The Semigroup Algebra and Representations of a Compact Simple Semigroup*

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INTRODUCTION

In this paper the author shows how to construct the semigroup algebra of a compact simple semigroup with at most an enumerable number of idempotents, and establishes certain properties of this algebra. The author further establishes a connection between the representations of such a semigroup and those of its semigroup algebra. In the final part of the paper the author investigates the problem of obtaining in a canonical manner, a representation of the semigroup $S$, starting from a representation of any one maximal group in $S$, and also establishes certain properties of this "induced representation" of $S$.

In so far as the author is aware, the problem of the construction of the semigroup algebra in the case of a compact simple semigroup has not been attacked before in its full generality. The usefulness of the group algebra in the case of a locally compact group is very well known. The semigroup algebra serves a similar purpose: it enables us to relate questions of representations of the semigroup (i.e., homomorphisms into operators on a Hilbert space) with questions of representations of a symmetric Banach algebra. The latter type of questions have been already thoroughly investigated.

While in the group case the interesting representations are representations into a group of unitary operators on a Hilbert space, in the case of our semigroups it is hardly possible to insist on dealing with unitary operators except as special cases. Rather we deal with operators on a Hilbert space, which have a locally unitary character (this will be made precise further on).

In the group case a certain measure (Haar measure) plays an important part in the group algebra. Similarly in the semigroup case, a certain measure (an idempotent measure) plays a vital role. With the help of such a measure

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we first introduce a notion of convolution \( f \ast g \) (Section 1) of functions \( f, g \) in \( L^1(d\mu) \) the linear space of complex valued functions on \( S \) integrable with respect to the measure. Defining the product \( f \cdot g \) by \( f \cdot g = f \ast g \), we make \( L^1(d\mu) \) an algebra. In Section 2 we introduce an involution in this algebra, and thus obtain a symmetric algebra (or algebra with involution) which we denote by \( L'(S) \). This section also deals with the question of uniqueness of the semigroup algebra. In Section 3 we construct approximate identities in \( L'(S) \), these have properties similar to those of approximate identities in the group case. In Section 4 we establish that this semigroup algebra is reduced and semisimple. In Section 5 we deal with representations of the semigroup, which have a locally unitary character and satisfy certain conditions. We succeed in establishing a relation between such representations of the semigroup and symmetric representations of the semigroup algebra. In the final section (Section 6) which is independent of the earlier sections we show that there exists a canonical way of obtaining a unitary representation (see Definition 5.1) \( ^*M^L \) of the semigroup \( S \), starting from a unitary representation \( L \) of any maximal group \( G_{ij} \) in \( S \). This "induced" representation \( ^*M^L \), restricted to any maximal group \( G_{s8} \) in \( S \), turns out to be equivalent to the original representation \( L \) of \( G_{ij} \).

**Notation.** Concerning the semigroups dealt with in this paper we shall adhere to current notation [1, 2]. We shall also use the notation followed in the author's paper [3]. Concerning normed algebras, the notation followed is frequently that used by Naimark [4].

**Basic Assumptions Throughout the Paper:** \( S \) is assumed to be a compact simple semigroup with at most an enumerable number of idempotents [1, 2]; \( \mu \) is assumed to be a regular Borel probability measure on \( S \) such that \( \Sigma(\mu) = S \) [5, 3] and such that \( \mu \) is idempotent on \( S \) [3].

Using such a measure we first consider \( L^1(d\mu) \) the linear space of complex valued functions \( f(s) \) on \( S \) such that \( \int_S |f(s)| \, d\mu(s) < \infty \). The next thing to do is to establish the notion of a convolution in \( L^1(d\mu) \) which will generalize the notion of a convolution of functions on a locally compact group.

1. **Convolution in \( L^1(d\mu) \)**

A regular Borel probability measure \( \mu \) on \( S \) with \( \Sigma(\mu) = S \) which is idempotent on \( S \), is known to be a product measure [5, 3, pp. 272–273 and further references]. Such a measure is also known to reduce to invariant measure (i.e., either Haar measure or the trivial null measure) on each maximal group \( G_{s8} \subset S \) [3, pp. 270–272].
Let \( f, g \in L^1(d\mu) \). Then we define the measures \( \nu_f, \nu_g \) by

\[
d\nu_f = f d\mu; \quad d\nu_g = g d\mu.
\]

Then the convolution \( \nu_f * \nu_g \), defined as in [3, p. 261], is a regular Borel measure on \( S \) (F. Riesz [6, p. 265]). We shall now prove the following theorem.

**Theorem 1.1.** \( \nu_f * \nu_g \) is absolutely continuous [7, p. 124] with respect to \( \mu \).

**Proof.** Suppose, without loss of generality, that both \( f \) and \( g \) are real valued. We can break up \( f \) and \( g \) into their positive and negative parts, as usual [7, p. 82]:

\[
f = f^+ - f^-
\]

\[
g = g^+ - g^-
\]

So, to begin with, suppose \( f \) and \( g \) to be both nonnegative.

Suppose \( f \) and \( g \) are both bounded. Let \( \mathcal{O} \) be any open set in \( S \). Let

\[
\mathcal{F} = \{ \varphi \mid \varphi \in C(S), 0 \leq \varphi \leq 1 \text{ and } \varphi = 0 \text{ outside } \mathcal{O} \}.
\]

Then

\[
(v_f * v_g)(\mathcal{O}) = \sup_{\varphi \in \mathcal{F}} \int_S \varphi(x) d(v_f * v_g)(x)
\]

\[
= \sup_{\varphi \in \mathcal{F}} \int_S \int_S \varphi(xy) f(x) g(y) d\mu(x) d\mu(y)
\]

\[
\leq M \sup_{\varphi \in \mathcal{F}} \int_S \varphi(x) d\mu(x)
\]

So if \( \mu(\mathcal{O}) = 0 \) then \( (v_f * v_g)(\mathcal{O}) = 0 \).

Next remove the restriction that \( f \) and \( g \) are bounded. Let \( \epsilon > 0 \). Then

\[
\exists M > 0 \quad \int_{f(x) > M} f(x) d\mu(x) < \epsilon; \quad \int_{g(y) > M} g(y) d\mu(y) < \epsilon.
\]
Hence, for \( \varphi \in \mathcal{F} \)

\[
\int \int_{S \times S} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y) = \int \int_{f(x) \leq M, \varphi(y) \leq M} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y)
\]

\[
+ \int \int_{f(x) > M, \varphi(y) \leq M} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y)
\]

\[
+ \int \int_{f(x) \leq M, \varphi(y) > M} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y)
\]

\[
\leq \int \int_{f(x) \leq M, \varphi(y) \leq M} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y) + K\varepsilon
\]

where \( K \) does not depend at all on the open set \( \mathcal{O} \). The last expression (\( \varphi \in \mathcal{F} \)) is

\[
\leq M^2 \int \int_{f(x) \leq M, \varphi(y) \leq M} \varphi(xy) \, d\mu(x) \, d\mu(y) + K\varepsilon
\]

\[
\leq M^2 \int \int_{S \times S} \varphi(xy) \, d\mu(x) \, d\mu(y) + K\varepsilon
\]

\[
\leq M^2 \int_{S} \varphi(x) \, d\mu(x) + K\varepsilon.
\]

Hence

\[
(\nu_{\alpha} \ast \nu_{\beta})(\mathcal{O}) \leq M^2 \mu(\mathcal{O}) + K\varepsilon
\]

which implies that

\[
\mu(\mathcal{O}) = 0 \Rightarrow (\nu_{\alpha} \ast \nu_{\beta})(\mathcal{O}) = 0.
\]

Next if \( E \) is any set in \( \mathcal{B}(S) \) (see [3, p. 259]), then \( \exists \) sequence of open sets \( \{\mathcal{O}_n\} \)

\[
E \subset \mathcal{O}_n, \ n = 1, 2, ...
\]

\[
\mu(\mathcal{O}_n) \rightarrow \mu(E) \quad \text{as} \quad n \rightarrow \infty,
\]

Hence

\[
(\nu_{\alpha} \ast \nu_{\beta})(E) \leq (\nu_{\alpha} \ast \nu_{\beta})(\mathcal{O}_n) \leq M^2 \mu(\mathcal{O}_n) + K\varepsilon.
\]

This implies that

\[
(\nu_{\alpha} \ast \nu_{\beta})(E) \leq M^2 \mu(E) + K\varepsilon.
\]
The theorem is thus proved for the case where \( f \) and \( g \) are both nonnegative. Let \( f \ast g \) denote the Radon-Nikodym derivative [7, p. 129] of \( \nu_f \ast \nu_g \) with respect to \( \mu \), in the case where \( f \) and \( g \) are both nonnegative.

Now if \( f \) and \( g \) are not necessarily nonnegative, we break up \( f \) and \( g \) into their positive and negative parts [7, p. 82]:

\[
\begin{align*}
  f &= f^+ - f^- \\
  g &= g^+ - g^-.
\end{align*}
\]

Now [7, p. 131]:

\[
  dv_f = (f^+ - f^-) \, d\mu = f^+ d\mu - f^- d\mu = dv_f^+ - dv_f^-.
\]

Similarly

\[
  dv_g = (g^+ - g^-) \, d\mu = g^+ d\mu - g^- d\mu = dv_g^+ - dv_g^-.
\]

The defining equation of \( \nu_f \ast \nu_g \) now becomes

\[
\begin{align*}
  \int_S \varphi(x) \, d(\nu_f \ast \nu_g)(x) &= \int_S \int_S \varphi(xy) \, dv_f^+(x) \, dv_g^+(y) + \int_S \int_S \varphi(xy) \, dv_f^-(x) \, dv_g^-(y) \\
  &\quad - \int_S \int_S \varphi(xy) \, f^+(x) g^+(y) \, d\mu(x) \, d\mu(y) - \int_S \int_S \varphi(xy) \, f^-(x) g^-(y) \, d\mu(x) \, d\mu(y) \\
  &= \int_S \varphi(x) \, d(\nu_f^+ \ast \nu_g^+)(x) + \int_S \varphi(x) \, d(\nu_f^- \ast \nu_g^-)(x) \\
  &\quad - \int_S \varphi(x) \, d(\nu_f^- \ast \nu_g^+)(x) - \int_S \varphi(x) \, d(\nu_f^+ \ast \nu_g^-)(x) \\
  &= \int_S \varphi(x) \, (f^+ \ast g^+)(x) \, d\mu(x) + \int_S \varphi(x) \, (f^- \ast g^-)(x) \, d\mu(x) \\
  &\quad - \int_S \varphi(x) \, (f^- \ast g^+)(x) \, d\mu(x) - \int_S \varphi(x) \, (f^+ \ast g^-)(x) \, d\mu(x) \\
  &= \int_s \varphi(x) \, (f \ast g)(x) \, d\mu(x),
\end{align*}
\]
where we have put

\[ f \ast g = (f^+ \ast g^+) - (f^- \ast g^+) - (f^+ \ast g^-) + (f^- \ast g^-). \]

This proves the theorem. Incidentally the argument used in the proof also shows that, for arbitrary \( f, f_1, f_2, g, g_1, g_2 \in L^1(d\mu) \):

\[
\begin{align*}
(f_1 + f_2) \ast g &= (f_1 \ast g) + (f_2 \ast g) \\
(f \ast (g_1 + g_2)) &= (f \ast g_1) + (f \ast g_2).
\end{align*}
\]

We can now formulate our first definition of the convolution of two functions in \( L^1(d\mu) \).

