Note

On the \((h, k)\)-domination numbers of iterated line digraphs\footnote{This research was supported by JSPS KAKENHI 21500017.}

Toru Hasunuma*, Mayu Otani

Institute of Socio-Arts and Sciences, The University of Tokushima, 1–1 Minamijosanjima, Tokushima 770-8502, Japan

A R T I C L E   I N F O

Article history:
Received 21 November 2011
Received in revised form 13 March 2012
Accepted 20 March 2012
Available online 9 April 2012

Keywords:
(h, k)-dominating set
Arc-disjoint Hamilton cycles
Iterated line digraphs
De Bruijn and Kautz digraphs
Fault-tolerance

A B S T R A C T

An \((h, k)\)-dominating set in a digraph \(G\) is a subset \(D\) of \(V(G)\) such that the subdigraph induced by \(D\) is \(h\)-connected and for every vertex \(v\) of \(G\), \(v\) is in-dominated and out-dominated by at least \(k\) vertices in \(D\). The \((h, k)\)-domination number \(\gamma_{h,k}(G)\) of \(G\) is the minimum cardinality of an \((h, k)\)-dominating set in \(G\). An \((h, k)\)-dominating set finds applications to fault-tolerant location problems of resources in communication networks and fault-tolerant virtual backbone in wireless networks.

Let \(G\) be a connected \(d\)-regular digraph and \(1 \leq k < d\). Let \(L^m(G)\) denote the \(m\)-iterated line digraph of \(G\). In this note, we show that \(\gamma_{h,k}(L^m(G)) = kd^{m-1}|V(G)|\) for all \(m \geq 2\) and \(0 \leq h \leq \min\{k, \left\lfloor \frac{d}{2} \right\rfloor\}\). From our results, the \((h, k)\)-domination numbers of \(d\)-ary (generalized) de Bruijn and Kautz digraphs are determined for \(0 \leq h \leq \min\{k, \left\lfloor \frac{d}{2} \right\rfloor\}\), which strengthen the previously known results on (generalized) de Bruijn and Kautz digraphs.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

In this note, a digraph may have loops, symmetric arcs, but not multiple arcs. Let \(G = (V, A)\) be a digraph and \(v \in V(G)\). The set of arcs incident to (respectively, incident from) \(v\) in \(G\) is denoted by \(A^-(v)\) (respectively, \(A^+(v)\)). The indegree \(deg^- v\) (respectively, outdegree \(deg^+ v\)) of \(v\) in \(G\) is \(|A^-(v)|\) (respectively, \(|A^+(v)|\)). The minimum degree \(\delta(G)\) of \(G\) is \(\min_{v \in V(G)} \min\{deg^- v, deg^+ v\}\). A digraph \(G\) is \(r\)-regular if \(deg^- v = deg^+ v = r\) for every vertex \(v\) of \(G\). The in-neighborhood \(N^- v\) (respectively, out-neighborhood \(N^+ v\)) of \(v\) is the set of vertices adjacent to (respectively, adjacent from) \(v\) in \(G\). The closed in-neighborhood \(N^- v\) (respectively, closed out-neighborhood \(N^+ v\)) of \(v\) is defined to be \(N^- v \cup \{v\}\) (respectively, \(N^+ v \cup \{v\}\)). A path (respectively, cycle) in this note means a directed path (respectively, directed cycle).

If there is an arc from \(u\) to \(v\), i.e., \((u, v) \in A(G)\), then we say that \(u\) is in-dominated by \(v\) and \(v\) is out-dominated by \(u\) in \(G\). Besides, we say that any vertex \(v\) is in-dominated and out-dominated by \(v\) itself. Let \(S \subseteq V(G)\). The subdigraph of \(G\) induced by \(S\) is denoted by \(S\). A digraph \(H\) is \((h, k)\)-connected if for any ordered pair of distinct vertices \(u\) and \(v\), there are at least \(h\) internally vertex-disjoint paths from \(u\) to \(v\) in \(H\). A \((h, k)\)-connected digraph is simply called a connected digraph. An\((h, k)\)-dominating set \(D\) in \(G\) is a subset \(D\) of \(V(G)\) such that \(|D|\) is \(h\)-connected, and for every vertex \(v\) of \(G\), \(v\) is in-dominated and out-dominated by at least \(k\) vertices in \(D\), i.e., \(|N^- [v] \cap D| \geq k\) and \(|N^+ [v] \cap D| \geq k\). The \((h, k)\)-domination number \(\gamma_{h,k}(G)\) of \(G\) is the minimum cardinality of an \((h, k)\)-dominating set in \(G\).

