interpretation of the Fokker–Planck formulation. Although knowing that this additional force will not alter expectation values of gauge invariant quantities it is disappointing to observe that due to its presence the standard—gauge fixed—QED action will never annihilate the Fokker–Planck operator on its right side due to the following reason: we recall that the bundle metric $h_{(A,\phi)}$ on the associated fiber bundle $\mathcal{A}\times\mathcal{F}\to\mathcal{A}\times_{\mathcal{G}}\mathcal{F}$ which is invariant under the corresponding group action gives rise to a natural connection γ, whose horizontal subbundle $\mathcal{H}$ is orthogonal to the corresponding group. The horizontal bundle $\mathcal{H}[\mathcal{A}\times\mathcal{F};\gamma]$ with respect to γ is defined by $\mathcal{H}[\mathcal{A}\times\mathcal{F};\gamma]\perp_h V(\mathcal{A}\times\mathcal{F})$, where the orthogonality is with respect to $h_{(A,\phi)}$. Elements of the vertical bundle $V(\mathcal{A},\mathcal{F})\to\mathcal{A}\times_{\mathcal{G}}\mathcal{F}$ are given in the form

$$\begin{aligned} Z_\xi(A,\phi) &= \frac{d}{dt}\bigg|_{t=0}\big(A^{\exp t\xi}, \phi^{\exp t\xi}\big) \\ &= (d\xi, -\xi\phi), \end{aligned} \tag{4.1}$$

where $\xi\in C^\infty(M;i\mathbf{R})$. The orthogonal span with respect to the vertical bundle fulfills

$$\begin{aligned} &d^*\tau_A + \frac{1}{2}(\bar{v}_\phi\phi - v_\phi\bar{\phi}) = 0, \\ &\tau_A\in T_A\mathcal{A},\ v_\phi\in T_\phi\mathcal{F} \end{aligned} \tag{4.2}$$

which follows from

$$\begin{aligned} 0 &= h_{(A,\phi)}\big((\tau_A, v_\phi), Z_\xi(A,\phi)\big) \\ &= h_{(A,\phi)}\big((\tau_A, v_\phi), (d\xi, -\xi\phi)\big) \\ &= \langle\tau_A, d\xi\rangle - \langle v_\phi, \xi\phi\rangle. \end{aligned} \tag{4.3}$$

Explicitly we can prove that $\gamma_{(A,\phi)}(\tau_A, v_\phi)\in C^\infty(P;i\mathbf{R})$

$$\begin{aligned} \gamma_{(A,\phi)}(\tau_A, v_\phi) &= \big(\Delta + |\phi|^2\big)^{-1} \\ &\quad\times\left[d^*\tau_A + \frac{1}{2}(\bar{v}_\phi\phi - v_\phi\bar{\phi})\right] \end{aligned} \tag{4.4}$$

defines a connection induced by $h_{(A,\phi)}$ in the principal bundle $\mathcal{A}\times\mathcal{F}\to\mathcal{A}\times_{\mathcal{G}}\mathcal{F}$ and is $U(1)$ invariant. Calculating its curvature $\Omega((\tau^1, v^1), (\tau^2, v^2))$ we find that it is nonvanishing and given by

$$\begin{aligned} \Omega &= \big(\Delta + |\phi|^2\big)^{-1}\big(v^2\bar{\phi} + \bar{v}^2\phi\big)\big(\Delta + |\phi|^2\big)^{-1} \\ &\quad\times\left[d^*\tau_A^1 + \frac{1}{2}\big(\bar{v}^1\phi - v^1\bar{\phi}\big)\right] \end{aligned}$$

$$- \left(\Delta + |\phi|^2\right)^{-1}\left(v^1\bar{\phi} + \bar{v}^1\phi\right)\left(\Delta + |\phi|^2\right)^{-1}$$
$$+ \left[d^*\tau_A^2 + \frac{1}{2}\left(\bar{v}^2\phi - v^2\bar{\phi}\right)\right]$$
$$\times \left(\Delta + |\phi|^2\right)^{-1}\left(v^1\bar{v}^2 - \bar{v}^1 v^2\right). \tag{4.5}$$