**Definition 1.2.** For given functions \( f, g \in L^1(d\mu) \), the Radon-Nikodym derivative of \( \nu_f \ast \nu_g \) with respect to the measure \( \mu \) is called the convolution of \( f \) and \( g \) and denoted by \( f \ast g \).

We could approach the problem of defining the convolution of \( f \) and \( g, f, g \in L^1(d\mu) \), in a somewhat different manner, as follows. Let \( \mu \) be as above. Let \( f, g \in L^1(d\mu) \). Then the integral

\[
\int_S \varphi(xy) f(y) \, d\mu(y),
\]

where \( \varphi \in C(S) \) and \( x \) is a fixed element in \( S \), defines a continuous linear functional on \( C(S) \). Hence (cf. [6, p. 265]) \( \exists \) regular Borel measure \( \mu_{f,x} \) on \( S \)

\[
\forall \varphi \in C(S), \int_S \varphi(xy) f(y) \, d\mu(y) = \int_S \varphi(y) \, d\mu_{f,x}(y).
\]

**Lemma 1.3.** The regular Borel measure \( \mu_{f,x} \) is absolutely continuous with respect to \( \mu \).

**Proof.** Let \( E \) be an open set in \( S \). Consider the equality

\[
\int_S \varphi(xy) f(y) \, d\mu(y) = \int_S \varphi(y) \, d\mu_{f,x}(y), \forall \varphi \in C(S).
\]

Let

\[
\mathcal{F} = \{ \varphi \in C(S) \mid 0 \leq \varphi \leq 1, \varphi = 0 \text{ outside } E \}.
\]

To begin with, we may assume that \( f \geq 0 \). Then from the above equality

\[
\sup_{\varphi \in \mathcal{F}} \int_S \varphi(xy) f(y) \, d\mu(y) = \sup_{\varphi \in \mathcal{F}} \int_S \varphi(y) \, d\mu_{f,x}(y) = \mu_{f,x}(E).
\]
But the left side $\leq \int_{x^{-1}G} f \, d\mu$. Since $\mu$ is idempotent, therefore $\mu$ is invariant measure (i.e., Haar measure or the trivial null measure) on each maximal group $G_{ab} \subseteq S$. The lemma thus follows in the case where $f \geq 0$. The proof is now completed by using a familiar argument, viz., for an arbitrary real valued $f \in L^1(d\mu)$, we consider separately the positive and negative parts of $f$.

Having established the last lemma, we could adopt the following definition (Definition A) as an alternative definition of the convolution $f \ast g$. However we shall not, in the sequel, adopt the possible Definition A. Rather, we shall formulate, a little further on, another definition (Definition 1.6) which is more satisfactory than Definition A. In the sequel we shall use both Definitions 1.2 and 1.6 and occasionally use a certain relation between Definitions 1.6 and A.

**DEFINITION A.** For fixed $x \in S$, let $\mu_{t,x}$ be as in the last lemma. Then by the generalized translate $A_{y}zf(y)$ of the function $f(y) \in L^1(d\mu)$ is meant the Radon-Nikodym derivative of the measure $\mu_{t,x} \cdot$ with respect to the measure $\mu(\cdot) : d\mu_{t,x}(y) = \{A_{y}zf(y)\} \, d\mu(y)$. (Here the subscript $y$ in "A$_{y}$" denotes that the operator $A_{y}^z$ acts on $f$ as a function of $y$.) The convolution $f \ast g$ of $f$ and $g$, where $f, g \in L^1(d\mu)$, is defined by:

$$(f \ast g)(y) = \int_S f(x) \{A_{y}zf(y)\} \, d\mu(y).$$

**COROLLARY 1.4.** The linear operator $A_{y}^z : f(y) \rightarrow A_{y}zf(y)$ is bounded:

$$\| A_{y}^z \| = \int_S |(A_{y}zf(y))| \, d\mu(y) \leq \int_{x^{-1}\mathcal{S}} |f(y)| \, d\mu(y) \leq \int_S |f(y)| \, d\mu(y).$$

Now let $f(x, y)$ be any continuous function on $S \times S$. Then for fixed $g \in L^1(d\mu)$, the double integral

$$\iint_{S \times S} f(x, xy) \, g(y) \, d\mu(x) \, d\mu(y)$$

defines a continuous linear functional on $C(S \times S)$, and hence there exists a unique regular Borel measure $\nu_{\phi}$ on the product Borel field $\mathcal{B}(S) \times \mathcal{B}(S)$ such that

$$\iint_{S \times S} f(x, xy) \, g(y) \, d\mu(x) \, d\mu(y) = \iint_{S \times S} f(x, y) \, d\nu_{\phi}(x, y).$$
Theorem 1.5. The measure \( \nu_0 \) is absolutely continuous with respect to the measure \( \mu \times \mu \).

Proof. Let \( E = A \times B \), where \( A \) and \( B \) are open sets in \( S \). As before we consider first the case where \( g \geq 0 \). Let

\[ \mathcal{F} = \{ f(x, y) \in C(S \times S) \mid 0 \leq f \leq 1 \text{ and } f = 0 \text{ outside } E \}. \]

Then

\[
\sup_{f \in \mathcal{F}} \iint_{S \times S} f(x, xy) g(y) \, d\mu(x) \, d\mu(y) = \sup_{f \in \mathcal{F}} \iint_{S \times S} f(x, y) \, dv_0(x, y) = v_0(E).
\]

Also

\[
\sup_{f \in \mathcal{F}} \iint_{S \times S} f(x, xy) g(y) \, d\mu(x) \, d\mu(y)
\leq \int_S \left[ \sup_{f \in \mathcal{F}} \int f(x, xy) g(y) \, d\mu(y) \right] \, d\mu(x)
\leq \int_A \left[ \int_{x^{-1}B} g(y) \, d\mu(y) \right] \, d\mu(x).
\]

Here \( x^{-1}B \) is defined in the usual way (cf. [3, p. 269]). Now let \( \varepsilon > 0 \). The integral \( \int g(y) \, d\mu(y) \) is uniformly absolutely continuous, i.e., \( \exists \delta > 0, \exists \varepsilon \) such that \( \mu(E) < \delta \Rightarrow \int g(y) \, d\mu(y) < \varepsilon \). Consider a fixed \( x \in S \), and let \( S_x \) be the minimal right ideal containing \( x \). Then (with the usual definition of \( S_x^{-1} \); cf. [3, p. 269])

\[ x^{-1}B = x^{-1}(B \cap S_x). \]

Consider one minimal left ideal \( S'_x \). Using the fact that \( \mu \) reduces to invariant measure (Haar measure or the trivial null measure) on each maximal group \( G_{\alpha} = S_x \cap S_{\beta} \), it follows easily that

\[ \mu(x^{-1}(B \cap S_x \cap S'_x)) = \mu(S'_x) \mu(B \cap S_x \cap S'_x). \]

So

\[
\mu(x^{-1}B) \leq \sum_j \mu(S'_x) \mu(B \cap S_x \cap S'_x)
\leq \mu(B) \sum_j \mu(S'_x)
\leq \mu(B).
\]

Choosing \( A \ni \mu(A) \leq \eta \) (where \( \eta \leq 1 \)) and \( B \ni \mu(B) < \delta \), the theorem
follows in the case where $g \geq 0$. The proof is now completed by considering, for real valued $g$, separately the positive and negative parts of $g$.

We can now formulate our alternative definition of the convolution $f \ast g$.

**Definition 1.6.** The Radon-Nikodym derivative of $\nu_y$ with respect to $\mu \times \mu$ is called the generalized translate of $g(y)$ and is denoted by $\mathcal{A}_y g(y)$ (where the subscript $y$ in the symbol ‘‘$\mathcal{A}_y$’’ denotes that the operator $\mathcal{A}_y$ acts on $g(y)$ as a function of $y$). The convolution of $f$ and $g$, where $f, g \in L^1(d\mu)$ is defined by

$$(f \ast g)(y) = \int_S f(x) \mathcal{A}_y g(y) \, d\mu(x).$$

The question of existence of the integral in Definition 1.6 is settled in the next theorem. Also the following question naturally arises: what is the relation between Definitions 1.6 and $A$? We shall not attempt a complete investigation of this question, but we shall be content with deducing just what relation is needed for our purpose in the sequel.

In Definition 1.6 take $f(x, y) = \varphi(x) \chi(y)$ where $\varphi(x)$, $\chi(y)$ are, respectively, continuous functions of $x$ and $y$ alone. Let $g \in L^1(d\mu)$. Then (Fubini)

$$\int_S \varphi(x) \chi(xy) g(y) \, d\mu(x) \, d\mu(y) = \int_S \varphi(x) \left\{ \int_S \chi(y) \mathcal{A}_y g(y) \, d\mu(y) \right\} \, d\mu(x).$$

On the other hand the left side of this equations (using Definition $A$)

$$= \int_S \varphi(x) \left\{ \int_S \chi(xy) g(y) \, d\mu(y) \right\} \, d\mu(x)$$

$$= \int_S \varphi(x) \left\{ \int_S \chi(y) A_y g(y) \, d\mu(y) \right\} \, d\mu(x).$$

Therefore for arbitrary continuous $\varphi, \chi$ on $S$,

$$\int_S \varphi(x) \left\{ \int_S \chi(y) \mathcal{A}_y g(y) \, d\mu(y) \right\} \, d\mu(x)$$

$$= \int_S \varphi(x) \left\{ \int_S \chi(y) A_y g(y) \, d\mu(y) \right\} \, d\mu(x).$$

It follows (Dunford and Schwartz [6, p. 265]) that for almost all $(\mu)x$,

$$\mathcal{A}_y g(y) = A_y g(y)$$

for almost all $(\mu)y$.