Dominating sets and its variations in undirected graphs or digraphs have extensively been studied [18], although compared with undirected graphs, there is a smaller number of results for dominating sets in digraphs [1, 2, 6, 7, 13, 21, 22]. The notion of \("(h, k)\)-dominating set" is related to three variations of a dominating set; “connected dominating set” [5], “\(k\)-tuple dominating set” [15] and “twinc dominating set” [6]. Namely, an \((h, k)\)-dominating set is corresponding to an \(h\)-connected \(k\)-tuple twin dominating set in a digraph. For undirected graphs, an \((h, k)\)-dominating set can be analogously defined as an
h-connected k-tuple dominating set.) A (0, 1)-dominating set is an ordinary (twin) dominating set and has an application to location problems of resources in communication networks. Also, a (1, 1)-dominating set finds an application to virtual backbone in wireless networks [27]. Now suppose that D is an (h, k)-dominating set in a digraph G, where h ≤ k. Then, for any S ⊆ V(G), if |S| < h (respectively, |S| < k), then D \ S is at least a (1, 1)-dominating set (respectively, (0, 1)-dominating set) in the digraph G − S obtained from G by deleting all the vertices in S. Thus, an (h, k)-dominating set can be applied to fault-tolerant location problems and fault-tolerant virtual backbone in networks. For undirected graphs, Shang et al. [25] studied (h, k)-dominating sets when h ≤ 2. For digraphs, Li et al. [21] investigated (1, 1)-dominating sets. Also, Araki [1, 2] studied (0, k)-dominating sets and (1, k)-dominating sets for de Bruijn and Kautz digraphs, and Shan et al. [24] studied (0, 1)-dominating sets in generalized de Bruijn digraphs.

For (u, v), (x, y) ∈ A(G), we say that (u, v) is adjacent to (x, y) in G if v = x. The line digraph L(G) of G is the digraph whose vertex set is the arc set of G and in which a vertex (u, v) is adjacent to a vertex (x, y) if and only if the arc (u, v) is adjacent to the arc (x, y) in G, i.e., V(L(G)) = A(G) and A(L(G)) = {((u, v), (v, w)) | (u, v), (v, w) ∈ A(G)}.

Theorem 2.3. For any arc-disjoint paths in L(G) of G.

Proof. Let D be an (h, k)-dominating set in L(G). By definition, for every vertex (u, v) of L(G), |N^−_{L(G)}((u, v)) ∩ D| ≥ k and |N^+_{L(G)}((u, v)) ∩ D| ≥ k. For each vertex v of L(G), let c(v) = |A^−_{L(G)}((v)) ∩ D| + |A^+_{L(G)}((v)) ∩ D|. Then, it holds that |D| = 1 2 ∑ v∈V(G) c(v), since each arc of D is counted twice in the sum ∑ v∈V(G) c(v).

Let |D| ≥ k and |A^−_{L(G)}((v)) ∩ D| ≥ k and |A^+_{L(G)}((v)) ∩ D| ≥ k, i.e., c(v) ≥ 2k. Consider the case that there is no loop at v. In this case, for any (u, v), (v, w) ∈ A(G), A^−_{L(G)}((v, w)) ∩ D = A^+_{L(G)}((v, w)) ∩ D = A^+_{L(G)}(v, w). Suppose that there are two arcs (u, v) and (v, w), both of which are not in D. Then, |A^−_{L(G)}((v, w)) ∩ D| = |N^−_{L(G)}((v, w)) ∩ D| = |N^+_{L(G)}((v, w)) ∩ D| ≥ k. Similarly, |A^+_{L(G)}((v, w)) ∩ D| = |N^+_{L(G)}((v, w)) ∩ D| ≥ k. Therefore, c(v) ≥ 2k. Suppose that every arc incident from v is in D. Then, |A^−_{L(G)}((v)) ∩ D| = deg^−_{L(G)}(v). Also, for any arc (v, w) of G, |A^+_{L(G)}((v, w)) ∩ D| = |N^+_{L(G)}((v, w)) ∩ D| = |N^+_{L(G)}((v, w)) ∩ D| − 1 ≥ k − 1. Thus, c(v) = |A^−_{L(G)}((v)) ∩ D| + |A^+_{L(G)}((v)) ∩ D| ≥ k − 1 + deg^−_{L(G)}(v) ≥ 2k. Similarly, if every arc incident to v is in D, then c(v) ≥ deg^−_{L(G)}(v) + k − 1 ≥ 2k. Therefore, for every vertex v of G, c(v) ≥ 2k. Hence, |D| = 1 2 ∑ v∈V(G) c(v) ≥ k|V(G)|.

