Error inequalities for a generalized trapezoid rule

Nenad Ujević

Department of Mathematics, University of Split, Teslina 12/III, 21000 Split, Croatia

Received 17 September 2004; received in revised form 23 March 2005; accepted 24 March 2005

Abstract

A generalized trapezoid rule is derived. Various error bounds for this rule are established.

MSC: 26D15; 65D30

Keywords: Quadrature rule; Generalization; Numerical integration; Error bounds

1. Introduction

In recent years a number of authors have considered error inequalities for some known and some new quadrature rules. Sometimes they have considered generalizations of these rules. For example, the well-known trapezoid quadrature rule is considered in [1–6] and some generalizations are given in [1] and [3]. In [1] we can find

\[ \int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] + \frac{(-1)^n}{n!} \int_a^b \left( t - \frac{a+b}{2} \right)^n f^{(n)}(t)\,dt. \]

For \( n = 1 \) we get the trapezoid rule,

\[ \int_a^b f(t)\,dt = \frac{b-a}{2} [f(a) + f(b)] - \int_a^b \left( t - \frac{a+b}{2} \right) f'(t)\,dt. \]

E-mail address: ujevic@pmfst.hr.
In this paper we consider a different generalization of the trapezoid rule and give corresponding error inequalities. Similar error inequalities are established in [5] and [7]. We also give a numerical example which shows that this generalization can be very effective.

2. Main results

Lemma 1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a function such that \( f^{(n-1)} \) is absolutely continuous. Then

\[
\int_a^b f(x) \, dx = \frac{f(a) + f(b)}{2} (b - a) - \sum_{i=1}^{m} \frac{2i(b-a)^{2i+1}}{2^i(2i+1)!} f^{(2i)} \left( \frac{a+b}{2} \right) + R(f),
\]

where \( m = \lfloor \frac{n-1}{2} \rfloor \), the integer part of \( (n-1)/2 \),

\[
R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t) \, dt
\]

and

\[
S_n(t) = \begin{cases} 
\frac{(t-a)^{n-1}}{n!} \left[ t + \frac{(n-2)a - nb}{2} \right], & t \in [a, \frac{a+b}{2}] \\
\frac{(t-b)^{n-1}}{n!} \left[ t + \frac{(n-2)b - na}{2} \right], & t \in \left( \frac{a+b}{2}, b \right]
\end{cases}
\]

Proof. We briefly sketch the proof. First we note that

\[
S_1(t) = t - \frac{a+b}{2},
S_2(t) = \frac{1}{2} (t - a)(t - b)
\]

are Peano kernels for the trapezoid quadrature rule, that is, we have

\[
\int_a^b S_2(t) f''(t) \, dt = - \int_a^b S_1(t) f'(t) \, dt = - \frac{f(a) + f(b)}{2} (b - a) + \int_a^b f(t) \, dt.
\]

Now it is not difficult to prove that (1) holds, for example, by induction. \( \square \)

Remark 2. If we introduce the notations

\[
P_n(t) = \frac{(t-a)^{n-1}}{n!} \left[ t + \frac{(n-2)a - nb}{2} \right],
Q_n(t) = \frac{(t-b)^{n-1}}{n!} \left[ t + \frac{(n-2)b - na}{2} \right]
\]

then we see that \( P_n \) and \( Q_n \) form Appell sequences of polynomials, that is

\[
P'_n(t) = P_{n-1}(t), \quad Q'_n(t) = Q_{n-1}(t), \quad P_0(t) = Q_0(t) = 1.
\]

Thus we can also use integration by parts to prove that (1) holds.
Lemma 3. The Peano kernels $S_n(t)$, $n > 1$, satisfy

\[
\int_a^b S_n(t) \, dt = 0, \quad \text{if } n \text{ is odd},
\]

\[
\int_a^b |S_n(t)| \, dt = \frac{n(b - a)^{n+1}}{2^n (n+1)!},
\]

\[
\max_{t \in [a,b]} |S_n(t)| = \frac{(n - 1)(b - a)^n}{2^n n!}.
\]