This enables us to use Lemma 1.3 and Corollary 1.4 and establish the next theorem.
**Theorem 1.7.** The convolution $f \ast g$ given by Definition 1.6 coincides with the convolution given by Definition 1.2.

**Proof.** For any function $\varphi \in C(S)$, and $f, g \in L^1(d\mu)$

$$\int_S \varphi(y)(f \ast g)(y) \, d\mu(y) = \int_S \int_S \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y)$$

$$= \int_S f(x) \left( \int_S \varphi(xy) g(y) \, d\mu(y) \right) \, d\mu(x)$$

$$= \int_S f(x) \left( \int_S \varphi(y) \mathcal{L}_y \ast g(y) \, d\mu(y) \right) \, d\mu(x)$$

$$= \int_S \varphi(y) \left( \int_S f(x) \mathcal{L}_y \ast g(y) \, d\mu(x) \right) \, d\mu(y)$$

where we have invoked a theorem of Tonelli [6, p. 194, Corr. 15] to justify the change in the order of integration. This proves the theorem.

**Note.** We may similarly define the generalized translates $\mathcal{L}_y \ast g(x)$, $\mathcal{L}_{y'} \ast g(x)$ by the following:

$$\forall \varphi \in C(S), \int_S \varphi(xy) g(x) \, d\mu(x) = \int_S \varphi(x) \{ \mathcal{L}_y \ast g(x) \} \, d\mu;$$

$$\forall f \in C(S \times S), \int_S \int_S f(xy, y) g(x) \, d\mu(x) \, d\mu(y)$$

$$= \int_S \int_S f(x, y) \{ \mathcal{L}_y \ast g(x) \} \, d\mu(x) \, d\mu(y).$$

It follows that for $f, g \in L^1(d\mu)$

$$(f \ast g)(x) = \int_S \{ \mathcal{L}_x \ast f(x) \} g(y) \, d\mu(y).$$

Having defined the convolution $f \ast g, f, g \in L^1(d\mu)$ we find that the following rules hold:

1. $\alpha(f \ast g) = \alpha f \ast g$ \quad $\alpha$ a constant.

2. $\| f \ast g \|_1 \leq \| f \|_1 \| g \|_1$. 
For (Dunford and Schwartz [1, p. 262]),

$$\| f \ast g \|_1 = \sup_{\| \varphi \| \leq 1} \left| \int_S \varphi(y)(f \ast g)(y) \, d\mu(y) \right|$$

$$= \sup_{\| \varphi \| \leq 1} \left| \int_S \int_S \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y) \right|$$

$$\leq \int_S |f(x)| |g(y)| \, d\mu(x) \, d\mu(y).$$

3. \((f \ast g) \ast h = f \ast (g \ast h)\).
4. \((f + g) \ast h = (f \ast h) + (g \ast h)\).
5. \(f \ast (g + h) = (f \ast g) + (f \ast h)\).

The verification of 3–5 is simple and therefore omitted.

Notation. The set of all functions in \(L^1(d\mu)\), with the product \(f \cdot g\) defined by:

\(f \cdot g = f \ast g\) (cf. Definitions 1.2 and 1.6) forms a Banach algebra (Naimark [4, p. 176]) which in the sequel we shall denote by \(L^1(S)\).

**2. Involution in \(L^1(S)\)**

Our next object is to define an involution in \(L^1(S)\). We first need the following theorem.

**Theorem 2.1.** The mapping \(L^1(d\mu) \to L^1(d\mu \times d\mu)\) defined by:

\(f(y) \to \alpha_{y}^* f(y)\), is one-one.

**Proof.** Suppose \(\alpha_{y}^* f(y) = 0\) a.e. This assumption implies that for almost all \(x\)

\(\alpha_{y}^* f(y) = 0\) for almost all \(y\).

Consider one such \(x\) for which \(\alpha_{y}^* f(y) = 0\) a.e. Let \(G_{a\beta}\) be a maximal group in \(S\), lying in the same minimal right ideal which contains \(x\). Let \(C \subset G_{a\beta}\) be a compact set and let \(\emptyset\) be an open set in \(S \ni \emptyset \supset C\) and \(\mu(\emptyset - C) < \eta\) where \(\eta > 0\). Then \(\alpha_{y}^* f(y) = 0\) a.e. implies that for any \(\varphi \in C(S)\)

\(\int_S \varphi(xy) f(y) \, d\mu(y) = 0\).

Let \(\varphi\) be a continuous function on \(S \ni \varphi = 1\) on \(C\), \(\varphi = 0\) outside \(\emptyset\) and
$0 \leq \varphi \leq 1$ (Urysohn). The last equation implies that for this particular function $\varphi$

$$0 = \int_{x^{-1}C} f(y) \, d\mu(y) - \int_{x^{-1}(\varnothing - C)} \varphi(xy) f(y) \, d\mu(y).$$

Therefore

$$\left| \int_{x^{-1}C} f(y) \, d\mu(y) \right| \leq \int_{x^{-1}(\varnothing - C)} \varphi(xy) f(y) \, d\mu(y) \leq \int_{x^{-1}(\varnothing - C)} |f(y)| \, d\mu(y).$$

Arguing as before, we see that $\mu(\varnothing - C) < \eta$ implies that

$$\mu(x^{-1}(\varnothing - C)) < \mu(S_\beta')\eta$$

where $S_\beta'$ is the minimal left ideal containing $G_{ab}$. Hence $\int_{x^{-1}(\varnothing - C)} |f(y)| \, d\mu(y)$ can be made as small as we please by choosing $\eta > 0$ sufficiently small. Hence

$$\int_{x^{-1}C} f(y) \, d\mu(y) = 0.$$

If $f \geq 0$, we obtain from the last equation that by choosing $C$ suitably, $f = 0$ a.e. on the minimal left ideal containing $G_{ab}$. This is true of every minimal left ideal in $S$. Hence the theorem is proved in the case where $f \geq 0$.

In the case of an arbitrary $f \in L^1(d\mu)$, we separate $f$ into its positive and negative parts: $f = f^+ - f^-$. We first show that

$$\mathcal{A}_\varphi f^+(y) = \{\mathcal{A}_\varphi f(y)\}^+$$
$$\mathcal{A}_\varphi f^-(y) = \{\mathcal{A}_\varphi f(y)\}^-.$$

To prove this we note that

$$\int_S \varphi(xy) f(y) \, d\mu(y) = \int_S \varphi(xy) \{f^+(y) - f^-(y)\} \, d\mu(y)$$

$$= \int_S \varphi(y)[\mathcal{A}_\varphi f^+(y) - \mathcal{A}_\varphi f^-(y)] \, d\mu(y).$$

The left side also

$$= \int_S \varphi(y) \mathcal{A}_\varphi f(y) \, d\mu(y)$$
$$= \int_S \varphi(y)[\{\mathcal{A}_\varphi f(y)\}^+ - \{\mathcal{A}_\varphi f(y)\}^-] \, d\mu(y).$$
Now recall that $f(y) \geq 0 \Rightarrow \partial_y^z f(y) \geq 0$ a.e. Let $C$ be a compact set on which $\partial_y^z f(y) > 0$ and $\mathcal{O}$ an open set $\ni \mathcal{O} \cap C$ and $\mu(\mathcal{O} - C) < \eta$ where $\eta > 0$. Let $\varphi$ be a continuous function on $S \ni \varphi = 1$ on $C$ and $\equiv 0$ outside $\mathcal{O}$, and $0 \leq \varphi \leq 1$. Since $\eta > 0$ is arbitrary we obtain that

$$\int_C \{\partial_y^z f(y)\}^+ \, d\mu(y) = \int_C \{\partial_y^z f^+(y) - \partial_y^z f^-(y)\} \, d\mu(y).$$

If $C$ were to contain a compact subset $C_1$ on which $\partial_y^z f^-(y) \neq 0$ then arguing as before we find that

$$\int_{C_1} \{\partial_y^z f(y)\}^+ \, d\mu(y) = \int_{C_1} \partial_y^z f^-(y) \, d\mu(y).$$

The left side is $> 0$, the right side $< 0$, which is impossible unless $\mu(C_1) = 0$. It follows that

$$\{\partial_y^z f(y)\}^+ = \partial_y^z f^+(y) \quad \text{a.e.}$$

$$\{\partial_y^z f(y)\}^- = \partial_y^z f^-(y) \quad \text{a.e.}$$

From this it follows that

$$\partial_y^z f(y) \to 0 \quad \text{a.e.} \Rightarrow \{\partial_y^z f(y)\}^+ = 0 \quad \text{a.e.} \quad \text{and} \quad \{\partial_y^z f(y)\}^- = 0 \quad \text{a.e.} \quad \Rightarrow \partial_y^z f^+(y) = 0 \quad \text{a.e.} \quad \text{and} \quad \partial_y^z f^-(y) = 0 \quad \text{a.e.}$$

This proves the theorem.

The following theorem is proved similarly.

**Theorem 2.2.** The mapping $L^1(d\mu) \to L^1(d\mu \times d\mu)$ defined by

$$f(y) \mapsto \partial_y^z \partial f(y),$$

is one-one.

It is convenient at this point to investigate the nature of the functions $\partial_y^z f(s')$ and $\partial_y^z \partial f(s')$ for a given continuous function $f$ on $S$.

We make use of the product space representation of $S$ [2]:

$$S \approx T \times X \times Y.$$ 

Let $s = (t, x, y), \ s' = (t', x', y').$ Let $\varphi_1(t), \varphi_2(x), \varphi_3(y)$ be continuous functions, respectively, of $t, x, y$ alone. By virtue of the fact that $\mu$ is a product measure on $S$ [3], and that

$$S \times S' \approx (T \times X \times Y) \times (T' \times X' \times Y')$$
and also $S \times S' \cong (T \times X' \times Y) \times (T' \times X \times Y)$ we obtain the following relation

$$\int_{(T \times X \times Y) \times (T' \times X' \times Y)} \varphi_1(tx'y't') \varphi_2(x') \varphi_3(y') f(t', x', y') \, d\chi(t') \, d\chi(t') \, d\alpha(x) \, d\beta(y)$$

$$\times \, d\alpha(x') \, d\beta(y')$$

$$= \int_{(T \times X \times Y) \times (T' \times X' \times Y)} \varphi_2(x') \varphi_3(y') \int_{T \times T} \varphi_1(tx'y't') f(t', x', y') \, d\chi(t) \, d\chi(t')$$

$$\times \, d\alpha(x) \, d\beta(y) \, d\alpha(x') \, d\beta(y').$$

Since finite sums of such products $\varphi_1(t)\varphi_2(x)\varphi_3(y)$ are uniformly dense in $C(S)$ (Stone-Weierstrass, cf. [6, p. 349]) it follows (after a slight change of notation that)

$$\mathcal{A}^s f(s') = f(\{txy\}^{-1}t', x', y).$$

Similarly

$$\mathcal{A}^s f(s') = f(t'\{xyt\}^{-1}, x, y').$$

An immediate consequence of this representation of $\mathcal{A}_s^s f(s')$ and $\mathcal{A}^s f(s')$ is the following lemma.