A digraph H is h-arc-connected if for any ordered pair of distinct vertices u and v, there are at least h arc-disjoint paths from u to v in H. For arc-connectivity on digraphs, Su proved the following theorem.

Theorem 2.2 (Su [26]). Let G be an h-arc-connected digraph. Let v be a vertex of G and ℓ an integer with ℓ ≤ h. Then for any ℓ disjoint pairs of nonloop arcs (((u_1, v), (v, w_1)), . . . , ((u_{ℓ}, v), (v, w_{ℓ}))) where {u_1, . . . , u_{ℓ}} ⊆ N^−(v) and {w_1, . . . , w_{ℓ}} ⊆ N^+(v), there exist ℓ arc-disjoint cycles C_1, . . . , C_ℓ such that (((u_i, v), (v, w_i))) ∈ A(C_i) for each i and G − ∪_{i=1}^{ℓ} A(C_i) is (h − ℓ)-arc-connected.

A subdigraph F of a digraph G is a k-factor of G if F is k-regular and V(F) = V(G). From Lemma 2.1 and Su’s Theorem, we have the following result.

Theorem 2.3. If G has an h-arc-connected k-factor F where 0 ≤ h ≤ k < δ(G), then γ_{h,k}(L(G)) = k|V(G)| and A(F) is a minimum (h, k)-dominating set in L(G).

Proof. Let (u, v), (x, y) ∈ A(F). Suppose that u ≠ x. Since F is h-arc-connected, there are at least h arc-disjoint paths from u to x in F. This means that there are h internally vertex-disjoint paths from (u, v) to (x, y) in (A(F))_{L(G)}. Next, suppose that v = x, i.e., (u, v) is adjacent to (x, y) in L(G). Since F is h-arc-connected, there are h − 1 disjoint pairs of nonloop arcs (((u_1, v), (v, y_1)), . . . , ((u_{h−1}, v), (v, y_{h−1}))) such that u_i ≠ u and y_i ≠ y for i = 1, . . . , h − 1. Thus, from Su’s Theorem, in
Theorem 2.4. Let $G$ be a $d$-regular digraph and $1 \leq k < d$. If there are $p$ arc-disjoint Hamilton cycles in $L(G)$, then $\gamma_{h,k}(L^2(G)) = kd|V(G)|$ for $0 \leq h \leq \min\{k,p\}$.

Proof. Suppose that there are $p$ arc-disjoint Hamilton cycles in $L(G)$. We show that $L(G)$ can be decomposed into $p$ Hamilton cycles and $d - p$ 1-factors.

For each vertex $v$ at which there is no loop in $G$, let $K(v) = (A^-_c(v) \cup A^+_c(v))_{L(G)}$. If there is a pair of symmetric arcs $(u, v), (v, u)$ at $v$, then $K(v)$ is redefined to be the digraph obtained from $(A^-_c(v) \cup A^+_c(v))_{L(G)}$ by deleting the arc $(u, v)$ for $v$. For each vertex $w$ at which there is a loop in $G$, $K(w)$ is similarly defined except for regarding $(w, v) \in A^-_c(v)$ and $(v, w) \in A^+_c(w)$ as distinct vertices. Then, since $G$ is $d$-regular, for every vertex $v$ of $G$, the underlying undirected graph of $K(v)$ is isomorphic to the complete bipartite graph $K_{d,d}$. Besides, it can be easily checked that $A(L(G)) = V_{even}(G)A(K(v))$ and $A(K(v)) \cap K(A(v)) = \emptyset$ for any two distinct vertices $u, v \in V(G)$. Note that for each vertex $w$ at which there is a loop in $G$, the loop $(w, v, (w, w))$ in $L(G)$ corresponds to an arc from $(w, v) \in A^-_c(v)$ to $(w, v) \in A^+_c(v)$ in $K(w)$.

