Higher order topological derivatives in elasticity

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The topological derivative provides the variation of a response functional when an infinitesimal hole of a particular shape is introduced into the domain. In this work, we compute higher order topological derivatives for elasticity problems, so that we are able to obtain better estimates of the response when holes of finite sizes are introduced in the domain. A critical element of our algorithm involves the asymptotic approximation for the stress on the hole boundary when the hole size approaches zero; it consists of a composite expansion that is based on the responses of elasticity problems on the domain without the hole and on a domain consisting of a hole in an infinite space. We present a simple example in which the higher order topological derivatives of the total potential energy are obtained analytically and by using the proposed asymptotic expansion. We also use the finite element method to verify the topological asymptotic expansion when the analytical solution is unknown.

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1. Introduction

Classically known as topological derivative, the first order topological derivative field indicates the variation of a response functional when an infinitesimal hole of radius \( r \) centered at location \( x \) is introduced in the body (Sokołowski and Zochowski, 1999). It originally found applications in the structural optimization community in the so-called bubble method (Eschenauer et al., 1994). In this method, holes are systematically nucleated in strategic locations to both lighten the structure and maintain load integrity. Once the holes are nucleated, traditional shape optimization methods enlarge and reconfigure them. This concept has more recently been combined with the fictitious domain finite element method to alleviate remeshing tasks that plague traditional shape optimization (Céa et al., 2000; Allaire et al., 2004; Mei and Wang, 2004). For example, Norato et al. (2007) combined the topological derivative with an implicit geometric modeler to percolate holes and move the boundary to obtain the optimal shape and topology. In Novotny et al. (2003), an alternative topological derivative computation is proposed that is based on shape sensitivity analysis, the so-called topological-shape sensitivity method; it is used to solve design problems in steady-state heat conduction. In other studies, the topological-shape sensitivity method was applied to calculate the topological derivative in elasticity problems (Novotny et al., 2007). The topological derivative has also been applied to inverse scattering problems. For example, in Feijoo (2004) it is used locate the boundaries of impenetrable scatters immersed in an otherwise homogeneous medium. Guzina and Bonnet (2004) similarly solve an inverse problem using the topological derivative to identify the locations of cavities embedded in an elastic solid. Likewise, the topological derivative is applied to detect and locate cracks in an inverse heat conduction problem (Amstutz et al., 2005) and to resolve inpainting problems, i.e., to identify the edges of a partially hidden image (Auroux and Masmoudi, 2006).

The first order topological asymptotic expansion, i.e., the expansion that includes the first order topological derivative, gives good estimates for the response functional when infinitesimal holes are introduced in the domain. However, to obtain estimates corresponding to the insertion of finite size holes, one should use higher order terms in the expansion. In Rocha de Faria et al. (2007), the topological-shape sensitivity method was extended to obtain the second order topological derivative for the total potential energy associated with the Laplace equation in two-dimensional problems with different types of boundary conditions. Unfortunately, their calculation disregarded some higher order terms, leading to a discrepancy in the second order topological derivative as pointed out by Bonnet (2007). However, despite disregarding those terms, Rocha de Faria et al. (2008) argued that in some cases their proposed “incomplete” second order topological asymptotic expansion provides a better estimate for the total potential energy than the first order topological asymptotic expansion. To clarify this inconsistency, the complete second order topological asymptotic expansion for the Laplace problem was presented in Rocha de Faria and Novotny (2009) along with their incomplete expansion showing, by means of numerical examples, that the difference is indeed small. Similar higher order topological derivatives were also
described in the context of two-dimensional potential problems by Bonnet (2009).

In this work, we utilize the topological-shape sensitivity method to obtain higher order topological derivatives for two-dimensional linear elasticity problems of homogeneous isotropic materials (for nonlinear examples, such as in contact problems, the reader may refer to Khudnev et al. (2009) and Sokolowski and Zochowski (2005)). Therefore, we must evaluate the shape sensitivity of an existing hole with respect to the hole radius. It has been shown that this sensitivity depends on the stress evaluated on the hole boundary (Haug et al., 1986). To obtain the higher order topological derivatives, we propose an algorithm to obtain an asymptotic expansion for the stress as the hole size approaches zero; it is based on the responses of elasticity problems on the domain without the hole and on a domain consisting of a hole in an infinite space (Kozlov et al., 1999). Without loss of generality, we limit our discussion to a single response functional, the total potential energy.

The remainder of this paper discusses the evaluation of higher order topological derivatives (Section 2) and the asymptotic analysis (Sections 3 and 4). An analytical example is presented in Section 5 and conclusions are drawn in Section 6. For completeness, we provide in the appendix details of the analytical solution for the infinite domain problem.

2. Topological derivative

We consider a domain $\Omega$ with boundary $\partial \Omega$ and outward normal vector $n$. When a small hole of radius $\epsilon$ is introduced with center at location $\hat{x}$, we denote the perturbed domain $\Omega_\epsilon(\hat{x})$ which has boundary $\partial \Omega_\epsilon(\hat{x}) = \partial \Omega \cup \partial B_\epsilon(\hat{x})$ where $\partial B_\epsilon(\hat{x})$ is the hole boundary (cf. Fig. 1).

The variation of a bounded response functional $\Psi$ due to this perturbation is expressed by the following topological asymptotic expansion:

$$
\Psi(\Omega_\epsilon(\hat{x})) = \Psi(\Omega) + \sum_{j=1}^n f_j(\epsilon) D_j^\epsilon \Psi(\hat{x}) + R(f_n(\epsilon)),
$$

(1)

where $D_j^\epsilon \Psi$ is the nonzero $j$th order topological derivative of $\Psi$ cf. Sokolowski and Zochowski (2001) and Nazarov and Sokolowski (2003) for the $n = 1$ case and Rocha de Faria (2007) for the $n > 1$ case. The gauge functions $f_j$ depend on the hole boundary conditions; they are functions of the hole size $\epsilon$, positive valued and monotonically tend to zero as $\epsilon$ tends to zero. These functions also satisfy

$$
\lim_{\epsilon \to 0} \frac{f_j(\epsilon)}{f_j(\epsilon)} = 0, \quad k > j \quad \text{and} \quad \lim_{\epsilon \to 0} \frac{R(f_n(\epsilon))}{f_n(\epsilon)} = 0,
$$

(2)

where $R$ is the remainder function. Here we denote

$$
\Psi^{(n)} \epsilon := \Psi(\Omega_\epsilon(\hat{x})) = \Psi(\Omega) + \sum_{j=1}^n f_j(\epsilon) D_j^\epsilon \Psi(\hat{x}),
$$

(3)

the $n$th order topological asymptotic expansion; it is an approximation to $\Psi(\Omega_\epsilon(\hat{x}))$ as $\epsilon \to 0$ which is accurate to $O(f_\epsilon^2)$, i.e., the error is $O(f_\epsilon^2)$.

From Eq. (1) we have the formal definition of the first order topological derivative,

$$
D_1^\epsilon \Psi(\hat{x}) = \lim_{\epsilon \to 0} \frac{\Psi(\Omega_\epsilon(\hat{x})) - \Psi(\Omega)}{f_1(\epsilon)},
$$

(4)

if it exists, cf. Sokolowski and Zochowski (1999). However, following the developments in Sokolowski and Zochowski (2001), Novotny et al. (2003) and Nazarov and Sokolowski (2003), we may interpret the above as the singular limit of the shape derivative $\frac{\delta}{\delta \epsilon} \psi(\Omega_\epsilon(\hat{x}))$ of the functional $\Psi$ with respect to the radius $\epsilon$ of a small hole centered $\hat{x}$, viz.

$$
D_1^\epsilon \Psi(\hat{x}) = \lim_{\epsilon \to 0} \frac{1}{f_1(\epsilon)} \frac{d}{d \epsilon} \Psi(\Omega_\epsilon).
$$

(5)

Without loss of generality, in this work we equate $\Psi$ to the total potential energy, i.e.,

$$
\Psi(\Omega) = \frac{1}{2} \int_{\partial \Omega} \nabla \mathbf{u} \cdot \mathbf{T} d\Omega - \int_{\partial \Omega} \mathbf{t}^\epsilon \cdot \mathbf{u} d\Omega,
$$

(6)

where $\mathbf{u}$ is the displacement vector, $\mathbf{T} = \mathcal{C} \nabla \mathbf{u}$ is the symmetric Cauchy stress tensor, $\mathcal{C}$ is the elasticity tensor for a linear elastic homogeneous isotropic material and $\mathbf{t}^\epsilon$ is the applied boundary traction on $\partial \Omega$. For simplicity we assume traction loading and zero body forces.