As a consequence [5–8] there does not exist (even locally) a manifold whose tangent bundle is isomorphic to this horizontal subbundle. Specifically this implies that any vector field along the gauge group cannot be written as a gradient with respect to the metric $h_{(A,\phi)}$. The total drift term—containing the extra vertical force term—thus can never arise as derivative of the standard gauge fixed QED action; the Fokker–Planck operator can never be annihilated on its right by the standard QED path integral density; an equivalence proof presently cannot be given.

4.2. *The induced field metric with flat connection*

The crucial observation in [5–8] is to consider a larger class of modified stochastic processes than considered so far, yet always keeping expectation values of gauge invariant observables unchanged: one introduces not only the extra vertical drift terms as discussed above but one also modifies the Wiener increments by specific extra terms and introduces extra so-called Ito-terms, correspondingly.

The idea is to view the new terms multiplying the Wiener increments as vielbeins giving rise to the inverse of a yet not specified metric on the space $\mathcal{A} \times \mathcal{F}$. The appearance of this metric induces a specific connection with a potentially analogous obstruction as discussed above. A necessary requirement to overcome this obstruction is therefore that the corresponding curvature has to vanish. The question how to find such a metric is reduced to the question how to find a flat connection.

Indeed, there exists a flat connection $\tilde{\gamma}$ in our bundle. This connection is the pull-back of the Maurer–Cartan form θ on $\mathcal{G}_0$ via the global trivialization χ^{-1} and $\mathrm{pr}_{\mathcal{G}}$

$$\tilde{\gamma} = \left(\chi^{-1*}\,\mathrm{pr}_{\mathcal{G}}^*\,\theta\right)(A,\phi) = \Delta^{-1}d^*, \tag{4.6}$$

where $\mathrm{pr}_{\mathcal{G}}^*$ is the projector $\Sigma \times \mathcal{G} \to \mathcal{G}$. The projector onto the horizontal subbundle $\tilde{\mathcal{H}}[\mathcal{A} \times \mathcal{F}; \tilde{\gamma}]$ with respect to $\tilde{\gamma}$ is given by

$$\tilde{\mathbf{P}} = \mathbf{1} - D_A\tilde{\gamma}. \tag{4.7}$$

We see that the horizontal subbundle $\tilde{\mathcal{H}}$ is orthogonal to the gauge orbits with respect to the induced field metric; in particular the gauge fixing surface is then orthogonal to the gauge orbits.

In the adapted coordinates the induced field metric is denoted by $\tilde{G} = \tilde{e}^\dagger\,\tilde{e}$. The just discussed orthogonality condition of the gauge fixing surface and the gauge orbit with respect to the induced field metric is transformed into simply

$$\left(\tilde{G}^{-1}\right)^{\Sigma\mathcal{G}} = \left(\tilde{G}^{-1}\right)^{\mathcal{G}\Sigma} = 0, \quad \text{where}$$
$$\tilde{G}^{-1} = \tilde{E}\,\tilde{E}^\dagger \quad \text{with } \tilde{E}\,\tilde{e} = \mathbf{1}. \tag{4.8}$$

This condition is fulfilled provided $\tilde{E}$ is defined as

$$\tilde{E} = \begin{pmatrix} E^\Sigma \\ e_{\mathcal{G}}^\dagger \end{pmatrix}. \tag{4.9}$$