Proof. A simple calculation gives

\[
\int_a^b S_n(t) \, dt = -\frac{(b - a)^{n+1}}{2^n (n+1)!} \left[ 1 - (-1)^{n+1} \right].
\]

From the above relation we see that (6) holds, since $1 - (-1)^{n+1} = 0$ if $n$ is odd. We have

\[
\int_a^b |S_n(t)| \, dt = \int_a^\frac{a+b}{2} |P_n(t)| \, dt + \int_\frac{a+b}{2}^b |Q_n(t)| \, dt
\]

\[
= \frac{n(b - a)^{n+1}}{2^n (n+1)!}.
\]

Finally, we have

\[
\max_{t \in [a,b]} |S_n(t)| = \max \left\{ \max_{t \in [a, \frac{a+b}{2}]} |P_n(t)|, \max_{t \in [\frac{a+b}{2}, b]} |Q_n(t)| \right\}
\]

\[
= \max \left\{ \left| P_n \left( \frac{a + b}{2} \right) \right|, \left| Q_n \left( \frac{a + b}{2} \right) \right| \right\}
\]

\[
= \frac{(n - 1)(b - a)^n}{2^n n!}.
\]

We introduce the notations

\[
I = \int_a^b f(t) \, dt,
\]

\[
F = \frac{f(a) + f(b)}{2} (b - a) - \sum_{i=1}^n \frac{2i(b - a) 2^{i+1}}{2^{2i} (2i+1)!} f^{(2i)} \left( \frac{a + b}{2} \right).
\]

Theorem 4. Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$, $n > 1$, is absolutely continuous and there exist real numbers $\gamma_n$, $\Gamma_n$ such that $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$, $t \in [a, b]$. Then

\[
|I - F| \leq \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{n}{2^n} (b - a)^{n+1} \quad \text{if } n \text{ is odd}
\]

and

\[
|I - F| \leq \frac{(b - a)^{n+1} n}{2^n (n+1)!} \| f^{(n)} \|_{\infty} \quad \text{if } n \text{ is even}.
\]
Proof. Let \( n \) be odd. From (2) and (6) we get
\[
R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t) \, dt = (-1)^n \int_a^b S_n(t) \left[ f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right] \, dt
\]
such that we have
\[
|R(f)| = |I - F| \leq \max_{t \in [a, b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^b |S_n(t)| \, dt.
\] (11)
We also have
\[
\max_{t \in [a, b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \leq \frac{\Gamma_n - \gamma_n}{2}.
\] (12)

From (11), (12) and (7) we get
\[
|I - F| \leq \frac{\Gamma_n - \gamma_n}{(n + 1)!} \frac{n}{2^{n+1}} (b - a)^{n+1}.
\]
Let \( n \) be even. Then we have
\[
|R(f)| = |I - F| \leq \int_a^b |S_n(t)| \, dt \, f^{(n)} \| f^{(n)} \|_\infty = \frac{(b - a)^{n+1} n}{2^n (n+1)!} \| f^{(n)} \|_\infty.
\]

Theorem 5. Let \( f : [a, b] \to R \) be a function such that \( f^{(n-1)}, n > 1, \) is absolutely continuous and let \( n \) be odd. If there exists a real number \( \gamma_n \) such that \( \gamma_n \leq f^{(n)}(t), t \in [a, b] \) then
\[
|I - F| \leq (T_n - \gamma_n) \frac{(n - 1)(b - a)^n}{2^n n!}.
\] (13)
where
\[
T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a}.
\]

If there exists a real number \( \Gamma_n \) such that \( f^{(n)}(t) \leq \Gamma_n, t \in [a, b] \) then
\[
|I - F| \leq (\Gamma_n - T_n) \frac{(n - 1)(b - a)^n}{2^n n!}.
\] (14)