Here we adopt the topological-shape sensitivity method to evaluate the topological derivatives. In this approach, a small hole of radius $\epsilon$ is presumed to exist at the location $\hat{x}$ (Fig. 1b). A shape sensitivity analysis is performed on Eq. (1) such that (Rocha de Faria, 2007)

$$
\frac{d}{d \epsilon} \Psi(\Omega_\epsilon) = \sum_{j=1}^n f_j(\epsilon) D_j^\epsilon \Psi(\hat{x}) + R'(f_n(\epsilon))f_n(\epsilon)
$$

$$
= \sum_{j=1}^n \left( f_j(\epsilon) D_j^\epsilon \Psi(\hat{x}) \right) + f_1(\epsilon) D_1^\epsilon \Psi(\hat{x})
$$

$$
+ \sum_{k=j+1}^n \left( f_k(\epsilon) D_k^\epsilon \Psi(\hat{x}) \right) + R'(f_n(\epsilon))f_n(\epsilon).
$$

(7)

Note that here we omit the $\hat{x}$ dependency of the domain $\Omega_\epsilon$ for notation simplicity. We adopt this notation for the equations that follow. Rearranging the above equation, taking the limit as $\epsilon \to 0$ and assuming that

$$
\lim_{\epsilon \to 0} \frac{R'(f_n(\epsilon))f_n(\epsilon)}{f_1(\epsilon)} = 0
$$

(8)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig_1}
\caption{Domains (a) without perturbation and (b) with a hole of size $\epsilon$ with center at location $\hat{x}$.}
\end{figure}
Remark 1. From Eq. (9) we see that in addition to the requirements of Eq. (2), the gauge function \( f_i \) must also be defined such that \( D_i^j \Psi \) is finite and nonzero. Indeed \( D_i^j \Psi \) must be finite since we assume that the response functional \( \Psi \) is always bounded. The choice of \( f_i \) is up to an \( O(1) \) constant, i.e., if \( f_i \) is a suitable function for Eq. (9), then \( \beta f_i \) is also suitable for any constant \( \beta > 0 \). However, the choice of \( \beta \) has no effect on the topological asymptotic expansion, cf. Eq. (1).

As just mentioned, the shape derivative \( \frac{\partial}{\partial X} \Psi(\Omega_t) \) corresponds to the shape variation of the domain \( \Omega_t \) with respect to the hole radius \( \epsilon \); this variation is prescribed via the so-called velocity field \( v \), i.e., the boundary variation

\[
v(x) = 0 \quad \text{on} \quad \partial \Omega,
\]

\[
v(x) = -n \quad \text{on} \quad \partial B.
\]

Therefore, the shape sensitivity results in an integral over the boundary \( \partial B \), as

\[
\frac{d}{d\epsilon} \Psi(\Omega_t) = - \int_{\partial B} \Sigma \cdot n \, d\partial B,
\]

where \( \Sigma \) denotes the energy momentum tensor

\[
\Sigma = \frac{1}{2} (\nabla u \cdot T) I - \nabla u^T \nabla \mu T,
\]

and the sub-index \( \epsilon \) denotes the quantities evaluated on the perturbed domain \( \Omega_t \). Similar sensitivity expressions are obtained for other functionals, cf. Haug et al. (1986).

To evaluate the limit in Eq. (9) we need the behavior of \( T \), as \( \epsilon \) approaches zero. Assuming traction free boundary conditions on the hole, the boundary value problem in the perturbed domain \( \Omega_t \) is stated as: find \( T \), such that

\[
\text{div} T = 0 \quad \text{in} \quad \Omega_t,
\]

\[
T \cdot n = 0 \quad \text{on} \quad \partial B,
\]

\[
T \cdot n = t^0 \quad \text{on} \quad \partial \Omega.
\]

3. Asymptotic analysis

To obtain an approximation for \( T \), valid for \( \epsilon \ll 1 \), we propose the following composite expansion (Fig. 2):

\[
T_t(x) = T(x) + \tilde{T}(y),
\]

where

\[
T(x) = F_0(\epsilon) T^0(x) + F_1(\epsilon) T^{1}(x) + F_2(\epsilon) T^{2}(x) + \cdots
\]

is denoted the outer stress and

\[
\tilde{T}(y) = F_0(\epsilon) \tilde{T}^0(y) + F_1(\epsilon) \tilde{T}^{1}(y) + F_2(\epsilon) \tilde{T}^{2}(y) + \cdots
\]

is denoted the inner stress; the latter uses the scaled variable \( y = x/\epsilon \). The gauge functions \( F_\epsilon(e) \) satisfy

\[
\lim_{\epsilon \to 0} \frac{F_{\epsilon}(\epsilon)}{F_\epsilon(e)} = 0.
\]

Note that the sum of the outer and inner stresses must satisfy the boundary conditions of Eq. (13), i.e.,

\[
T_t \cdot n = T \cdot n + \tilde{T} \cdot n \biggr|_{y=x/\epsilon} = t^0, \quad x \in \partial \Omega
\]

and

\[
T_t \cdot n = T \cdot n + \tilde{T} \cdot n \biggr|_{y=x/\epsilon} = 0, \quad x \in \partial B.
\]

In the following, we describe the boundary value problems for \( T^0 \) and \( \tilde{T}^0 \).

The outer stress \( T^0 \) satisfies the boundary value problem in the unperturbed domain (Fig. 2), i.e., find \( T^0 \) such that

\[
\text{div} T^0 = 0 \quad \text{in} \quad \Omega,
\]

\[
T^0 \cdot n = t^0 \quad \text{on} \quad \partial \Omega.
\]

The prescribed traction \( t^0 \) is defined such that the boundary condition of Eq. (18) is satisfied. In general the outer stress \( T^0 \) does not satisfy the traction free boundary condition on the hole of the perturbed domain, cf. Eq. (19). Moreover, for existence of the solution, the resultant vector of the external forces must be zero, i.e., (Muskhelishvili, 1953)

\[
\int_{\partial \Omega} t^0 \cdot d\partial \Omega = 0.
\]

The inner stress \( \tilde{T}^0 \) is used to annihilate the traction \( T^0 \cdot n \) on the hole boundary introduced by the outer stress. It is expressed in terms of the stretched coordinate \( y \), which for larger values corresponds to points \( x \) within \( O(\epsilon) \) distance from \( x \). Hence for the corresponding boundary value problem, we solve an infinite domain problem in which the stress decays away from the hole. More precisely, the inner stress \( \tilde{T}^0 \) satisfies the boundary value problem

\[
\text{div} \tilde{T}^0(\rho, \hat{\rho}) = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \mathbb{B}_1,
\]

\[
\tilde{T}^0(\rho, \hat{\rho}) \cdot n(\rho, \hat{\rho}) = t^0(\rho, \hat{\rho}) \quad \text{on} \quad \partial \mathbb{B}_1,
\]

\[
\tilde{T}^0(\rho, \hat{\rho}) \to 0 \quad \text{at} \quad \rho \to \infty.
\]

Fig. 2. Composite expansion expressed as the sum of responses on a domain without the hole \( \Omega \) and local (scaled) infinite domain with a hole \( \mathbb{R}^2 \setminus \mathbb{B}_1 \).
where the point with position vector \( \mathbf{y} = x/\epsilon \) is now identified by the position vector \( \mathbf{r}/\epsilon \), with respect to the cylindrical coordinate system with origin at \( \mathbf{x} \) and axis vectors \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \), cf. Eq. (3). Note that \( \rho = r/\epsilon \), where \( \hat{r}, \hat{\theta} \) is the position vector of the point \( \mathbf{x} \) with respect to the same cylindrical coordinate system. We also denote \( B_1 \) as the hole of radius \( \rho = 1 \) with boundary \( \partial B_1 \) and normal vector \( \mathbf{n} = -\hat{e}_3 \). In the same manner as the outer problem, the traction \( \mathbf{T}^0 \) is defined such that the boundary condition of Eq. (19) is satisfied and also satisfy global equilibrium cf. Eq. (21).