**Lemma 2.3.** If $f(\cdot)$ is a continuous function on $S$ then so are the functions $\mathcal{A}_s^s f(s')$ and $\mathcal{A}^s f(s')$ on $S \times S$.

**Remark.** We note in passing that the above representations for $\mathcal{A}_s^s f(s')$ and $\mathcal{A}^s f(s')$ would be valid for arbitrary $f \in L^1(d\mu)$ provided the symbols

$$f(\{txy\}^{-1}t', x', y), f(t'\{xyt\}^{-1}, x, y')$$

represented measurable functions of $(t, x, y; t', x', y')$.

We now turn to the process of defining an involution $f \to f^*$ in $L^1(S)$. We shall first do this for characteristic functions of Baire sets and then by approximation, to arbitrary functions in $L^1(S)$.


The criterion we adopt for defining $f^*$, for a given $f \in L^1(S)$, is the fulfillment of the following relation:

$$\mathcal{O}_{y} f^*(y) = \mathcal{O}_{x} f^*(x)$$

for almost all $(\mu \times \mu)$ points $(x, y) \in S \times S$. One reason for doing this is simply the fact that this relation is certainly valid in the case of a group with the usual definition of involution on a group. Further if this relation were to be true then the adjoint $A_n^*$ of the operator $A_n$ to be defined later (in Section 4) for certain functions $\xi \in L^2(d\mu)$ would be easily identified with $A_n^*$.

The arguments used in Lemma 2.3 yield

**Lemma 2.3(a).** If $f$ is a Baire measurable function in $L^1(S)$ then

$$\mathcal{O}_{s} f(s') = f(t'y^{-1}, x, y)$$

with $s = (t, x, y)$, $s' = (t', x', y')$. Hence if $f$ is the characteristic function of a Baire set in $S$, then $\mathcal{O}_{y} f(y)$ is also the characteristic function of a Baire set in $S \times S$. From this it follows that if $f$ is the characteristic function of a Borel set in $S$, then $\mathcal{O}_{y} f(y)$ is also the characteristic function of a Borel set in $S \times S$.

On the other hand, for any Borel function $f \in L^1(S)$ there exists a sequence $\{f_n\}$ of Baire functions $f_n \to f$ in $L^1(\mu \times \mu)$, and there exists a subsequence $\{f_{n_k}\}$ of $f_{n_k} \to \mathcal{O}_{y} f(y)$ almost all $(\mu \times \mu)$ points $(x, y)$. Hence

**Lemma 2.4(a).** If, for a function $f(y) \in L^1(S)$, $\mathcal{O}_{y} f(y)$ is the characteristic function of a Borel set in $S \times S$, then $f(y)$ must also be the characteristic function of a Borel set in $S$.

Next, using Theorem 2.1 we obtain

**Lemma 2.4(b).** The mapping $f(y) \to \mathcal{O}_{y} f(y)$ induces a map of Borel sets in $S$ into Borel sets in $S \times S$ (modulo sets of measure 0) such that disjoint Borel sets in $S$ are mapped into disjoint Borel sets in $S \times S$, and that the measures $\mu(E)$ and $(\mu \times \mu)(F)$ of corresponding Borel sets $E \subset S$, $F \subset S \times S$, are equal.

Before continuing with the process of defining $f^*$ for arbitrary given $f \in L^1(S)$, we shall make here some remarks concerning the points $(t'y^{-1}, x, y')$ associated with pairs of points $(s, s') \in S \times S$, with $s = (t, x, y)$, $s' = (t', x', y')$ (see above). We notice that the point
(t’{xyt}^{-1}, x, y') is a continuous mapping $T(s, s')$ of the point $(s, s') \in S \times S$ into $S$:

$$T(s, s') = (t’{xyt}^{-1}, x, y').$$

Hence if $f$ is a Baire measurable function in $L^1(S)$

$$\mathcal{A}_{s'}^s f(s') = f(T(s, s')) = f \cdot T(s, s'),$$

where $a \circ b$ denotes the composite of $a$ with $b$, $a$ and $b$ being functions. Similarly for arbitrary Baire measurable function $g \in L^1(S)$

$$\mathcal{A}_{s'}^s g(s') = g((txy)^{-1} t', x', y) = g \cdot T_1(s, s'),$$

where $T_1$ is a continuous mapping of $S \times S$ into $S$:

$$T_1(s, s') = ((txy)^{-1} t', x', y).$$

Now, using a familiar notation (cf. Wallace [2]), we can assert the following.

**Lemma 2.5.** (a) $T_1(s, s) = \eta(s); T(s, s) = \eta(s)$ where $\eta(s)$ denotes the identity of the unique maximal group to which $s$ belongs. (b) $T(s, s')T(s', s) = \eta(s'); T(s', s')T(s, s') = \eta(s)$.

We shall prove only the first assertion of (a); the others follow similarly. Using now a somewhat different but again familiar notation (cf. Wallace [2], also [3]) let $s = e_2 e_1 (t, e_1, e_2)$, where $t$ is an element of the maximal group $T = eSe$, $e$ being an idempotent in $S$, $e_1$ is an idempotent in the minimal right ideal $S_1 = eS$, and $e_2$ an idempotent in the minimal left ideal $S_2 = eS$. Here $e := e_{ij}$ is the identity of the group $G_{ij} = S_i S_j' = eSe$. Let $e_1$ be the identity of the maximal group $G_{ij} \subseteq S_i$, and $e_2$ the identity of the maximal group $G_{ai} \subseteq S_i$. Then $\exists$ elements $x_{ib} \in G_{ip}$ and $y_{aj} \in G_{aj}$, $\exists x_{ib} y_{aj} = e_{ij}$. It follows that $y_{aj} x_{ib} = e_{aj}$ where $e_{ab}$ the identity of the group $G_{ab} = S_a S_b'$ which contains the element $s$. Then, denoting by $\theta(x)$ the inverse of $x$ in the unique maximal group to which $x$ belongs [2], we find:

$$e_1 e_2 = \theta(x_{ib}) x_{ib} y_{aj} \theta(y_{aj})$$

$$= \theta(x_{ib}) e_{ij} \theta(y_{aj})$$

$$= (\theta(x_{ib}) e_{ij}) (e_{ij} \theta(y_{aj})).$$

And

$$(e_1 e_2)^{-1} = \theta(e_1 e_2)$$

$$= (e_{ij} \theta(y_{aj}))^{-1} (\theta(x_{ib}) e_{ij})^{-1}$$

$$= (e_{ij} y_{aj}) (x_{ib} e_{ij}) = e_{ij} e_{aj} e_{ij}.$$

Then

$$T_1(s, s) = e_2 e_{ij} e_{aj} e_{ij} e_1$$

$$= e_2 e_{aj} e_1 = e_{aj} e_1 = e_{ab}.$$
where we have used the fact that every idempotent in a minimal right ideal is a left identity for all elements of that right ideal, and correspondingly, every idempotent in a minimal left ideal is a right identity for all elements of that left ideal.

To return to the process of defining $f^*$ for a given $f \in L^1(S)$, let $E$ be a Baire set in $S$, and $f$ its characteristic function. Let $F$ be the Baire set in $S \times S$ of which $\mathcal{O}^\mu_x f(s')$ is the characteristic function. Let

$$E^* = \{ s_0 \in S : s_0 = T(s, s'), (s, s') \in F \}.$$ 

Then $E^*$ is a Baire set. Since $\mu(E) = (\mu \times \mu)(F)$, and $\mu(E^*) = (\mu \times \mu)(F)$, it follows that $\mu(E) = \mu(E^*)$. We make $E^*$ correspond to $E$, and the characteristic function $f^*$ of $E^*$ (modulo a set of measure 0) correspond to $f$. This correspondence is then extended to a correspondence which maps the characteristic function $f$ of a Borel set $E$ into the characteristic function $f^*$ of a Borel set $E^*$ and $\| f \|_1 = \| f^* \|_1$. Also the induced correspondence which maps a Borel set $E$ into its corresponding Borel set $E^*$ (modulo a set of measure 0), preserves measure; $\mu(E) = \mu(E^*)$; and maps disjoint Borel sets into disjoint Borel sets (modulo sets of measure 0). These considerations yield the following.

**LEMMA 2.6(a).** If $f$ is the characteristic function of a Borel set $E$, then $\exists \ | f^*, f^*$ also being the characteristic function of a Borel set $E^*$, such that

$$\mathcal{O}^\mu_y f(y) = \mathcal{O}^\mu_x f^*(x).$$

for almost all $(\mu \times \mu)$ points $(x, y) \in S \times S$, and $\mu(E) = \mu(E^*)$.

**LEMMA 2.6(b).** If $f$ is a simple function (i.e. a finite linear combination of characteristic functions of disjoint Borel sets), then $\exists \ |$ simple function $f^*$ such that

$$\mathcal{O}^\mu_y f(y) = \mathcal{O}^\mu_x f^*(x)$$

for almost all $(\mu \times \mu)$ points $(x, y)$, and $\| f \|_1 = \| f^* \|_1$.

**LEMMA 2.6(c).** If $f$ is any function in $L^1(S)$ then $\exists$ sequence $(f_n)$ of simple functions $\exists \| f_n - f \|_1 \rightarrow 0$. Then $\| f_m - f_n \|_1 \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore $\| f_m^* - f_n^* \|_1 \rightarrow 0$ as $m, n \rightarrow \infty$, and hence $\exists f^* \exists \| f_n^* - f^* \|_1 \rightarrow 0$. This $f^*$ satisfies

$$\mathcal{O}^\mu_y f(y) = \mathcal{O}^\mu_x f^*(x).$$
for almost all \((\mu \times \mu)\) points \((x, y)\), and \(\|f\|_1 = \|f^*\|_1\). Also \(f^*\) does not depend on the choice of the particular sequence \(\{f_n\}\) approximating to \(f\).

To sum up, we have established the following.