Let $C_1, \ldots, C_p$ be arc-disjoint Hamilton cycles in $L(G)$. For each $v \in V(G)$, let $K'(v)$ be the digraph obtained from $K(v)$ by deleting all the arcs in $K'(v) \cap (\cup_{i \in S_p} A(C_i))$. For each Hamilton cycle in $C_i$, $A(K'(v)) \cap A(G)$ is a matching with $d$ arcs. Thus, the underlying undirected graph of $K'(v)$ is a $(d - p)$-regular bipartite graph. Any $r$-regular bipartite graph with $2r$ vertices can be decomposed into $r$ matchings with $r$ edges. Then, for each $v \in V(G)$, suppose that $K'(v)$ is decomposed into $(d - p)$ matchings $M_1(v), M_2(v), \ldots, M_{d-p}(v)$. For each $1 \leq i \leq d - p$ and each vertex $(u, v)$ of $L(G)$, there exist exactly one arc incident to $(u, v)$ in $M_i(v)$ and exactly one arc incident from $(u, v)$ in $M_i(v)$. Therefore, the subdigraph $D_h$ of $L(G)$ induced by $\cup_{v \in V(G)} A(M_i(v))$ is a 1-factor of $L(G)$ such that $D_{1,2}, \ldots, D_{d-p}$ are arc-disjoint each other.

Constructing a union of $k$ factors from $p$ Hamilton cycles and $d - p$ 1-factors, for any $0 \leq h \leq \min\{k, p\}$, an $h$-arc-connected $k$-factor of $L(G)$ can be obtained. Thus, from Theorem 2.3, $\gamma_{h,k}(L^2(G))$ is determined to be $kd|V(G)|$ for $0 \leq h \leq \min\{k, p\}$.

Let $G$ be an Eulerian digraph. A 2-path in $G$ is a path of length 2 and denoted by a sequence of vertices $(u, v, w)$ in $G$. For a 2-path $(u, v, w)$ and an Euler tour $E$ in $G$, we then write $(u, v, w)$ in $E$. Let $V_4(G)$ be the set of vertices with indegree (outdegree) at least two in $G$, i.e., $V_4(G) = \{v \in V(G) \mid \deg^+_v \geq 2, \deg^-_v \geq 2\}$. For an Euler tour $E$ in $G$, let $T(v; E) = \{(u, v), (v, w)\}$ if $(u, v, w) \in E$. Let $E_1, E_2$ be Euler tours in $G$. If for every vertex $v \in V_4(G)$, $T(v; E_1) \cap T(v; E_2) = \emptyset$, then we say that $E_1$ and $E_2$ are compatible. Fleischner and Jackson presented the following theorem.

Theorem 2.5 (Fleischner and Jackson [12]). If $G$ is an Eulerian digraph and each vertex of $V_4(G)$ has indegree (outdegree) at least 2, then there are $\left\lceil \frac{d}{2} \right\rceil$ pairwise compatible Euler tours in $G$.

It can be easily checked that compatible Euler tours in $G$ correspond to arc-disjoint Hamilton cycles in $L(G)$ if $V_4(G) = V(G)$. Thus, the next corollary is obtained.

Corollary 2.6. Let $G$ be a connected $d$-regular digraph. Then, there are $\left\lceil \frac{d}{2} \right\rceil$ arc-disjoint Hamilton cycles in $L(G)$.

Fleischner and Jackson assume in their paper that a digraph has no loop, while in this note, a digraph may have a loop. However, Corollary 2.6 also holds for digraphs with loops. To see this, given a $d$-regular digraph $G$ with loops, we slightly modify $G$ to have no loop. Let $V_i$ be the set of vertices at which there is a loop in $G$. For each vertex $v \in V_i$, delete the loop $(v, v)$ and add a new vertex $v'$ with a pair of symmetric arcs $(v, v')$ and $(v', v)$. Let $G'$ be the resulting digraph. From Theorem 2.5, there are $\left\lceil \frac{d}{2} \right\rceil$ pairwise compatible Euler tours in $G'$. Note that for each vertex $v \in V_i$, $(v, v') \notin V_4(G)$. Thus, there are $\left\lceil \frac{d}{2} \right\rceil$ arc-disjoint Hamilton cycles in $L(G)$ such that they are arc-disjoint each other, except for the arcs in $\{(v, v'), (v', v)\}$ for $v \in V_i$. Each arc in $\{(v, v'), (v', v)\}$ is used in all the Hamilton cycles $C_1, C_2, \ldots, C_{\left\lceil \frac{d}{2} \right\rceil}$ in $L(G)$ such that they are arc-disjoint each other.