To solve the inner boundary value problem of Eq. (22), we use the Muskhelishvili complex potentials method (Muskhelishvili, 1953). The solution method, described in Appendix A, gives the inner stress

\[
\mathbf{T}^0 (\rho, \vartheta) = \sum_{k=1}^{\infty} \frac{1}{\rho^k} \mathbf{g}^{(k)} (k, \vartheta),
\]

where the functions \( \mathbf{g}^{(k)} \) depend on the boundary tractions \( \hat{e} \).

### 3.1. Boundary condition for the outer problem

To determine the boundary tractions for the outer problems we combine Eqs. (15), (16) and (18) to give

\[
\begin{align*}
F_0 (\epsilon) \mathbf{T}^0 (x/\epsilon) \mathbf{n} + & F_1 (\epsilon) \mathbf{T}^1 (x/\epsilon) \mathbf{n} + F_2 (\epsilon) \mathbf{T}^2 (x/\epsilon) \mathbf{n} + \cdots \\
+ F_0 (\epsilon) \mathbf{T}^0 (x/\epsilon) \mathbf{n} + & F_1 (\epsilon) \mathbf{T}^1 (x/\epsilon) \mathbf{n} + F_2 (\epsilon) \mathbf{T}^2 (x/\epsilon) \mathbf{n} + \cdots \\
= & \mathbf{T}^0 (x),
\end{align*}
\]

where \( \mathbf{n} \) is evaluated at \( x \) unless specifically indicated otherwise. Knowing the general form of the inner stress \( \mathbf{T}^0 (x/\epsilon) \) we now examine its behavior for \( x \in \partial \Omega \). Hence Eq. (23) is rewritten as

\[
\mathbf{T}^0 (r/\epsilon, \vartheta) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{h}^{(k)} (r, \vartheta),
\]

where

\[
\mathbf{h}^{(k)} (r, \vartheta) = \frac{\mathbf{g}^{(k)} (k, \vartheta)}{r^k}
\]

are all \( O(1) \) quantities with respect to \( \epsilon \).

From Eqs. (24) and (25) we define the gauge functions \( F_k \) such that

\[
F_k (\epsilon) = \epsilon^k,
\]

and hence Eqs. (15) and (16) become

\[
\begin{align*}
\mathbf{T} (x) = & \mathbf{T}^0 (x) + \epsilon \mathbf{T}^1 (x) + \epsilon^2 \mathbf{T}^2 (x) + \cdots, \\
\mathbf{T} (y) = & \mathbf{T}^0 (y) + \epsilon \mathbf{T}^1 (y) + \epsilon^2 \mathbf{T}^2 (y) + \cdots.
\end{align*}
\]

Combining Eqs. (24), (25) and (27) gives

\[

t^0 (\mathbf{x} n + \epsilon \mathbf{T}^1 (\mathbf{x}) \mathbf{n} + \epsilon^2 \mathbf{T}^2 (\mathbf{x}) \mathbf{n} + \\
+ \epsilon^3 \mathbf{h}^{(1)} (\mathbf{x}, \vartheta) \mathbf{n} + \epsilon^3 \mathbf{h}^{(2)} (\mathbf{x}, \vartheta) \mathbf{n} + \cdots \\
+ \epsilon^3 \mathbf{h}^{(1)} (\mathbf{x}, \vartheta) \mathbf{n} + \epsilon^3 \mathbf{h}^{(2)} (\mathbf{x}, \vartheta) \mathbf{n} + \cdots \\
+ \epsilon^3 \mathbf{h}^{(1)} (\mathbf{x}, \vartheta) \mathbf{n} + \epsilon^3 \mathbf{h}^{(2)} (\mathbf{x}, \vartheta) \mathbf{n} + \cdots \\
+ \epsilon^3 \mathbf{h}^{(1)} (\mathbf{x}, \vartheta) \mathbf{n} + \epsilon^3 \mathbf{h}^{(2)} (\mathbf{x}, \vartheta) \mathbf{n} + \cdots + \mathbf{T}^0 (\mathbf{x}),
\]

where \( \mathbf{h}^{(k)} (\mathbf{x}, \vartheta) = \mathbf{x} \in \partial \Omega \) is a boundary point.

Collecting like-wise powers of \( \epsilon \) yields the outer traction boundary conditions of Eq. (20), i.e.,

\[
\begin{align*}
t^0 (x) = & t^0 (x), \\
t^1 (x) = & 0, \\
t^2 (x) = & h^2 (x, \vartheta) \mathbf{n}, \\
\vdots & \\
t^j (x) = & - \sum_{k=0}^{j} h^j_{2, k} (x, \vartheta) \mathbf{n}, \quad j \geq 2.
\end{align*}
\]

### Remark 2.

From Eqs. (26) and (30), we observe that as \( \epsilon \) increases, i.e., the hole position moves away from the boundary \( \partial \Omega \), the outer traction \( t^j (x) \) for \( j \geq 2 \) decreases due to the \( 1/\epsilon^m \) terms with \( m = 2, 3, \ldots \). Hence the contribution of the outer solutions \( t^j (x) \) for \( j \geq 2 \) to the composite expansion of Eq. (14) decreases.

### 3.2. Boundary condition for the inner problem

To determine the boundary tractions for the inner problems we combine Eqs. (19) and (28) to give

\[

t^0 (x) \mathbf{n} + \epsilon \mathbf{T}^1 (x) \mathbf{n} + \epsilon^2 \mathbf{T}^2 (x) \mathbf{n} + \\
+ \mathbf{T}^0 (x) \mathbf{n} + \epsilon \hat{T}^1 (x) \mathbf{n} + \epsilon^2 \hat{T}^2 (x) \mathbf{n} + \cdots = 0,
\]

for \( x \in \partial B \). Since \( B \) is a hole with small radius \( \epsilon \), we expand \( T^0 (x) \) about \( x \), i.e., the center of the hole, using \( x - \hat{x} = -\epsilon n \) where \( n(x) \) is the normal to \( \partial B \) to obtain

\[

t^0 (x) \mathbf{n} - \epsilon \frac{d}{dx} T^0 (x) \mathbf{n} + \epsilon^2 \frac{d^2}{dx^2} T^0 (x) \mathbf{n} + \\
+ \mathbf{T}^0 (x) \mathbf{n} - \epsilon \frac{d}{dx} \mathbf{T}^1 (x) \mathbf{n} + \epsilon^2 \frac{d^2}{dx^2} \mathbf{T}^1 (x) \mathbf{n} + \\
+ \epsilon^2 \mathbf{T}^2 (x) \mathbf{n} - \epsilon^2 \frac{d}{dx} \mathbf{T}^2 (x) \mathbf{n} + \epsilon^2 \frac{d^2}{dx^2} \mathbf{T}^2 (x) \mathbf{n} + \\
+ \cdots + \mathbf{T}^0 (\rho, \vartheta) \mathbf{n} + \epsilon \hat{T}^1 (\rho, \vartheta) \mathbf{n} + \epsilon^2 \hat{T}^2 (\rho, \vartheta) \mathbf{n} + \cdots = 0.
\]

### Fig. 3. Global and local coordinate systems.
The hole boundary point with position vector \( \mathbf{x} \in \partial B \), is above described by the position vector \( \rho \mathbf{e}_\rho(\vec{\theta}) \in \partial B_1 \) where \( \rho = 1 \) and \( \mathbf{e}_\rho = - \mathbf{n} \).