**Theorem 2.7.** To each function \(f \in L^1(d\mu)\), \(\exists f^* \in L^1(d\mu)\) satisfying

(i) \(\|f\|_1 = \|f^*\|_1\)
(ii) \((f + g)^* = f^* + g^*\)
(iii) \((af)^* = \bar{a} f^*\)
(iv) \(\mathcal{A}_y f(x) = \mathcal{A}_y f^*(y)\)
(v) \((f^*)^* = f\)
(vi) \((f \cdot g)^* = g^* \cdot f^*\).

Only property (vi) remains to be proved. The proof of (vi) is based on certain results that are proved later in this paper but do not make any use of property (vi) of the involution.

In the proof of Theorem 4.3, we construct an algebra of bounded operators \(A_n\) on a Hilbert space \(L^2(d\mu)\), for \(a \in L^1(S)\). The mapping \(a \rightarrow A_n\), for \(a \in L^1(S)\), preserves sums, products (i.e., convolutions) and \(A_n^* = A_n^*\) and is further one-one. So let \(a, b \in L^1(S)\). Then

\[
A_{(a \cdot b)}^* = (A_{a \cdot b})^* = (A_a A_b)^* = A_b^* A_a^* = A_b^* A_a = A_{a^* b^*}.
\]

Hence

\[
(a \cdot b)^* = b^* \cdot a^*.
\]

This proves (vi). Thus Theorem 2.7 is completely proved.

Having defined the concept of an involution in \(L^1(S)\), if we look back at the contention proved after Lemma 2.3, we see that the involution which we have defined in \(L^1(S)\) does generalize the notion of an involution on a group.

From the proof of Theorem 2.7 we also find that to each single-point set \(\{s\}\), \(s \in S\), \(\exists\) single-point set \(\{s^*\}\), \(s^* \in S\), defined uniquely up to a set of measure 0, such that \(\{s^*\} \rightarrow \{s\}\) up to a set of measure 0, \(d\mu(s^*) = d\mu(s)\), and \(a^*(s) = \overline{a(s^*)}\) for almost all \(s \in S\), \(a \in L^1(S)\).
DEFINITION 2.8. The algebra $L^1(S)$ with the involution $f ightarrow f^*$ (cf. Theorem 2.7) forms a symmetric Banach algebra (see Naimark [4, p. 184]). We adjoin an identity $e$ to $L^1(S)$ if necessary. The extended algebra thus obtained is called the semigroup algebra (semigroup ring) of $S$ and denoted by $R(S)$.

In this definition we notice that we have indulged in a slight abuse of language, and, before proceeding to the next section, a remark concerning the uniqueness of the symmetric Banach algebra $L^1(S)$ would be in order. The only way in which nonuniqueness could arise is through the non-uniqueness of the measure $\mu$. So let $\mu, \nu$ be two measures on $S$, each fulfilling our earlier assumptions (see Introduction), and let $L^1_\mu(S)$, $L^1_\nu(S)$ be the symmetric Banach algebras constructed as above with the respective measures $\mu, \nu$. We recall here the property of such measures (cf. [3, pp. 270–273]). We shall assume, for the remaining part of this section only, that neither $\mu$ nor $\nu$ reduces to the trivial null measure on any of the maximal groups $G_{ab}^+$ of $S$. With these assumptions about $\mu$ and $\nu$, we see that sets of $\mu$-measure 0 have $\nu$-measure 0, and vice-versa.

Let $f \in L^1_\mu(S)$. Let $U$ denote the mapping $L^1_\mu(S) \rightarrow L^1_\nu(S)$ defined as follows. For $x$ belonging to the maximal group $G_{ij}$ define

$$(Uf)(x) = f(x)^{\frac{\mu(G_{ij})}{\nu(G_{ij})}}.$$ 

Then $Uf \in L^1_\nu(S)$ and $\int_S f \, d\mu = \int_S Uf \, d\nu$. We can similarly define the mapping $V : L^1_\nu(S) \rightarrow L^1_\mu(S)$ as follows. Let $f \in L^1_\nu(S)$. For $x \in G_{ij}$, let

$$(Vf)(x) = f(x)^{\frac{\nu(G_{ij})}{\mu(G_{ij})}}.$$ 

Then $Vf \in L^1_\mu(S)$ and $\int_S f \, d\nu = \int_S Vf \, d\mu$. Clearly the mappings $U$, $V$ are 1-1 and linear. Also $UV$, $VU$ are, respectively, the identity mappings on $L^1_\nu(S), L^1_\mu(S)$. Further, the mapping $U$ (as also $V$) preserves the convolution, the involution and the norm. This is proved as follows.

First consider the convolution. We note that $\forall \varphi \in C(S)$, and $f, g \in L^1_\mu(S)$:

$$\int \int_{S \times S} \varphi(xy)(Uf)(x)(Ug)(y) \, dv(x) \, dv(y) = \int \int_{S \times S} \varphi(x)(Uf * Ug) \, dv(x).$$ 

The left side also equals

$$\int \int_{S \times S} \varphi(xy) f(x) g(y) \, d\mu(x) \, d\mu(y) = \int_S \varphi(x)(f * g)(x) \, d\mu(x).$$
On the other hand,

$$\int_S \varphi(x) U(f \ast g)(x) \ d\nu(x) = \int_S \varphi(x)(f \ast g)(x) \ d\mu(x).$$

This shows that

$$U(f \ast g) = (Uf) \ast (Ug).$$

Next consider the involution. Let $g \in L^1(d\mu)$. Then $\forall \varphi \in C(S)$

$$\int_S \varphi(xy) g(x) \ d\mu(x) = \int_S \varphi(x) \mathcal{A}_\pi^* g(x) \ d\mu(x).$$

However the left side of this equation also equals

$$\int_S \varphi(xy)(Ug)(x) \ d\nu(x) = \int_S \varphi(x) \mathcal{B}_\pi^* (Ug)(x) \ d\nu(x)$$

where $\mathcal{A}_\pi^* f(x)$, $\mathcal{B}_\pi^* f(x)$ are defined for a function $f \in L^1(d\nu)$ in exactly the same way respectively, as $\mathcal{A}_\pi f(x)$, $\mathcal{B}_\pi f(x)$ are defined for a function $f \in L^1(d\mu)$.

Comparing the last two equations we see that for almost all $y$ (fixed) the operator $U$ maps $\mathcal{A}_\pi^* g(x)$ (as a function of $x$) onto $\mathcal{B}_\pi^* (Ug)(x)$. Again, for $\varphi \in C(S), g \in L^1(d\mu)$,

$$\int_S \varphi(x) \mathcal{A}_\pi^* g(y) \ d\mu(x) = \int_S \mathcal{A}_\pi^* \varphi(y) g(x) \ d\mu(x)$$

$$= \int_S \mathcal{A}_\pi^* \varphi(y) Ug(x) \ d\nu(x)$$

$$= \int_S \mathcal{B}_\pi^* \varphi(y) Ug(x) \ d\nu(x)$$

$$- \int_S \varphi(x) \mathcal{B}_\pi^* (Ug)(y) \ d\nu(x).$$

This shows that for almost all $y$ (fixed), the operator $U$ maps $\mathcal{A}_\pi^* g(y)$ (as a function of $x$) onto the function $\mathcal{B}_\pi^* (Ug)(y)$.

Now let $g \in L^1(d\mu)$, and let $g^*$ the unique function corresponding to $g$ as in Theorem 2.7. Then $\forall \varphi \in C(S)$

$$\int_S \varphi(xy)g^*(x) \ d\mu(x) = \int_S \varphi(x) \mathcal{A}_\pi^* g^*(x) \ d\mu(x). \quad (1)$$
The left side also equals
\[ \int_S \varphi(xy)(Ug^*)(x) \, d\nu(x) = \int_S \varphi(x) \mathcal{B}_\alpha^*(Ug^*)(x) \, d\nu(x). \quad (2) \]

On the other hand
\[ \int_S \varphi(xy) \overline{g(y)} \, d\mu(y) = \int_S \varphi(y) \mathcal{A}_\alpha^* \overline{g(y)} \, d\mu(y) \]
\[ = \int_S \varphi(y) \mathcal{A}_\alpha^* g^*(x) \, d\mu(y) \quad (3) \]

and the left side of this equation equals
\[ \int_S \varphi(xy)(Ug^*)(y) \, d\nu(y) = \int_S \varphi(y) \mathcal{B}_\alpha^* (Ug^*)(y) \, d\nu(y) \]
\[ = \int_S \varphi(y) \mathcal{B}_\alpha^* (Ug^*)(x) \, d\nu(y). \quad (4) \]

Comparing the equations (1)–(4) we see that the operator \( U \) maps \( \mathcal{A}_\alpha^* g^*(x) \), on the one hand onto \( \mathcal{B}_\alpha^* (Ug^*)(x) \) and on the other hand, also onto \( \mathcal{B}_\alpha^* (Ug^*)(x) \). If we look back at the remarks in the last paragraph, and recall Theorem 2.2, we conclude that
\[ Ug^* = (Ug)^*. \]

Finally the norm-preserving property of \( U \) (as also of \( V \)) is obvious.

To sum up we have proved the following theorem.

**Theorem 2.9.** Let \( \mu, \nu \) be two regular Borel idempotent probability measures on \( S \), with \( \Sigma(\mu) = S, \Sigma(\nu) = S \), and such that neither \( \mu \) nor \( \nu \) reduces to the trivial null measure on any of the maximal groups \( G_\alpha \) of \( S \). Let \( L_\mu^*(S), L_\nu^*(S) \) be the semigroup algebras constructed as above with the measures \( \mu, \nu \) respectively. Then there exists an isomorphism \( U \) of \( L_\mu^*(S) \) onto \( L_\nu^*(S) \), which satisfies, for \( f, g \in L_\mu^*(S) \):

\( Uf \in L_\nu^*(S); \int_S f \, d\mu = \int_S Uf \, d\nu; \)

(1) \( U(f + g) = Uf + Ug; \)

(2) \( U(f^* g) = (Uf)_\mu^* (Ug) \) where the symbol \( "^*\mu" \) denotes the convolution defined with the help of the measure \( \mu \);

(3) \( U(f^*) = (Uf)^*, \) where, again, the involutions are defined in the respective algebras;

(4) \( \|f\|_{1,\mu} = \|Uf\|_{1,\nu}. \)
Our next object is to construct approximate identities in \(L'(S')\). Using the cartesian product space representation \(S \approx T \times X \times Y\) (cf. [2, 3]), we let

\[
Z_u(e_2 e_3) = Z_u(t)
\]

where \(Z_u(t)\) denotes an approximate identity in the compact group \(T\) (see Naimark [4, p. 373]).