To complete our discussion we also have to specify the vertical drift term; it is related to the gradient of $S_{\mathcal{G}}$, where we chose

$$S_{\mathcal{G}}[g] = \frac{1}{2\lambda}\left\langle d^*g^*\theta^{U(1)}, d^*g^*\theta^{U(1)}\right\rangle, \tag{4.10}$$

where λ is a positive constant and where $\theta^{U(1)}$ is the Maurer–Cartan form on $U(1)$. Note that in the original variables we obtain the standard background-gauge fixing term

$$\left(\chi^{-1*}\,\mathrm{pr}_{\mathcal{G}}^*\,S_{\mathcal{G}}\right)(A,\phi)$$
$$= S_{\mathcal{G}}\left(\omega(A)\right) = \frac{1}{2\lambda}\left\langle d^*(A - A_0), d^*(A - A_0)\right\rangle. \tag{4.11}$$

Summarizing we have

$$d\Psi = \left[-\tilde{G}\frac{\delta S_{\mathrm{tot}}}{\delta\Psi} + \frac{\delta\tilde{G}}{\delta\Psi}\right]ds + \tilde{\zeta}, \tag{4.12}$$

where

$$S_{\mathrm{tot}} = S_{\mathrm{inv}} + S_{\mathcal{G}} \quad \text{and} \quad d\tilde{\zeta}\,d\tilde{\zeta}^\dagger = 2\tilde{G}^{-1}\,ds. \tag{4.13}$$

4.3. *The equivalence proof*

It is easy now to prove for QED the equivalence of the stochastic quantization scheme with the path integral quantization. For the formulation in terms of the adapted coordinates $\Psi = \{B, \psi, g\}$ the associated Fokker–Planck equation is derived in straightforward manner

$$\frac{\partial\rho[\Psi, s]}{\partial s} = L[\Psi]\,\rho[\Psi, s], \tag{4.14}$$

where the Fokker–Planck operator $L[\Psi]$ is appearing in just factorized form

$$L[\Psi] = \frac{\delta}{\delta\Psi}\tilde{G}^{-1}\left[\frac{\delta S_{\mathrm{tot}}[\Psi]}{\delta\Psi} + \frac{\delta}{\delta\Psi}\right]. \tag{4.15}$$

Due to the positivity of $\tilde{G}$ the fluctuation dissipation theorem applies and the equilibrium Fokker–Planck distribution $\rho^{\mathrm{eq}}[\Psi]$ obtains by direct inspection as

$$\rho^{\mathrm{eq}}[\Psi] = \frac{e^{-S_{\mathrm{tot}}[B,\psi,g]}}{\int_{\Sigma\times\mathcal{G}_0}\mathcal{D}B\,\mathcal{D}\psi\,\mathcal{D}g\,e^{-S_{\mathrm{tot}}[B,\psi,g]}} = \frac{e^{-S_{\mathrm{inv}}[B,\psi]}\,e^{-S_{\mathcal{G}}[g]}}{\int_{\Sigma}\mathcal{D}B\,\mathcal{D}\psi\,e^{-S_{\mathrm{inv}}[B,\psi]}\int_{\mathcal{G}_0}\mathcal{D}g\,e^{-S_{\mathcal{G}}[g]}}. \tag{4.16}$$

This result is completely equivalent to the standard background-gauge fixed QED path integral prescription. The additional *finite* contributions of the gauge degrees of freedom always cancel out when evaluated on gauge invariant observables.

Similarly, in terms of the original variables $\Phi = \{A,\phi\}$ the Fokker–Planck equilibrium distribution is given by the standard background-gauge fixed path integral density

$$\rho^{\mathrm{eq}}[\Phi] = \frac{e^{-S_{\mathrm{tot}}[A,\phi]}}{\int_{\mathcal{A}}\mathcal{D}A\,\mathcal{D}\phi\,e^{-S_{\mathrm{tot}}[A,\phi]}} = \frac{e^{-S_{\mathrm{inv}}[A,\phi]-S_{\mathcal{G}}(\omega(A))}}{\int_{\mathcal{A}}\mathcal{D}A\,\mathcal{D}\phi\,e^{-S_{\mathrm{inv}}[A,\phi]-S_{\mathcal{G}}(\omega(A))}}. \tag{4.17}$$

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