Collecting like-wise powers of \( \epsilon \) yields the inner traction boundary condition of Eq. (22), i.e.,

\[
\mathbf{t}^{(j)}(\vec{\theta}) = - \left( T^{(j)}(\mathbf{x}) \mathbf{n} + \sum_{k=1}^{j} \left( -1 \right)^k \frac{d^k}{d\mathbf{x}^k} T^{(j-k)}(\mathbf{x}) \right) \mathbf{n} \mathbf{n} \mathbf{n} = 0.
\] (33)

4. Algorithm

We now simplify the shape sensitivity of Eq. (11) using the traction free boundary condition on the hole, i.e., \( T_r \mathbf{n} = 0 \). Hence we obtain

\[
\frac{d}{d\epsilon} \psi(\Omega_0) = - \int_{\partial B} \mathbf{n} \cdot \mathbf{n} d\partial B,
\]

where \( E \) is the Young’s modulus and we use the cylindrical coordinate system cf. Fig. 3 to express the integral and stress

\[
T_r = T_r^{(m)} \mathbf{e}_r \mathbf{e}_r + T_r^{(m)} \mathbf{e}_\theta \mathbf{e}_\theta + T_r^{(m)} \mathbf{e}_z \mathbf{e}_z.
\] (36)

To evaluate the topological derivative of Eq. (9), we approximate the stress \( T_r(\mathbf{x}) \) for \( \mathbf{x} \in \partial B_e \) using the composite expansion of Eq. (14), i.e.,

\[
T_r(\mathbf{x}) = T_r^{(m)}(\mathbf{x}) + O(\epsilon^{m+1}),
\] (37)

where

\[
T_r^{(m)}(\mathbf{x}) = \sum_{j=0}^{m} \epsilon^j \left( T_r^{(j)}(\mathbf{x}) + \vec{\mathbf{T}}(\mathbf{y}) \right).
\] (38)

with \( \mathbf{y} = \mathbf{x}/\epsilon \). As in Eq. (32), we additionally approximate \( T^{(j)}(\mathbf{x}) \) using Taylor expansion about \( \mathbf{x} \) with \( \mathbf{x} - \mathbf{x} = - \epsilon \mathbf{n} \).

The algorithm to compute \( D_r^{(j)} \psi \) is hence:

\[
m = 1, \quad j = 0.
\]

WHILE \( m \leq n \) DO

- Determine the outer stress \( T^{(j)}(\mathbf{x}) \) by solving Eq. (20) with \( \mathbf{t}^{(j)}(\mathbf{x}) \) given by Eq. (30);
- Determine the inner stress \( \vec{\mathbf{T}}(\mathbf{y}) \) by solving Eq. (22) with \( \mathbf{t}^{(j)}(\mathbf{x}) \) given by Eq. (33);
- IF \( j = 0 \) THEN

\[
T^{(k)}(\mathbf{x}) = \left( T^{(k)}(\mathbf{x}) + \vec{\mathbf{T}}(\mathbf{y}) \right)
\] (39)

ELSE

\[
T^{(k)}(\mathbf{x}) = T^{(k-1)}(\mathbf{x}) + \epsilon^j \left( T^{(k)}(\mathbf{x}) + \vec{\mathbf{T}}(\mathbf{y}) \right)
\] (40)

ENDIF

\[ j = j + 1 \]

- Evaluate

\[
T_r = T^{(j)}(\mathbf{x}) + O(\epsilon^{j+1})
\] (41)

\[
dt = \int_{\partial B_r} \mathbf{n} \cdot \mathbf{n} d\partial B + \sum_{i=1}^{m-1} f_i(\epsilon) D_r^{(j)} \psi(\mathbf{x}).
\] (42)

- IF \( dt \neq 0 \) THEN

\[
\rightarrow \text{choose } f_m(\epsilon) \text{ according to Remark 1;}
\]

\[
\rightarrow \text{evaluate } D_r^{(j)} \psi, \text{ combining Eqs. (9) and (35), i.e.,}
\]

\[
D_r^{(m)} \psi(\mathbf{x}) = - \lim_{\epsilon \to 0} \frac{1}{f_m(\epsilon)} \left( \int_0^{2\pi} \frac{r^m}{2E} \epsilon d\theta + \sum_{i=1}^{m-1} f_i(\epsilon) D_r^{(j)} \psi(\mathbf{x}) \right)
\] (43)

\[
\rightarrow m = m + 1;
\] ENDIF

END WHILE

The logical test “If \( dt \neq 0 \)” is included in the algorithm to account for the degenerate situation in which the \( j \)th order composite stress expansion \( T^{(j)} \) would render a zero topological derivative, i.e., it does not completely determine the next term on the topological asymptotic expansion of Eq. (1). We encounter this situation in our examples.

5. Analytical example

In order to verify the proposed asymptotic expansion, we present one simple example which consists of a circular domain \( \Omega \) of radius \( R \) such that \( \partial \Omega = \{(r, \theta) \mid r = R\} \) where we use the cylindrical coordinates \( x = r \mathbf{e}_r(\theta) \) (Fig. 4a). The domain is subject to the non-uniform traction \( \mathbf{t}^{(0)} = (1 + \cos(3\theta)) \mathbf{e}_x(\theta) + \sin(2\theta) \mathbf{e}_y(\theta) \) MPa over \( \partial \Omega \).

The response for this circle problem satisfies the boundary value problem

\[
\text{div} \mathbf{T} = \mathbf{0} \quad \text{in} \quad \Omega,
\]

\[
\mathbf{T}_n = 0 \quad \text{on} \quad \partial \Omega,
\] (44)

and has an analytical expression cf. Eqs. (57) and (58) and Muskhelishvili (1953), from which the total potential energy of the unperturbed domain is found to be

\[
\psi := \psi(\Omega) = \frac{\pi R^2 (-87 + 37\nu)}{48E}.
\] (45)

For the given parameter values \( E = 1 \) GPa, \( \nu = 0.3 \) and \( R = 1 \) m, the total potential energy is \( \psi = 4.96764 \) kNm and the Von Mises stress distribution is depicted on the deformed configuration in Fig. 4b.

5.1. Topological derivative for a hole introduced at the center

We first evaluate the total potential energy when a small hole of radius \( \epsilon \) is introduced at the center. The perturbed domain \( \Omega_r \), is now defined by two concentric circles of radii \( R \) and \( \epsilon \), such that \( \partial B_\epsilon = \{(r, \theta) \mid r = \epsilon\} \). cf. Fig. 5. The response of this ring problem satisfies the boundary value problem of Eq. (13) and is also available analytically.

Using the analytical response, the total potential energy of the perturbed domain is hence given by (Muskhelishvili, 1953)

\[
\psi_r := \psi(\Omega_r) = \frac{\pi R^2 (-87 + 37\nu)}{48E} - \frac{2\pi \epsilon^2}{E} + \frac{13\pi \epsilon^4}{2R^2 E} + \frac{8\pi \epsilon^6}{3R^2 E} - \frac{195\pi \epsilon^8}{4R^2 E} + o(\epsilon^8).
\] (46)
5.1.1. Topological derivative obtained from the analytical solution for the ring problem

To evaluate the topological asymptotic expansion for the total potential energy we need to evaluate $T_\epsilon$ on the boundary $\partial B_\epsilon$ as $\epsilon$ approaches zero. From the analytical response we obtain

$$T_\epsilon^{(1)}(\epsilon, \theta) = 2 - \frac{6\epsilon}{R} \cos(3\theta) + \frac{2\epsilon^2}{R^2} \left(1 + 2 \cos(2\theta)\right) + \frac{6\epsilon^3}{R^4} \cos(3\theta) + O(\epsilon^6).$$ (47)

And upon substituting Eq. (47) into Eq. (43), we find that to satisfy Eqs. (1) and (2) we require $f_\epsilon(\epsilon) = \pi \epsilon^{2k}$. Finally, computing the limit as $\epsilon$ approaches zero gives

$$D_\epsilon^{(1)} = \frac{2}{E},$$ (48)

$$D_\epsilon^{(2)} = \frac{13}{2R^2 E},$$ (49)

and

$$D_\epsilon^{(3)} = \frac{8}{3R^3 E}.$$ (50)

And hence from Eq. (3) we obtain the following topological asymptotic expansions:

$$\Psi^{(1)}_\epsilon = \frac{\pi R^2 (-87 + 37\epsilon)}{48E} - \frac{2\pi \epsilon^2}{E},$$ (51)

$$\Psi^{(2)}_\epsilon = \frac{\pi R^2 (-87 + 37\epsilon)}{48E} - \frac{2\pi \epsilon^2}{E} - \frac{13\pi \epsilon^4}{2R^2 E},$$ (52)

$$\Psi^{(3)}_\epsilon = \frac{\pi R^2 (-87 + 37\epsilon)}{48E} - \frac{2\pi \epsilon^2}{E} - \frac{13\pi \epsilon^4}{2R^2 E} + \frac{8\pi \epsilon^6}{3R^3 E},$$ (53)

which are seen to agree with Eq. (46).