Now let \(f \in \mathcal{L}'(S)\). Then \(\exists g \in \mathcal{C}(S) \exists \| f - g \|_1 < \epsilon\). Further \(g\) can be approximated uniformly by a finite linear combination

\[
g_1 = \sum_{i=1}^{m} g_1^{(i)}(t) g_2^{(i)}(e_1) g_3^{(i)}(e_2),
\]

where \(g_1^{(i)}, g_2^{(i)}, g_3^{(i)}\) are, respectively, continuous functions of \(t, e_2, e_1\) only (Stone-Weierstrass; cf. [6, p. 349]). So

\[
\| f - g_1 \|_1 = \| f - g + g - g_1 \|_1 \leq \| f - g \|_1 + \| g - g_1 \|_1 < 2\epsilon.
\]

**Lemma 3.1.** \(Z_u * g_1\) approximates \(g_1\) in \(L^1\)-norm.

**Proof of Lemma:**

\[
\| Z_u * g_1 - g_1 \|_1 = \sup_{|\psi| \leq 1} \left| \int_S \varphi(y) [(Z_u * g_1)(y) - g_1(y)] d\mu(y) \right|
\]

\[
= \sup_{|\psi| \leq 1} \left| \int_{S \times S} \varphi(xy) [\int_S \psi(z) g_1(z') g_1(y) - g_1(y)] d\mu(z) d\mu(y) \right|
\]

where in the last equality, we have used the idempotence of \(\mu\) and also Definition 1.2 of the convolution of two functions.

Now for any \(g_1^{(i)} \in \mathcal{C}(T), g_2^{(i)} \in \mathcal{C}(X), g_3^{(i)} \in \mathcal{C}(Y)\), we find

\[
\sup_{|\psi| \leq 1} \left| \int_{S \times S} \varphi(xy) \left[ Z_u(x) g_1^{(i)}(t') g_2^{(i)}(e_1') g_3^{(i)}(e_2') - g_1^{(i)}(t') g_2^{(i)}(e_1') g_3^{(i)}(e_2') \right] d\mu(x) d\mu(y) \right|
\]

\[
= \sup_{|\psi| \leq 1} \left| \int_{T \times X \times Y \times T \times X \times Y} \varphi(e_2 e_1' e_2' e_1') g_1^{(i)}(e_2') g_2^{(i)}(e_1') g_3^{(i)}(e_1) \left[ Z_u(t) - g_1^{(i)}(t') \right] d\mu(x) d\mu(y) \right|
\]

\[
\leq \int_{T \times X \times Y \times T \times X \times Y} g_2^{(i)}(e_2') g_3^{(i)}(e_1') \left[ \sup_{|\psi| \leq 1} \varphi(te_2 e_1') \right] d\mu(x) d\mu(y) \times \| Z_u(t) g_1^{(i)}(t') - g_1^{(i)}(t') \|_1 \times d\lambda(e_2) d\beta(e_1') d\beta(e_1)
where we have used the idempotent properties of Haar measure (cf. 8). The last expression

\[ \int \int \int \sum_{t't'} \phi(t't')(Z_u(t)g_1(t') - g_1(t')) dX(t') dX(t') d\alpha(e_2') d\beta(e_1') \]

where \( dX(t) \) denotes the \( L^1(dx) \)-norm on the compact group \( T \), \( dx \) being normalized Haar measure on \( T \) such that \( \chi(T) = 1 \). Since \( Z_u(t) \) is an approximate identity on \( L^1(T; dx) \) it follows that the last term on the right is less than, say, \( \epsilon \).

where \( \parallel g_{2}^{(i)} \parallel_{1,T} \parallel g_{3}^{(i)} \parallel_{1,T} \) denote the \( L^1(dx), L^1(d\beta) \) norms of \( g_{2}^{(i)}, g_{3}^{(i)} \) respectively. By suitable choice of \( U \) it follows that

\[ \parallel Z_u * f - f \parallel < \epsilon. \]

Summing up, we can now assert the following result.

**Theorem 3.2.** The algebra \( L^1(S) \) contains a set \( \{ Z_u \} \) which approximates the identity in \( L^1 \)-norm:

\[ \parallel Z_u * f - f \parallel_1 \to 0 \]

\[ \parallel f * Z_u - f \parallel_1 \to 0. \]

4. \( R(S) \) is a Reduced Algebra

Our next object is to show that \( R(S) \) is a reduced algebra (cf. Naimark [4, p. 259]).

**Definition 4.1.** Let \( \delta^2(S) \) be the class of continuous functions \( \xi \) on \( S \)

\[ \delta^2(t_1 t_2) = \delta(t_1). \]
**Lemma 4.2.** If \( x \in S \) and \( \xi \in \mathcal{E}_2(S) \) then \( \mathcal{A}_y \xi(y) \in L^2(d\mu) \) and
\[
\| \xi \|_2^2 = \| \mathcal{A}_y \xi \|_2^2.
\]

**Proof of Lemma:** We have already seen before that for \( \xi \in \mathcal{E}_2(S) \), \( \mathcal{A}_y \xi(y) \in C(S) \) (as a function of \( y \)). The verification of the statement
\[
\| \xi \|_2 = \| \mathcal{A}_y \xi \|_2
\]
is straightforward.

**Notation.** We shall denote by \( L^2(d\mu) \) the smallest closed subspace in \( L^2(d\mu) \) spanned by the functions in \( \mathcal{E}_2(S) \).

**Theorem 4.3.** The semigroup algebra \( R(S) \) is a reduced algebra.

**Proof.** Let \( a(y) \in L^1(d\mu) \) and \( \xi(y) \in L^2(d\mu) \). Then we define an operator \( A_a \) on \( L^2(d\mu) \) by:
\[
(A_a \xi)(y) = \int_S a(x) \mathcal{A}_y \xi(y) \, d\mu(x).
\]
Clearly \( A_a \xi \in L^2(d\mu) \). Further, using Minkowski’s inequality it follows that
\[
\| A_a \xi \|_2 \leq \int_S |a(x)| \| \mathcal{A}_y \xi(y) \|_2 \, d\mu(x)
\]
\[
= \int_S |a(x)| \| \xi \|_2 \, d\mu(x)
\]
\[
= \| a \|_1 \| \xi \|_2.
\]
Hence \( A_a \) is bounded and \( \| A_a \| \leq \| a \|_1 \). Also it is easily verified that the mapping
\[
a \rightarrow A_a
\]
is a nontrivial representation of \( L^1(d\mu) \).

From this we obtain a nontrivial representation of \( R(S) \), by setting
\[
\lambda e + a \rightarrow \lambda I + A_a.
\]
Hence for \( \xi \in L^2(d\mu) \)
\[
f(\lambda e + a) = \lambda \langle \xi, \xi \rangle + \langle A_a \xi, \xi \rangle
\]
is a positive functional (cf. Naimark [4, p. 187]) on \( R(S) \).

Now suppose \( f(a \ast a) = 0 \) for arbitrary positive functional \( f \). Then, in particular \( \| A_a \xi \|_a = 0 \) for all \( \xi \in L^2(d\mu) \). Hence, for any \( \xi \in L^2(d\mu) \)
\[
A_a \xi = a \cdot \xi = 0,
\]
i.e.,
\[
\int_S \{ \mathcal{A}_{y^*} a(s) \} \cdot \xi(s') \, d\mu(s') = 0
\]
for almost all \( s \in S \). Take \( \xi \) to be the characteristic function of a special Borel set, \( \exists \xi \in L^2(d\mu) \) and \( \xi = 0 \) outside one minimal left ideal. Consider first \( s \ni a(s) > 0 \) (i.e., \( a^-(s) = 0 \)). Recall that (Theorem 2.2)

\[
\{(\mathcal{M}_{\xi}^* a(s))\}^t = \mathcal{M}_{\xi}^* a^+(s), \quad \text{and} \quad \{(\mathcal{M}_{\xi}^* a(s))\}^\circ = \mathcal{M}_{\xi}^* a^-(s).
\]

Taking such special functions \( \xi \) we find first that \( \mathcal{M}_{\xi}^* a^+(s) = 0 \) for almost all \( s' \) and then \( \mathcal{M}_{\xi}^* a^-(s) = 0 \) for almost all \( s' \). Hence (Theorem 2.2) \( a^-(s) \rightharpoonup 0 \), \( a^+(s) \rightharpoonup 0 \) a.e., i.e. \( a(s) \rightharpoonup 0 \) a.e.

The remaining part of the argument is exactly as in the group case (see Naimark [4, p. 375]). Thus the theorem is proved.

In the course of the proof of the above theorem we also proved the following.

**Theorem 4.4.** If \( a(y) \in L^1(d\mu; S) \), \( \xi(y) \in L^\infty(d\mu; S) \) then \( a \cdot \xi \in L^\infty(d\mu) \).

**Theorem 4.5.** The semigroup algebra \( R(S) \) is semisimple. (Naimark [4, p. 164]).

5. **"Unitary" Representations of \( S \) and Their Relationship with the Representations of \( R(S) \)**

**Definition 5.1.** By a unitary representation of \( S \) we mean a homomorphism \( T \) of \( S \) into a semigroup of operators on a Hilbert space \( H \), such that \( T \) restricted to any maximal compact group \( G_{ab} \subset S \) is a unitary representation into a group of operators on \( T_sH \), for any fixed \( s \in G_{ab} \). The space \( H \) is called the representation space of \( T \).

**Definition 5.2.** A unitary representation \( T \) of \( S \) is said to be measurable if the function \( \varphi(s) = (T_s\xi, \eta) \) is measurable for arbitrary vectors \( \xi, \eta \in H \).

**Definition 5.3.** A unitary representation of \( S \) is said to be continuous on \( S \) at \( s_0 \) if

\[
\| T_s\xi - T_{s_0}\xi \| \to 0
\]

as \( s \to s_0 \), for any vector \( \xi \in H \).

Suppose \( x \to A_x \) is a representation of the semigroup algebra \( R(S) \) in the Hilbert space \( H \). Let \( N \) be the space of all vectors \( \xi \in H \ni A_s\xi = 0 \) for all \( a \in L^1(S) \). Then \( N \) is an invariant space. For suppose \( x = \lambda e + \alpha_1 \), \( \alpha_1 \in L^1(S) \). Then

\[
a x = a(\lambda e + \alpha_1) = \lambda a + \alpha a_1 \in L^1(S).
\]
Hence
\[ A_u A_x \xi = A_{ux} \xi = 0 \]
i.e.,
\[ \xi \in N \Rightarrow A_x \xi \in N \forall x \in R(S). \]

So \( N \) is an invariant subspace. In this subspace \( N \) a representation has the trivial form
\[ A_{\lambda} = \lambda I \]
\( I \) being the identity operator on \( H \). This trivial representation is called the degenerate representation.