Suppose that $G$ is a connected $d$-regular digraph. Let $m \geq 2$. Since $L^{-2}(G)$ is also a connected $d$-regular digraph, from Corollary 2.6, there are $\left\lceil \frac{d}{2} \right\rceil$ arc-disjoint Hamilton cycles in $L^{m-1}(G)$. Thus, by Theorem 2.4, the $(h, k)$-domination number of $L^m(G)$ is determined to be $kd|V(L^{m-2}(G))|$ for $0 \leq h \leq \min\{k, \left\lceil \frac{d}{2} \right\rceil\}$. Consequently, we have the following theorem.

Theorem 2.7. Let $G$ be a connected $d$-regular digraph and $1 \leq k < d$. Then, $\gamma_{h,k}(L^m(G)) = kd^{m-1}|V(G)|$ for all $m \geq 2$ and $0 \leq h \leq \min\{k, \left\lceil \frac{d}{2} \right\rceil\}$. 

3. Applications

3.1. De Bruijn and Kautz digraphs

The complete symmetric digraph $K_n^*$ is the digraph with $n$ vertices such that for any ordered pair of distinct vertices $u$ and $v$, there is an arc from $u$ to $v$. The complete digraph $K_n^*$ is obtained from $K_n^*$ by adding a loop to each vertex. The de Bruijn digraph $B(d, D)$ and the Kautz digraph $K(d, D)$ can be defined to be $L^{D-1}(K_d^1)$ and $L^{D-1}(K_d^*(D-1))$, respectively.

When $d$ is odd, it is not difficult to decompose $K_d^*$ into $d - 1$ Hamilton cycles. When $d$ is even, Tillson [28] proved that $K_d^2$ can also be decomposed into $d - 1$ Hamilton cycles except for $d = 4, 6$. For $d = 4, 6$, it can be easily checked that $K_d^2$ (respectively, $K_d^*$) can be decomposed into 2 (respectively, 4) Hamilton cycles and a 1-factor. Thus, from Theorem 2.3, $\gamma_{h,k}(B(d, 2)) = k(d + 1)$ for $0 \leq h < k < d$. Since $K_d^2$ is decomposed into $K_d^*$ and the 1-factor consisting of $d$ loops, $\gamma_{h,k}(B(d, 2)) = k(d - 1)$ for $0 \leq h < k < d$ (respectively, $0 \leq h \leq \min\{k, d - 2\}$ if $d \neq 4, 6$ (respectively, $d = 4, 6$).

For $D \geq 3$, applying Theorem 2.7 to $K_d^*$ and $K_{d+1}^*$, we have the next theorem.

**Theorem 3.1.** Let $d$ and $k$ be integers such that $d \geq 2$ and $1 \leq k < d$. Then,

\[
\begin{align*}
\gamma_{h,k}(B(d, D)) &= kd^{D-1} & \text{for } D \geq 3 \text{ and } 0 \leq h \leq \min\left\{k, \left\lfloor \frac{d}{2} \right\rfloor \right\}, \\
\gamma_{h,k}(K(d, D)) &= k(d^{D-1} + d^{D-2}) & \text{for } D \geq 3 \text{ and } 0 \leq h \leq \min\left\{k, \left\lfloor \frac{d}{2} \right\rfloor \right\}.
\end{align*}
\]

Araki [1,2] showed that $\gamma_{0,k}(B(d, D)) = \gamma_{1,k}(B(d, D)) = k(d^{D-1} - d^{D-2})$. Thus, Theorem 3.1 strengthens Araki’s results for connectivity. Since any connected regular digraph has an Euler tour, Araki’s results in fact are obtained as corollaries of Theorem 2.4.

In some special cases, the connectivity condition in Theorem 3.1 can further be improved. Verrall [29] showed that $K_d^2$ has $d - 1$ (respectively, $d - 2$) compatible Euler tours if $d$ is odd (respectively, even). That is, $L(K_d^2)$ has $d - 1$ (respectively, $d - 2$) arc-disjoint Hamilton cycles if $d$ is odd (respectively, even). Therefore, from Theorem 2.4, $\gamma_{h,k}(B(d, 3)) = kd^2$ for $0 \leq h < k < d$ (respectively, $0 \leq h \leq \min\{k, d - 2\}$ if $d$ is odd (respectively, even). Also, Rowley and Bose [23] presented a construction of $d - 1$ arc-disjoint Hamilton cycles in $B(d, D)$ when $d$ is a power of two. Again, from Theorem 2.4, $\gamma_{h,k}(B(d, D)) = kd^{D-1}$ for $D \geq 3$ and $0 \leq h < k < d$ when $d = 2^t$ for some $t > 0$.