Fig. 6a shows the analytical total potential energy curve $\Psi_\epsilon$, cf. Eq. (46) as a function of the normalized hole size $\epsilon/R$, as well as the topological asymptotic expansions $\Psi^{(1)}_\epsilon$, $\Psi^{(2)}_\epsilon$ and $\Psi^{(3)}_\epsilon$. As expected, for larger hole sizes $\epsilon$ the higher order topological asymptotic expansions $\Psi^{(2)}_\epsilon$ and $\Psi^{(3)}_\epsilon$ give better estimates for the total potential energy than the first order expansion $\Psi^{(1)}_\epsilon$. More importantly perhaps, in engineering applications for which the analytical solution is unavailable, the second order topological derivative can be used to provide a range of $\epsilon$ over which the first order topological derivative provides a reasonable approximation. For this example, we see that the first order topological derivative gives good results for $\epsilon/R < 0.15$, cf. Fig. 6b.

Fig. 7a depicts the difference between the analytical solution $\Psi_\epsilon$ and the $n$th topological asymptotic expansion, i.e., $\Psi^{(n)}_\epsilon$. For small radius sizes $\epsilon$ we note that the $\Psi^{(n)}_\epsilon$ estimate gives larger errors than the $\Psi^{(1)}_\epsilon$ and the $\Psi^{(2)}_\epsilon$ estimates. We also observe that for holes with small radii $\epsilon$ the error is reduced when the order $n$ of the topological asymptotic expansion increases, as expected. However, this trend is not necessarily true for holes with larger $\epsilon$. Indeed, as shown in Fig. 7b, for $\epsilon/R > 0.15$ the $\Psi^{(2)}_\epsilon$ estimate gives smaller errors than the $\Psi^{(3)}_\epsilon$ estimate. Comparing Eqs. (46), (52) and (53), we can write

$$\Psi_\epsilon - \Psi^{(2)}_\epsilon = \frac{8\pi \epsilon^6}{3R^3 E} - \frac{195\pi \epsilon^8}{4R^4 E} + o(\epsilon^8)$$ (54)

and

$$\Psi_\epsilon - \Psi^{(3)}_\epsilon = -\frac{195\pi \epsilon^8}{4R^4 E} + o(\epsilon^8).$$ (55)

We observe that the positive term in Eq. (54) yields smaller errors in the $\Psi^{(2)}_\epsilon$ estimate when compared to the $\Psi^{(3)}_\epsilon$ estimate for larger hole sizes. This explains the “kink” in Fig. 7b which is attributed to the sign change of Eq. (54), as depicted in Fig. 7a.
5.1.2. Topological derivative obtained from the composite expansion of $T_r$

In general, the analytical response is unavailable; rather we use the composite expansion of Eq. (14) to approximate it. Here we employ the algorithm introduced in Section 4 to evaluate the stress $T_r$, and the first and second order topological derivatives for the ring example.

- $j = 0$

In the $j = 0$ outer problem, the boundary traction is given by, cf. Eq. (30) and Fig. 5.

$$
i^{(0)}(\theta) = i^t(\theta) = (1 + \cos(3\theta))e_r(\theta) + \sin(2\theta)e_\theta(\theta),$$

which gives

$$
T^{(0)} = T^{(0)}_n \otimes e_r + T^{(0)}_m e_\theta \otimes e_r + T^{(0)}_o e_\theta \otimes e_r + T^{(0)}_e e_\theta \otimes e_r,
$$

where

$$
T^{(0)}_n(r, \theta) = 1 + \frac{3\pi}{2R} \cos(3\theta) - \frac{r^3}{2R^3} \cos(3\theta),
$$

$$
T^{(0)}_m(r, \theta) = -\frac{3\pi}{2R} \sin(3\theta) + \frac{r^2}{2R^2} \sin(2\theta) + \frac{3\pi^2}{2R^3} \sin(3\theta),
$$

$$
T^{(0)}_o(r, \theta) = 1 - \frac{3\pi}{2R} \cos(3\theta) + \frac{r^2}{2R^2} \cos(2\theta) + \frac{5\pi^3}{2R^3} \cos(3\theta).
$$

For a hole with center located at any given position $\bar{x}$, we have on the hole boundary $n = -e_r$, cf. Fig. 3. According to Eq. (33), the boundary traction for the $j = 0$ inner problem is defined as

$$
i^{(0)}(\bar{\theta}) = -\frac{T^{(0)}(\bar{x})}{r} n = T^{(0)}(\bar{x}) e_r
$$

$$
= T^{(0)}_n(\bar{x})(e_r \otimes e_r) e_r + T^{(0)}_m(\bar{x})(e_\theta \otimes e_r) e_r
$$

$$
+ T^{(0)}_o(\bar{x})(e_\theta \otimes e_r) e_r + T^{(0)}_e(\bar{x})(e_\theta \otimes e_r) e_r.
$$

For now, we consider the special case of a hole with center located at $\bar{x} = 0$ so that on $\partial B_r$ we have $n = -e_r$ and $\bar{\theta} = \theta$ (cf. Fig. 5) which gives

$$
i^{(0)}(\theta) = T^{(0)}_n(0, \theta)(e_r) e_r + T^{(0)}_m(0, \theta)(e_\theta) e_r = e_r(\theta).
$$

The inner stress is obtained from Eq. (23) and has cylindrical components

$$
\overline{T}^{(0)}_n(\rho, \theta) = -\frac{1}{\rho^2},
$$

$$
\overline{T}^{(0)}_m(\rho, \theta) = 0,
$$

$$
\overline{T}^{(0)}_o(\rho, \theta) = \frac{1}{\rho^2}.
$$

Therefore, using Eq. (39) we obtain
which gives the following stress component approximation:

\[ T_e^{(0)}(0, \theta) + \bar{T}_w^{(0)}(1, \theta) = 2, \]  

(62)

where \( \mathbf{x} \in \mathbb{E}_B \) and its square

\[ (T^{(0)}_e(\mathbf{x}))^2 = 4 + O(\epsilon). \]  

(63)

We now evaluate the first order topological derivative from Eq. (43), i.e.,

\[ D^{(1)}_i \Psi(0) = -\lim_{\epsilon \to 0} \left\{ \frac{1}{f_1}(\epsilon) \int_0^{2\pi} \frac{4 + O(\epsilon)}{2\epsilon} \, \epsilon \, d\theta \right\}, \]  

(65)

where we see that \( dt \neq 0 \). Hence from Remark 1, we require that \( f_1(\epsilon) = \pi \epsilon^2 \) and the above equation gives

\[ D^{(1)}_i \Psi(0) = -\frac{1}{4\pi E} \lim_{\epsilon \to 0} \left( \frac{4 \epsilon e^2}{\epsilon} \right) + \lim_{\epsilon \to 0} \left( \frac{O(\epsilon^2)}{\epsilon} \right) \, d\theta = -\frac{2}{E}. \]  

(66)

Combining Eqs. (3) and (66), we have

\[ \Psi^{(1)} = \Psi - \frac{2 \pi \epsilon^2}{E}, \]  

(67)

which agrees with our analytical result of Eq. (51).