**Convention.** We shall assume henceforth that \( N = \{0\} \). In this case we shall say that the given representation \( x \rightarrow A_x \) does not contain the degenerate representation.

The following theorem may now be proved.

**Theorem 5.4.** To each symmetric representation \( x \rightarrow A_x \) of the semigroup algebra \( R(S) \) not containing the degenerate representation, there corresponds a continuous unitary representation \( s \rightarrow T_s \) of the semigroup \( S \) which satisfies the following conditions: let \( H \) be the smallest closed subspace containing all the subspaces \( T_s H', s \in S \), where \( H' \) is the representation space of \( A \); then for any given vectors \( \xi, \eta \in H \),

(i) \( (T_s \xi, T_s \eta) = \langle T_s \xi, T_s \eta \rangle \) whenever \( s, s' \) belong to the same minimal left ideal in \( S \);

(ii) \( (T_x^* \xi, \eta) = (T_x \xi, \eta) \) for almost all \( x \in S \);

(iii) \( (T_{T(z,v)} \xi, \eta) = (\xi, T_{T(z,v)} \eta) \);

(iv) \( (T_{T(z,v)} \xi, \eta) = (T_{T(z,v)} \xi, T_\eta) \) for any idempotent \( e \) in the minimal left ideal containing \( T_{T(z,v)} \).

Conversely, let \( s \rightarrow T_s \) be a continuous unitary representation of \( S \), with representation space \( H' \), which satisfies the conditions (i) and (iv). Then to this representation \( T \) there corresponds a symmetric representation \( x \rightarrow A_x \) of the semigroup algebra \( R(S) \), and if, further, for every \( \alpha, \beta, \mu(G_{ab}) \neq 0 \) then this representation \( x \rightarrow A_x \) does not contain the degenerate representation. A continuous unitary representation \( T \) of \( S \) satisfying (i)–(iv), and the corresponding representation \( x \rightarrow A_x \) on the subspace \( H \) defined above are connected by the relation:

\[ A_{\lambda e + \alpha} = \lambda I + \int_S a(s) T_s d\mu(s). \]

(\textit{Note:} For integrals of operator functions see Naimark [4, p. 152]. For
the definition and general properties of symmetric representations of a symmetric Banach algebra see Naimark [4, p. 239].

PROOF. Let \( x \rightarrow A_x \) be a symmetric representation of the algebra \( R(S) \). Without loss of generality we shall assume that this is a cyclic representation of \( R(S) \) in \( H' \). Let \( \xi_0 \) be a cyclic vector of this cyclic representation.

Let \( H_1 \) be the set of all elements in \( H' \) of the form \( \xi = A_x \xi_0 \), where

\[
a = \sum_{i=1}^{m} a_i^{(t)}(t)a_i^{(x)}(x)a_i^{(y)}(y),
\]

\( a_i^{(t)}(t), a_i^{(x)}(x), a_i^{(y)}(y) \) being, respectively, continuous functions of \( t, x, y \) alone. We shall denote by \( C'(S) \) the collection of all such special functions \( a \) on \( S \). \( H_1 \) is dense in \( H' \). For otherwise there would exist a vector \( \xi_1 \neq 0 \) orthogonal to \( H_1 \), i.e., \( (\xi_1, A_x \xi_0) = 0 \) \( \forall a \in C'(S) \). Setting \( a = b^* \cdot x \) where \( b \in C'(S) \) and \( x \in R(S) \), we obtain

\[
(\xi_1, A_x \xi_0) = (\xi_1, A_{b^*} \xi_0) = 0.
\]

Thus the vector \( A_b \xi_1 \) is orthogonal to all vectors \( A_x \xi_0, x \in R(S) \). Since \( \xi_0 \) is a cyclic vector, the vectors \( A_x \xi_0 \) form a set dense in \( H' \). Hence \( A_b \xi_1 = 0 \) \( \forall b \in C'(S) \). This implies that \( \xi_1 \in N \), which contradicts our assumption that \( N = \{0\} \). Hence we conclude that \( H_1 \) is dense in \( H' \).

Suppose \( a(s) \in C'(S) \). Set \( a_{\xi_0} = a \) \( \cdot \xi_0(\cdot) \)

\[
- W_{\xi_0} a(s).
\]

Now define the operator \( T_a \) in \( H_1 \) by setting

\[
T_a \xi = A_{a \xi_0},
\]

for \( \xi = A_x \xi_0, a \in C'(S) \). To show that this gives an unambiguous definition of \( T \), we have to show that if \( \xi = A_b \xi_0 \), for some \( b \in H' \), then \( A_{b_0} \xi_0 = A_{a_0} \xi_0 \), or \( A_{(b_0-a_0)} \xi_0 = 0 \), i.e., we have to show that if \( a, b \in C'(S) \) and \( A_{a-b} \xi_0 = 0 \) then \( A_{(a-b))_0} \xi_0 = 0 \).

To prove this let \( Z_U(t) \) be an approximate identity (constructed before) in \( L^1(S) \), and let \( a \in C'(S) \cap I \), where \( I \) is the closed ideal of functions \( x \) in \( L^1(S) \exists A_x \xi_0 = 0 \). Then

\[
(Z_{U \xi_0})_{\xi_0} = \alpha_{a \xi_0} Z_{t \xi_0} (\cdot) = Z_{U \xi_0(t \xi_0)} (t \xi_0)^{-1} t
\]

and

\[
(Z_U)_{\xi_0} \cdot a \in I.
\]
However

\[ ((Z_U)_{s_0} \cdot a)(s') = \int_S \mathcal{A}_{s_0} Z_U(s) \mathcal{A}_{s_0}^* a(s') \, d\mu(s) \]

\[ = \int_S Z_U((t_0 x_0 y_0)^{-1} t') a((txy)^{-1} t', x', y') \, d\mu(s) \]

\[ = \int_S Z_U(t) a((txy)^{-1} (t_0 x_0 y_0)^{-1} t', x', y) \, d\mu(s) \]

\[ = \int_S Z_U(s) \mathcal{A}_{s_1}^* a(s_1) \, d\mu(s) \]

\[ \rightarrow a(s_1) = \mathcal{A}_{s_0}^* a(s') \]

where (see Section 2)

\[ s_1 = T_1(s_0, s') = ((t_0 x_0 y_0)^{-1} t', x', y_0) \]

and

\[ T_1(s, s_1) = ((txy)^{-1} (t_0 x_0 y_0)^{-1} t', x', y). \]

This shows that \( a \in C'(S) \cap I \Rightarrow a_{s_0} \in C'(S) \cap I. \)

This implies that the definition of \( T \) on \( H_1 \) is unambiguous. Since \( H_1 \) is dense in \( H' \), \( T \) has a unique extension to the entire space \( H' \).

Next let \( s_0 \) be a fixed element in \( S \). Suppose \( s_0 \in G_{a\beta} \). Let

\[ H_0 = T_{s_0} H'. \]

Clearly the set of vectors \( \{ A_{a_{s_0}}, \xi_0 \} \) where \( a \in C'(S) \), is dense in \( H_0 \).

Next we assert that for any \( s_0 \in G_{a\beta} \) and any \( s_0' \in S_{\beta} \) (the minimal left ideal containing \( G_{a\beta} \)), and vectors

\[ T_{s_0} \xi = A_{a_{s_0}} \xi_0, \quad T_{s_0} \gamma = A_{b_{s_0}} \xi_0, \]

where \( a, b \in C'(S) \), the following must be true:

Contention: \((T_{s_0} s_0, T_{s_0} \xi, T_{s_0} s_0, T_{s_0} \gamma) = (T_{s_0} \xi, T_{s_0} \gamma)\). This contention follows from the next lemma, the proof of which is a simple verification and is omitted.

**Lemma 5.5.** For \( a, b \in C'(S), s_0 \in G_{a\beta}, \) and \( s_0' \in S_{\beta} \)

\[ b_{s_0}^* \cdot a_{s_0} = b_{s_0}^* \cdot a_{s_0}. \]

The last lemma implies the following proposition.
PROPOSITION 5.6. The representation $T$ with representation space $H'$, restricted to a maximal group $G_{a\beta}$, is a unitary representation on the Hilbert space $T_{s_0}H'$, where $s_0$ is any fixed element in $G_{a\beta}$.

Now we would like to see how this representation $T$ acts on the different spaces $T_sH'$. First consider one minimal left ideal any $S_\beta'$, and two different maximal groups $G_{a\beta}$, $G_{a'\beta} \subset S_\beta'$, with respective identities $e$, $e'$. Let $s \in G_{a\beta}$, $s' \in G_{a'\beta}$, $s$, $s'$ correspond to each other in the canonical isomorphism (see 3, p. 261) that exists between $G_{a\beta}$ and $G_{a'\beta}$, i.e.,

$$s = es', s' = e's.$$ Then

$$T_sH' = T_e's'H' = T_e'T_s'H'; T_sH' = T_{e'}s'H' = T_{e'}T_s'H'$$

which means that the spaces $T_sH'$, $T_{e'}s'H'$ are isomorphic to each other. Next let $S_{\beta'}'$ be another minimal left ideal, and consider the groups $G_{a\beta}'$, $G_{a'\beta}' \subset S_{\beta'}'$, with respective identities $e_1$, $e_2$. Let $s_1 \in G_{a\beta}' \ni s_1 = se_1$. Then

$$T_{s_1}H' = T_{s_1}H' = T_{s_1}H'.$$

Hence for $s \in G_{a\beta}$, $s_1 \in G_{a\beta'}$, $T_s$ transforms $T_{s_1}H'$ onto itself. Further

$$T_e \cdot T_{s_1}\xi = T_{es_1}\xi = T_{es_1}\xi = T_{es_1} \cdot T_{s_1}\xi = T_{s_1} \cdot T_{s_1}\xi$$

which means that the spaces $T_{s_1}H'$, $T_{s_1}H'$ are isomorphic to each other. Next let $S_{\beta'}'$ be another minimal left ideal, and consider the groups $G_{a\beta}'$, $G_{a'\beta}' \subset S_{\beta'}'$, with respective identities $e_1$, $e_2$. Let $s_1 \in G_{a\beta}' \ni s_1 = se_1$. Then

$$T_{s_1}H' = T_{s_1}H' = T_{s_1}H'.$$

Hence for $s \in G_{a\beta}$, $s_1 \in G_{a\beta'}$, $T_s$ transforms $T_{s_1}H'$ onto itself. Further

$$T_{s_1}H' = T_{s_1}H' = T_{s_1}H'.$$

Hence

$$\| T_s \cdot T_{s_1}\xi \| = \| T_{s_1} \cdot T_s \xi \| = \| T_{s_1} \cdot T_s \xi \|.$$
for \( x, x' \) belonging to the same minimal left ideal, implies that (with the above notation of this paragraph) \( \| T_x \xi \| \leq \| T_x' \xi \|. \) Hence
\[
\| T_x' \xi \| \leq \| T_x \xi \|.
\]
i.e.,
\[
\| T_x' \xi \| \leq \| T_x \xi \|.
\]
Similarly
\[
\| T_x' \xi \| \leq \| T_x \xi \|.
\]
Also
\[
\| T_x \cdot T_x \xi \| = \| T_x \xi \|; \quad \| T_x \cdot T_x \xi \| = \| T_{xx} \cdot T_x \xi \| \leq \| T_{xx} \| \cdot \| T_x \xi \|
\]
implies that \( \| T_x \| \leq 1 \). The strict inequality cannot hold, and therefore \( \| T_x \| = 1 \). This is true for all idempotents in \( S \). Further, for all \( s \in S \), \( \| T_s \| \leq 1 \) and again the strict inequality cannot hold, and hence on the subspace \( H \),
\[
\| T_s \| = 1 \quad \forall s \in S.
\]