3.2. Generalized de Bruijn and Kautz digraphs

The generalized de Bruijn and Kautz digraphs are digraphs which generalize the de Bruijn and Kautz digraphs to have any number of vertices, respectively. Let $n \geq d \geq 2$. The generalized de Bruijn digraph $G_b(n, d)$ and the generalized Kautz digraph $G_k(n, d)$ are defined as follows: $V(G_b(n, d)) = V(G_k(n, d)) = \{0, 1, \ldots, n - 1\}$, $A(G_b(n, d)) = \{(x, y) \mid y = dx + i \pmod n, 0 \leq i < d\}$, and $A(G_k(n, d)) = \{(x, y) \mid y = -dx - i \pmod n, 1 \leq i \leq d\}$.

When $d$ divides $n$, i.e., $d|n$, the generalized de Bruijn and Kautz digraphs are known to be line digraphs. That is, $G_b(n, d)$ (respectively, $G_k(n, d)$) is isomorphic to $L(G_b(\frac{n}{d}, d))$ (respectively, $L(G_k(\frac{n}{d}, d))$). Thus, from Theorem 2.7, the next results are obtained.

**Proposition 3.2.** For $1 \leq k < d$ and $0 \leq h \leq \min\{k, \left\lfloor \frac{d}{2} \right\rfloor\}$,

\[
\begin{align*}
\gamma_{h,k}(G_b(n, d)) &= \frac{kn}{d} & \text{if } d^2|n, \\
\gamma_{h,k}(G_k(n, d)) &= \frac{kn}{d} & \text{if } d^2|n.
\end{align*}
\]

It has been shown by Du and Hwang [10] and Du et al. [9] that the generalized de Bruijn and Kautz digraphs are Hamiltonian, except for $G_b(n, 2)$ where $n$ is odd, and $G_k(n, 2)$ where $n$ is odd and is not a power of three. Therefore, from Theorem 2.3, we have the following results.

**Proposition 3.3.**

\[
\begin{align*}
\gamma_{1,1}(G_b(n, d)) &= \frac{n}{d} & \text{if } d|n, \\
\gamma_{1,1}(G_k(n, d)) &= \frac{n}{d} & \text{if } d|n,
\end{align*}
\]

except for $G_b(n, 2)$ where $\frac{n}{2}$ is odd, and $G_k(n, 2)$ where $\frac{n}{2}$ is odd and is not a power of three.

Shan et al. [24] proved that $\gamma_{0,1}(G_b(n, d)) = \frac{n}{d}$ if $d|n$. Thus, Proposition 3.3 extends the result of Shan et al. except for the case that $d = 2$ and $\frac{n}{2}$ is odd.
4. Concluding remarks

In this note, we have shown that the \((h, k)\)-domination number of an iterated line digraph \(L^m(G)\) of a connected \(d\)-regular digraph \(G\) is equal to \(kd^{m-1}|V(G)|\) for all \(m \geq 2\) and \(0 \leq h \leq \min\{k, \lfloor \frac{d}{2} \rfloor\}\). In order to strengthen our results for connectivity, we need to construct compatible Euler tours in \(G\), i.e., arc-disjoint Hamilton cycles in \(L(G)\), as many as possible. Fleischner and Jackson [12] conjectured that there are \(d-2\) compatible Euler tours in any \(d\)-regular digraph, which still remains open.

Knor and Niepel [20] investigated distance independent dominating sets in iterated line undirected graphs. It would be interesting to study \((h, k)\)-dominating sets in iterated line undirected graphs. Guha and Khuller [14] and Klasing and Laforest [19] showed hardness results for \((1, 1)\)-dominating sets and \((0, k)\)-dominating sets in undirected graphs, respectively, and they also proposed approximation algorithms. Besides, Li et al. [21] and Shang et al. [25] presented approximation algorithms for \((1, 1)\)-dominating sets in digraphs and \((h, k)\)-dominating sets where \(h \leq 2\) in undirected graphs, respectively. It would be also interesting to further investigate the \((h, k)\)-domination problem from an algorithmic point of view.

Acknowledgments

The authors are grateful to the anonymous referees for their helpful comments and suggestions that improved the paper.

References