**• i = 1**

In the \( j = 1 \) outer problem, the boundary traction is given by Eq. (30) as

\[ T^{(1)} = 0, \]  

(68)

which according to the boundary value problem of Eq. (20) gives

\[ T^{(1)}(\mathbf{x}) = 0. \]  

(69)

The \( j = 1 \) inner problem has the boundary traction, cf. Eq. (33)

\[ T^{(1)}(\mathbf{x}) = \left( T^{(1)}_e(\mathbf{x}) + \frac{dT^{(0)}_e(\mathbf{x})}{d\theta} \right) \mathbf{e}_r + \left( T^{(1)}_w(\mathbf{x}) + \frac{dT^{(0)}_w(\mathbf{x})}{d\theta} \right) \mathbf{e}_\theta = \frac{3}{2R} \cos(3\theta) \mathbf{e}_r + \frac{3}{2R} \sin(3\theta) \mathbf{e}_\theta. \]  

(70)

Hence from Eq. (23) the inner stress has cylindrical components

\[ \bar{T}^{(1)}_e(\rho, \theta) = \frac{6 \cos(3\theta)}{R \rho^3} - \frac{15 \cos(3\theta)}{2R \rho^3}, \]

\[ \bar{T}^{(1)}_w(\rho, \theta) = \frac{6 \sin(3\theta)}{R \rho^3} - \frac{9 \sin(3\theta)}{2R \rho^3}, \]  

(71)

Therefore, via Eq. (40) we obtain

\[ \left( \bar{T}^{(1)}_e(\mathbf{x}) \right)_{\rho_0} = \left( \frac{6 \cos(3\theta)}{2R} - \frac{6 \cos(3\theta)}{R} \right) - \epsilon \left( \frac{3 \cos(3\theta)}{2R} \right) = 2 - 6 \frac{\epsilon}{R} \cos(3\theta), \]  

(72)

which gives the following stress component approximation:

\[ T^{(1)}_e(\mathbf{x}) = 2 - \frac{6 \epsilon}{R} \cos(3\theta) + O(\epsilon^2). \]  

(73)

where \( \mathbf{x} \in \mathbb{E}_B \) and its square

\[ (T^{(1)}_e(\mathbf{x}))^2 = 4 - \frac{24 \epsilon}{R} \cos(3\theta) + O(\epsilon^2). \]  

(74)

We now use Eq. (43) to evaluate the second order topological derivative, i.e.,

\[ D^{(2)}_i \Psi(0) = -\lim_{\epsilon \to 0} \left\{ \frac{1}{f_2}(\epsilon) \int_0^{2\pi} \left( \frac{T^{(1)}_e(\mathbf{x})}{2\epsilon} \right) \frac{\epsilon}{2\epsilon} \, d\theta \right\}. \]  

(75)

From Remark 1 we require \( f_2(\epsilon) = \pi \epsilon^3 \). However, Eq. (75) gives

\[ dt = -\frac{24}{3 \pi R} \cos(3\theta) \, d\theta = 0. \]  

(76)

Therefore, we see that the second order topological derivative \( D^{(2)}_i \Psi(0) \) cannot be determined from the composite expansion of Eq. (73). In other words, to obtain the third term in the topological asymptotic expansion of Eq. (3) we need to compute the \( O(\epsilon^2) \) term on the composite expansion \( T_e \) by solving the \( j = 2 \) problem. Moreover, we observe that the topological asymptotic expansion of the total potential energy is not a function of \( \epsilon^2 \), as expected from Eq. (46).

**Remark 3.** Eq. (75) could be expressed as

\[ D^{(2)}_i \Psi(0) = -\frac{1}{2E} \lim_{\epsilon \to 0} \left\{ \frac{1}{f_2}(\epsilon) \int_0^{2\pi} \left( \frac{24 \epsilon^2}{R} \cos(3\theta) + \frac{36 \epsilon^3}{R^3} \cos^2(3\theta) \right. \right. \]

\[ \left. + \epsilon \epsilon^3 + O\left( \epsilon^4 \right) \right) \, d\theta. \]  

(77)

where \( \epsilon \) is a constant that depends on the missing \( O(\epsilon^2) \) term on the composite expansion of \( T^{(1)}_e \) cf. Eq. (73). And since \( \epsilon(3 \cos(3\theta) \epsilon \epsilon^3 = 0 \), we see that if we assign \( f_2(\epsilon) = \pi \epsilon^3 \) then Eq. (77) gives

\[ D^{(2)}_i \Psi(0) = -\frac{1}{2E} \int_0^{2\pi} \frac{36 \epsilon^3}{4 \pi R} \cos^2(3\theta) \, d\theta - c_1 = -\frac{9}{2R^2} - c_1, \]  

(78)

where \( c_1 \) depends on the \( O(\epsilon^2) \) contribution to \( \epsilon \epsilon^3 \), which is a attributed to the outer stress \( T^{(2)}_e \) and hence it is expressed in terms of quantities on the boundary \( \partial \Omega \). Therefore, Eq. (78) does not determine \( D^{(2)}_i \Psi \), since \( c_1 \) is unknown. According to Remark 2, we know that the contribution of the outer stress \( T^{(2)}_e \) is minimal when the hole is located far away from the boundary \( \epsilon \partial \Omega \), as discussed in Bonnet (2007), Rocha de Faria et al. (2008) and here in the text surrounding Fig. 9. In this work, we denote the expansion \( \epsilon \epsilon^2 \) the “incomplete” second order topological asymptotic expansion obtained from Eqs. (3) and (78)

\[ \epsilon \epsilon^2 = \epsilon - \frac{2 \pi \epsilon^2}{E} - \frac{9 \pi \epsilon^4}{2R^2 E}, \]  

(79)

which does not agree with our analytical result of Eq. (52).

**• i = 2**

In the \( j = 2 \) outer problem, the boundary traction is given by Eq. (30) as

\[ \mathbf{r}^{(2)} = -h^{(2)}_e(\hat{\theta}) \mathbf{n} = -\frac{1}{R} \mathbf{e}_r, \]  

(80)

which yields the components
Hence the \( j = 2 \) inner problem has the boundary traction, cf. Eq. (33)

\[
\mathbf{t}^{(2)} = -\left( \mathbf{T}^{(2)}(\mathbf{x}) - \frac{d}{d\mathbf{x}} \mathbf{T}^{(1)}(\mathbf{x}) \right) n + \frac{1}{2} \frac{d^2}{d\mathbf{x}^2} \mathbf{T}^{(0)}(\mathbf{x}) n n
\]

\[
= \left( \mathbf{T}^{(2)}(\mathbf{x}) + \frac{1}{2} \frac{d^2}{d\mathbf{x}^2} \mathbf{T}^{(0)}(\mathbf{x}) \right) \mathbf{e}_r,
\]

\[
= \left( \frac{1}{R^2} \mathbf{e}_r + 0 \mathbf{e}_r \right) + \left( 0 \mathbf{e}_r + \frac{1}{2} \frac{d^2}{d\mathbf{x}^2} \mathbf{T}^{(0)}(\mathbf{x}) \right) \mathbf{e}_t,
\]

which gives the inner stress components, cf. Eq. (23)

\[
\mathbf{T}^{(2)}(\rho, \theta) = 2 \cos(2\theta) - 2 \cos(2\theta) - \frac{1}{R^2} \rho^2,
\]

\[
\mathbf{T}^{(2)}(\rho, \theta) = \sin(2\theta) - 2 \sin(2\theta) - R^2 \rho^2,
\]

\[
\mathbf{T}^{(2)}(\rho, \theta) = \cos(2\theta) + \frac{1}{R^2} \rho^2.
\]

Therefore, we obtain from Eq. (40)

\[
\left( \mathbf{T}^{(2)}(\mathbf{x}) \right) = \left( \mathbf{T}^{(1)}(\mathbf{x}) \right) + \epsilon^2 \left( \mathbf{T}^{(0)}(\mathbf{x}) \right) + \frac{\epsilon^2}{2} \frac{d^2}{d\mathbf{x}^2} \mathbf{T}^{(0)}(\mathbf{x})
\]

\[
= \left( \mathbf{T}^{(1)}(\mathbf{x}) \right) + \epsilon^2 \left( \frac{1}{R^2} + \frac{2 \cos(2\theta)}{R^2} + \frac{1}{R^2} \right) + \frac{\epsilon^2}{2} \frac{4 \cos(2\theta)}{R^2}
\]

\[
= 2 - \frac{6 \epsilon}{R} \cos(3\theta) + \frac{2 \epsilon^2}{R^2} \left( 1 + 2 \cos(2\theta) \right),
\]

which yields the following stress component approximation:

\[
T^{(2)}(\mathbf{x}) = 2 - \frac{6 \epsilon}{R} \cos(3\theta) + \frac{2 \epsilon^2}{R^2} \left( 1 + 2 \cos(2\theta) \right) + O(\epsilon^3),
\]

where \( \mathbf{x} \in \mathcal{B}_\epsilon \), and its square

\[
\left( T^{(2)}(\mathbf{x}) \right)^2 = 4 - \frac{24 \epsilon}{R} \cos(3\theta) + \frac{36 \epsilon^2}{R^2} \cos^2(3\theta)
\]

\[
+ \frac{8 \epsilon^2}{R^2} \left( 1 + 2 \cos(2\theta) \right) + O(\epsilon^3).
\]