Proof. We have
\[
|R(f)| = |I - F| = \left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) \, dt \right|
\]
since (6) holds. Then we have
\[
\left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) \, dt \right| \leq \max_{t \in [a, b]} |S_n(t)| \int_a^b (f^{(n)}(t) - \gamma_n) \, dt
\]
\[
= \frac{(n - 1)(b - a)^n}{2^n n!} \left[ f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n (b - a) \right]
\]
\[
= \frac{(n - 1)(b - a)^n}{2^n n!} (T_n - \gamma_n).
\]
In a similar way we can prove that (14) holds. □
Remark 6. Note that we can apply the estimations (9) and (10) only if $f^{(n)}$ is bounded. On the other hand, we can apply the estimation (13) if $f^{(n)}$ is unbounded above and we can apply the estimation (14) if $f^{(n)}$ is unbounded below.

3. A numerical example

Here we consider the integral (special function) $\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt$ and apply the summation formula (1) to this integral. We get the summation formula $\text{Si}(x) = F(x) + R(x)$, where

$$ F(x) = \frac{x}{2} \left( 1 + \frac{\sin x}{x} \right) - \sum_{i=1}^{m} \frac{2i^2 (i+1)!}{2^{2i}(2i+1)!} f^{(2i)} \left( \frac{X}{2} \right) $$

and $f(t) = (\sin t)/t$. We calculate the derivatives $f^{(j)}(t)$ as follows. We have

$$(g(t)h(t))^{(j)} = \sum_{k=0}^{j} \left( \begin{array}{c} j \\ k \end{array} \right) g^{(k)}(t)h^{(j-k)}(t).$$

If we choose $g(t) = \sin t$ and $h(t) = 1/t$ then we get

$$f^{(j)} \left( \frac{x}{2} \right) = \sum_{i=0}^{\left\lfloor \frac{j}{2i} \right\rfloor} \left( \begin{array}{c} j \\ 2i \end{array} \right) (-1)^{j-2i+1} \frac{(j-2i-1)!2^{j-2i}}{x^{j-2i+1}} \cos \frac{x}{2}$$

$$+ \sum_{i=0}^{\left\lfloor \frac{j-1}{2i} \right\rfloor} \left( \begin{array}{c} j \\ 2i \end{array} \right) (-1)^{j-i} \frac{(j-2i)!2^{j-2i+1}}{x^{j-2i+1}} \sin \frac{x}{2},$$

We now compare the summation formula (15) with the known compound formula (for the trapezoid rule),

$$\int_0^x f(t) \, dt = \frac{h}{2} [f(0) + f(x)] + h \sum_{i=1}^{n-1} f(x_i) + R(x),$$

where $x_i = ih$, $h = x/n$.

Let us choose $x = 1$. The “exact” value is $\text{Si}(1) = 0.946083070367$.

If we choose $m = 2$ in (15) and $n = 100$ in (16) then we get $\text{Si}(1) \approx 0.946080675618$ and

$\text{Si}(1) \approx 0.946080560625$, respectively.

If we choose $m = 3$ in (15) and $n = 8200$ in (16) then we get $\text{Si}(1) \approx 0.946083078954$ and

$\text{Si}(1) \approx 0.946083069999$, respectively.

If we choose $m = 4$ in (15) and $n = 32000$ in (16) then we get $\text{Si}(1) \approx 0.946083070347$ and

$\text{Si}(1) \approx 0.946083070342$, respectively.

All calculations are done in double precision arithmetic. The first approximate results (derived from (15)) are obtained much faster than the second approximate results (derived from (16)). The same is valid if we use some quadrature rule of higher order, for example Simpson’s rule. This is a consequence of the fact that we have to calculate the function $\sin t$ many times when we apply the compound formula and we have only to calculate $\cos(x/2)$ and $\cos(x/2)$ when we apply the summation formula.

Similar summation formulas can be obtained for the integrals (special functions) $\int_0^x (e^t - 1)/t \, dt$, $\int_0^x [(\cos t - 1)/t] \, dt$, $\int_0^x \exp(-t^2) \, dt$, etc.
References