We now use Eqs. (43) and (86) to evaluate the second order topological derivative, i.e.,

\[
D^{(2)}_e \Psi(0) = -\lim_{\epsilon \to 0} \left\{ \frac{1}{f(\epsilon)} \left( \int_0^{2\pi} \left( \frac{2E}{2} \right) \epsilon d\theta + f(\epsilon) D^{(1)}_e \Psi(0) \right) \right\}
\]

\[
= -\lim_{\epsilon \to 0} \left\{ \frac{1}{f(\epsilon)} \int_0^{2\pi} \left( \frac{\epsilon^4}{2R^2 E} (8 + 36 \cos^2(3\theta)) + O(\epsilon^4) \right) d\theta \right\},
\]

where we used the fact that \( \int_0^{2\pi} \cos(k\theta) d\theta = 0 \) for \( k \in \mathbb{N} \). From Eq. (87) we see that \( dt \neq 0 \). Moreover, from Remark 1 we require \( f_2(\epsilon) = \pi \epsilon^4 \) and hence Eq. (87) gives

\[
D^{(2)}_e \Psi(0) = -\frac{4}{2R^2 E} \frac{9}{2R^2 E} = -\frac{13}{2R^2 E}.
\]

Combining Eqs. (3), (66) and (88) yields the second order topological asymptotic expansion

\[
\Psi^{(2)} = \Psi - \frac{2\pi \epsilon^2}{E} - \frac{13\pi \epsilon^4}{2R^2 E},
\]

which agrees with our analytical result of Eq. (52).
The incorporation of the $O(\varepsilon^2)$ term in the composite expansion of $T_\epsilon$, and consequently the computation of the second order topological asymptotic expansion, may be impractical from the computational point of view. Indeed its evaluation requires the solution of two outer problems; the $j=0$ problem in which the traction $t_P$ is applied and the $j=2$ problem in which the traction $h_\hat{\varepsilon}^{(0)} (\hat{r},\hat{\theta}) n$ is applied. The $j=0$ problem is what the engineer sees, i.e., the domain without the hole subject to the traction boundary condition. Whereas the $j=2$ problem is a new problem which is dependent on the hole location, here $x = 0$. Thus, for each hole location of interest, a $j=2$ problem must be solved. On the other hand, the “incomplete” topological derivative expansion $\psi^{(2)}_\epsilon$ depends only on the $j=0$ problem. And as the influence of the external boundary $\partial{\Omega}$ disappears, i.e., as the hole position $x$ moves away from the boundary, the error of the $\psi^{(2)}_\epsilon$ approximation decreases, cf. Remarks 2 and 3.

Fig. 11. Meshed domains for holes of different sizes $\epsilon = \{0.1,0.2,0.3\}$.

Fig. 12. Topological asymptotic expansions when a hole is introduced at $\hat{r} = 0.3$ and different angles $\hat{\theta}$.
Fig. 8 compares the topological asymptotic expansions. In this example, the $\Psi^{(2)}_e$ expansion provides a better estimate for the total potential energy than the $\Psi^{(1)}_e$ expansion, perhaps because the hole position $x = 0$ is far from the boundary $\partial \Omega$. Unfortunately, we cannot expect the same behavior for every problem. The $r = 0$ curve in Fig. 9 shows that the relative error in $\Psi^{(2)}_e$ is not very large for smaller hole sizes, e.g. when $\epsilon/R < 0.4$ the error is smaller than 3%. Of course, the $\Psi^{(2)}_e$ expansion yields the best estimate, as depicted in Fig. 8.

5.2. Topological derivative for a hole introduced at location $x = (r, \hat{\theta})$ inside the domain $\Omega$

Here we evaluate the total potential energy when a small hole of radius $\epsilon$ is introduced at a prescribed location $x = (r, \hat{\theta})$, as shown in Fig. 10. Rather than using an analytical expression to obtain $\Psi^{(1)}_e$, here we verify the topological asymptotic expansion computations using the finite element software ABAQUS (ABAQUS, 2005) to calculate $\Psi^{(1)}_e$ on domains with holes of different sizes $\epsilon$ located at different positions $(r, \hat{\theta})$. Indeed we introduce finite size holes $\epsilon = \{0.1, 0.2, 0.3\}$ as shown in Fig. 11.

First we evaluate the total potential energy for holes at different angles $\hat{\theta}$ and $r = 0.3$, i.e., a point inside and somewhat distant from the boundary $\partial \Omega$. One can see from Fig. 12 that the $\Psi^{(1)}_e$ expansion gives good estimates for $\epsilon/R < 0.15$. On the other hand, the $\Psi^{(2)}_e$ expansion is able to give good estimates for larger size holes, i.e., $\epsilon/R < 0.25$. We also note that the “incomplete” $\Psi^{(2)}_e$ expansion gives a better estimate than the first order topological asymptotic expansion $\Psi^{(1)}_e$. Moreover, it can be used to determine the range where $\Psi^{(1)}_e$ is valid, i.e., $\epsilon/R < 0.15$. From Fig. 9 we also observe that for $\epsilon/R < 0.4$ the relative error in $\Psi^{(2)}_e$ is smaller than 4% when the hole is introduced at the angle $\hat{\theta} = 0$.

We now evaluate the total potential energy for holes at $r = 0.6$. In this case, the hole position is getting closer to the boundary $\partial \Omega$, especially for holes with large radius. One can see from Fig. 13 that $\Psi^{(1)}_e$ continues to give good estimates for $\epsilon/R < 0.15$. However, $\Psi^{(2)}_e$ no longer gives good estimates for larger size holes, e.g. $\epsilon/R > 0.2$. This observation can be explained by the closer proximity of the hole to the domain boundary. Notably, the “incomplete” $\Psi^{(2)}_e$ expansion does not introduce a substantial improvement over the $\Psi^{(1)}_e$ expansion. Therefore, it cannot be used to limit the range where $\Psi^{(1)}_e$ is valid. Indeed, as previously mentioned on Remarks 2 and 3, the larger holes located at $r = 0.6$ are close to the boundary $\partial \Omega$ so that disregarding the terms on the composite expansion corresponding to the outer problem $j = 2$ generates significant errors, cf. Fig. 9 for the $\hat{\theta} = 0$ case. We observe that when $\epsilon/R = 0.4$ (i.e., the largest hole size for the location $r = 0.6$) the relative error in
$\Psi^{(2)}$ is 18%, a much bigger value when compared to holes away from the domain boundary.

In the context of topology optimization, the topological derivative level set determines the location where holes should be percolated in each step of the optimization process. In a classical optimization problem, we want to minimize the compliance subject to a volume constraint. Naturally, the nucleation of holes increases compliance. Thus we want to nucleate holes such that this increase is minimal, i.e., we want to nucleate holes at the position $\hat{x}$ such that $|\Psi - \Psi|_1$ is minimized.

Fig. 14 shows the level set that includes first and second order topological derivatives, i.e.,

\[
\frac{\Psi(\Omega) - \Psi(\Omega_0)}{\pi \epsilon^2} = D(\Psi)(\hat{x})
\]

and

\[
\frac{\Psi(\Omega) - \Psi(\Omega_0)}{\pi \epsilon^2} = D(\Psi)(\hat{x}) + \epsilon^2 D^{(2)}(\Psi)(\hat{x}).
\]

In these plots, the arrows point to the locations $\hat{x}$ in which the nucleated holes of radii $\epsilon = [0.05, 0.1, 0.2, 0.3]$ yield the minimal compliance increase. We see that including the second order topological derivative in the topological asymptotic expansion of the total potential energy changes the nucleation position. This additional feature may improve the convergence of a topology optimization algorithm that only considers first order topological derivatives.

Recall that to obtain the level set of Eq. (91), we must solve the $j = 2$ problem for each hole location $\hat{x}$. Indeed, the $j = 2$ outer problem is subject to the boundary traction

\[
\mathbf{t}(\hat{x}) = -\frac{g(\hat{x})}{\partial n} - \mathbf{n},
\]

where $\hat{r}$, the distance from the center of the hole $\hat{x}$ to the boundary location $x$, changes for each location $\hat{x}$ where $D^{(2)}(\Psi)$ is evaluated. When implemented with the finite element method, this requires an additional loading for each $\hat{x}$, i.e., each node or Gauss point location. Fortunately, since the unperturbed domain $\Omega$ does not change, only one stiffness matrix assembly and factorization is required.

Fig. 14. Contour plots of $\frac{\Psi - \Psi}{\pi \epsilon^2}$ using left: first order, center: second order and right: "incomplete" second order expansions for the introduction of holes with radius $\epsilon = [0.05, 0.1, 0.2, 0.3]$ (from top to bottom).
The boundary tractions of the $j = 2$ inner problem depends on the stress of the $j = 2$ outer problem and also on the second derivative of the stress of the $j = 0$ outer problem, both evaluated at $X$, i.e.,

$$
\mathbf{t}^{(2)} = -\left( T^{(2)}(X) + \frac{1}{2} \frac{d^2}{\partial \mathbf{x}^2} T^{(0)}(X)[n,n] \right) n.
$$

In the finite element method, the computation of this stress derivative is not straightforward; higher order finite elements or stress recovery procedures may be required to obtain the desired level of accuracy.

From the previous considerations, we note that the solution of the $j = 2$ problem might be impractical from the computational point of view. To avoid this extra computation, one could base their estimates on the “incomplete” second order topological derivative, i.e.,

$$
\Psi^{(2)} - \Psi^{(1)} = D_1^{(2)} \Psi^{(1)}(X) + \epsilon^2 D_2^{(2)} \Psi^{(1)}(X).
$$

Hole nucleation sites using this “incomplete” expansion also appear in Fig. 14. As depicted, the differences between the first order and incomplete expansion are subtle for this example.

We next express the hole boundary condition of Eq. (22) using $\Psi^{(1)}$ and $\Psi^{(2)}$. Fortunately these problems have analytical solutions, as was previously observed. We observe that the results obtained via the topological derivative are in good agreement with the analytical/numerical ones, especially for small size holes. As expected the $\Psi^{(2)}$ estimate gives better results than $\Psi^{(1)}$. Moreover, $\Psi^{(2)}$ can be used to provide the range of $\epsilon$ over which the $\Psi^{(1)}$ estimate gives reasonable approximations.

In order to calculate the $\Psi^{(2)}$ estimate, we need to compute the $O(\epsilon^2)$ term on the composite expansion of $T_r$. This calculation may be impractical in most engineering problems, since it requires the solution of a boundary value problem for each position $X$. However, the contribution of this term is minimal when the hole position $X$ is distant from the domain boundary $\partial \Omega$. In this case, the use of the “incomplete” $\Psi^{(2)}$ estimate, which is based on the $O(\epsilon)$ expansion of $T_r$, appears to give reasonable results.

### 6. Conclusions

In this linear elasticity work, we propose an algorithm to evaluate the composite expansion for the stress field $T_r$ at the boundary of a hole when the radius $\epsilon$ approaches zero. This approximated stress field is subsequently used to evaluate the higher order topological derivatives based on the topological-shape sensitivity method. Here we adopt the total potential energy as the response function $\Psi^{(1)}$ and compare the $jth$ order topological asymptotic expansion $\Psi^{(j)}$ with analytical results $\Psi^{(j)}$, and also numerical results $\Psi^{(j)}$ obtained using the commercial software ABAQUS. We observe that the results obtained via the topological derivative are in good agreement with the analytical/numerical ones, especially for small size holes. As expected the $\Psi^{(2)}$ estimate gives better results than larger hole radii $\epsilon$ than $\Psi^{(1)}$. Moreover, $\Psi^{(2)}$ can be used to provide the range of $\epsilon$ over which the $\Psi^{(1)}$ estimate gives reasonable approximations.

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### Appendix A. Solution of the inner problem

Using the Muskhelishvili complex potentials method (Muskhe- lishvili, 1953), the inner stress

$$
\bar{T} = \bar{T}_r \mathbf{e}_r \otimes \mathbf{e}_r + \bar{T}_\theta \mathbf{e}_r \otimes \mathbf{e}_\theta + \bar{T}_\rho \mathbf{e}_\rho \otimes \mathbf{e}_\rho + \bar{T}_\phi \mathbf{e}_\phi \otimes \mathbf{e}_\phi
$$

is defined in terms of two complex valued potentials $\varphi$ and $\psi$ expressed in terms of the complex number $z = \rho e^{i\theta}$ such that [Eqs. (39.4) and (39.5) in Muskhelishvili (1953)]

$$
\bar{T}_r = 2 \left( \varphi' + \varphi'' \right),
$$

$$
\bar{T}_\theta = 2 \left( \psi' + \psi'' \right),
$$

$$
\bar{T}_\rho = 2 \left( \rho \varphi' + \varphi'' \right),
$$

$$
\bar{T}_\phi = 2 \left( \rho \psi' + \psi'' \right).
$$

where $\rho$ is the magnitude of $z$ and $\theta$ is the angle between the radial bases $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

### Fig. A.1. Principal stress visualization.
\[ \sum_{k=0}^{\infty} \frac{(1 + k) a_k e^{-ik\theta}}{\lambda^2} + \sum_{k=0}^{\infty} \frac{b_k e^{ik\theta} - b_0 e^{i\theta} - b_1 e^{i\theta}}{\lambda^2} \]

\[ = -\sum_{k=2}^{\infty} b_{k-2} e^{-ik\theta} = -\sum_{k=-\infty}^{\infty} A_k e^{ik\theta}. \quad (A.12) \]

Matching the \( e^{ik\theta} \) terms and using Eqs. (A.5), (A.6) and (A.9) gives

\[ a_0 = b_0 = a_1 = b_1 = 0, \]

\[ b_2 = A_0, \]

\[ a_k = -A_k, \quad k \geq 2, \]

\[ b_k = -(k-1)A_{k-2} + A_{k+2}, \quad k \geq 3, \quad (A.13) \]

where

\[ A_k = \frac{1}{2\pi} \int_0^{2\pi} \left( f_1(\theta) - i f_2(\theta) \right) e^{-ik\theta} d\theta. \quad (A.14) \]

To obtain the inner stress \( \mathbf{T} \) at any point of the domain we replace Eq. (A.3), now with the known coefficients \( a_k \) and \( b_k \) of Eq. (A.13), into Eq. (A.2) which gives

\[ \mathbf{T}^{(\rho)}(\rho, \theta) = \sum_{k=0}^{\infty} 2\rho \mathbf{g}^{(k)}(\rho, \theta), \quad (A.15) \]

where the cylindrical components of \( \mathbf{g} \) are

\[ g_{rr}(k, \theta) = (2 + k) \left( \cos(k\theta) \text{Re}\{a_k\} + \sin(k\theta) \text{Im}\{a_k\} \right) - \cos((2-k)\theta) \text{Re}\{b_k\} - \sin((2-k)\theta) \text{Im}\{b_k\} \]

\[ g_{\theta\theta}(k, \theta) = (2 - k) \left( \cos(k\theta) \text{Re}\{a_k\} + \sin(k\theta) \text{Im}\{a_k\} \right) + \cos((2-k)\theta) \text{Re}\{b_k\} + \sin((2-k)\theta) \text{Im}\{b_k\} \]

\[ g_{r\theta}(k, \theta) = -k \cos(k\theta) \text{Im}\{a_k\} - \sin(k\theta) \text{Re}\{a_k\} + \sin((2-k)\theta) \text{Re}\{b_k\} - \cos((2-k)\theta) \text{Im}\{b_k\}. \quad (A.16) \]

References