Next we turn to the continuity of the representation \( T \). We first need the following lemma.

**Lemma 5.7.** For \( a \in C'(S) \): \( \| a_{s_0} - a_{s_0'} \|_1 \rightarrow 0 \) as \( s_0 \rightarrow s_0' \).

The proof of this lemma is omitted. Now, since every representation of a symmetric Banach algebra is continuous (Naimark [4, p. 241]), it follows that
\[
\| A_{a_{s_0}} - A_{a_{s_0'}} \| \rightarrow 0 \quad \text{as} \quad s_0 \rightarrow s_0'.
\]
Hence
\[
\| A_{a_{s_0}} \xi_0 - A_{a_{s_0}} \xi_0 \| \rightarrow 0 \quad \text{as} \quad s_0 \rightarrow s_0'.
\]
i.e.,
\[
\| T_{s_0} \xi - T_{s_0'} \xi \| \rightarrow 0 \quad \text{as} \quad s_0 \rightarrow s_0'.
\]
where \( \xi = A_{a_{s_0}} \xi_0 \in H_1 \). Since \( H_1 \) is dense in \( H' \), it follows that \( T \) is continuous on the entire space \( H' \).

Next we consider how the initial representation of \( R(S) \) is obtained from the representation \( T \) of \( S \) (which was just constructed above). Suppose \( a_{s_0} \in L^1(S) \). We set
\[
B_{a_{s_0}} = \int a_{s_0}(s) T_s d\mu(s).
\]
We shall show that \( A_{a_{s_0}} = B_{a_{s_0}} \). Set
\[
f(a) = (A_{a_{s_0}}, \xi_0), \quad a \in L^1(S)
\]
For $\xi, \eta \in H$, $\xi = A_\alpha \xi_0$, $\eta = A_\beta \eta_0$, we have

$$(B_{a_{\alpha}} \xi, \eta) = \int a_0(s_1) (T_s \xi, \eta) \, d\mu(s_1)$$

$$= \int a_0(s_1) (T_s A_{a_{\alpha}} \xi_0, A_a \xi_0) \, d\mu(s_1)$$

$$- \int a_0(s_1) (A_{a_{\alpha}} A_{a_{\alpha}} \xi_0, \xi_0) \, d\mu(s_1)$$

$$= \int a_0(s_1) f(b^* \cdot a_{\alpha}) \, d\mu(s_1).$$

Now $f$ is a continuous functional in $L^1(S)$, and hence the last expression is equal to

$$f \left( b^* \cdot \int a_0(s_1) a_{\alpha}, d\mu(s_1) \right) = f(b^* \cdot (a_0 \cdot a))$$

$$= (A b^* \cdot (a_{\alpha} \cdot a), \xi_0)$$

$$= (A_{a_{\alpha}} A_{a_{\alpha}} \xi_0, A_\beta \xi_0)$$

$$= (A_{a_{\alpha}}, \eta).$$

Since the vectors $\xi, \eta$ under consideration are dense in $H$; therefore $B_{a_{\alpha}} = A_{a_{\alpha}}$. We thus conclude:

$$A_{a_{\alpha}} = \int_S a_0(s) T_s \, d\mu(s),$$

$$A_{\lambda \alpha + \alpha} = \lambda I + \int_S a_0(s) T_s \, d\mu(s).$$

To prove that $T$ satisfies (ii), let $\xi, \eta \in H$, and $a \in L^1(S)$. Then

$$\langle A_{a\eta}, \xi \rangle = \langle \xi, A_{a\eta} \rangle = \langle A_{a^* \xi}, \eta \rangle = \langle A_{a^* \xi}, \eta \rangle$$

$$= \int_S a^*(s) (T_s \xi, \eta) \, d\mu(s) = \int_S a^*(s) (\xi, T_s^* \eta) \, d\mu(s)$$

$$= \int_S a(s) (T_s \eta, \xi) \, d\mu(s).$$

That is, for any $a \in L^1(S)$

$$\langle A_{a\eta}, \xi \rangle = \int_S a(s) (T_s \eta, \xi) \, d\mu(s)$$

$$= \int_S a(s) (T_s^* \eta, \xi) \, d\mu(s).$$
However the left side also equals

$$\int_S a(s)(T_s \eta, \xi) \, d\mu(s).$$

Hence

$$(T_s^* \eta, \xi) = (T_s \eta, \xi) \quad \text{for almost all } s;$$

or, what amount to the same thing

$$(T_s^* \eta, \xi) = (T_s^* \eta, \xi) \quad \text{for almost all } s.$$

To prove that $T$ satisfies (iii), let $\xi, \eta \in H$, and define, for any $x \in S$:

$$f(x) = (T_x \xi, \eta) \quad \text{(1)}$$

$$g(x) = (T_x \eta, \xi).$$

Then

$$g(x) = (\eta, T_x^* \xi) = (T_x^* \xi, \eta) = (T_x^* \xi, \eta)$$

$$\Rightarrow f(x^*) = f^*(x).$$

for almost all $x$. It follows that (cf. Theorem 2.7)

$$\mathcal{A}^* \overline{f(y)} = \mathcal{A}^* g(x).$$

Since $f, g$ are continuous functions (cf. Lemma 2.6)

$$\mathcal{A}^* \overline{f(y)} =: (T_{T(x,y)} \xi, \eta),$$

$$\mathcal{A}^* \overline{g(x)} = (T_{T(x,y)}^* \xi, \eta).$$

Hence (1)–(3) imply that

$$(T_{T(x,y)} \xi, \eta) = (T_{T(x,y)}^* \xi, \eta)$$

or: given any vectors $\xi, \eta \in H$

$$(T_{T(x,y)} \xi, \eta) = (\xi, T_{T(x,y)} \eta).$$

Now, for any 2 points $s_1, s_2$ lying in the same minimal right ideal, the spaces $T_{s_1} H', T_{s_2} H'$ coincide with each other. It follows (cf. remarks following
Proposition 5.6) that for any \(x, y\) and any idempotent \(e\) lying in the same minimal left ideal which contains \(T(y, x)\):

\[
(T_{T(x, y)}\xi, \eta) = (\xi, T_{T(y, x)}\eta) = (\xi, T_{T(y, x)} \cdot T_e \eta) = (T_{T(x, y)}\xi, T_e \eta).
\]

Hence

\[
(T_{T(x, y)}\xi, \eta) = (T_{T(x, y)}\xi, T_e \eta).
\]

This proves the first part of the theorem.

Now for the converse. Let \(T\) be a continuous unitary representation of \(S\), with representation space \(H'\), and having the properties (i) and (iv) in the statement of the theorem. Let \(H\) be the smallest closed subspace containing all the closed subspaces \(T_s H', s \in S\). It is enough to consider the given representation \(T\) on \(H\). Repeating the arguments immediately after Proposition 5.6 above, it follows that \(\forall s \in S, \|T_s\| = 1\) on \(H\). Now for \(a \in L^1(S)\), and \(\xi, \eta \in H\), set

\[
(A_a \xi, \eta) = \int_S a(s)(T_s \xi, \eta) \, d\mu(s).
\]

Since

\[
\left| \int_S a(s)(T_s \xi, \eta) \, d\mu(s) \right| \leq \|\xi\| \cdot \|\eta\| \cdot \|a\|_1
\]

the operator \(A\) thus defined in \(H\) is bounded. Also it is easy to see that the correspondence \(a \to A_a\) is a representation of \(L^1(S)\).

To prove that the representation \(a \to A_a\) thus obtained is a symmetric representation, we need only verify that the representation \(T\) satisfies the property (ii). For this we retrace the steps in the proof of the first part of the theorem. Let \(\xi, \eta \in H\) and define \(f, g\) as in (1). We now have to show that \(g = f^*\). Let \(x, y \in S\). We recall (Lemma 2.5(b)) that

\[
T(x, y)T(y, x) = \eta(y)
\]

\[
T(y, x)T(x, y) = \eta(x).
\]

These relations imply that for \(\xi, \eta\) belonging to any of the spaces \(T_{T(x, y)} H', T_{T(y, x)} H'\) we must have

\[
(T_{T(x, y)}\xi, \eta) = (\xi, T_{T(y, x)} \eta).
\]

Now let \(\xi', \eta' \in H\). Any vector \(\xi \in H\) is the limit of a sequence \(\{\xi_j\}\), where each \(\xi_j\) is the sum of a finite number \(\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{j,i_j}\) from \(T_{s_j} H', \ldots, T_{s_{i_j}} H'\). Hence it suffices to consider the case where \(\xi' = T_{s'} \xi\), where \(s'\) belongs to
the minimal left ideal containing $T_{\tau(x,y)}$ and $\xi \in T_{\tau(x,y)}H'$, and $\gamma' = T_{\tau}^* \eta$
where $e^*\eta$ belongs to the minimal left ideal containing $T_{\tau(x,z)}$ and $\eta \in T_{\tau(x,z)}H'$.
In this case
\[
(T_{\tau(x,y)}\xi', \gamma') \cdot (T_{\tau(x,y)}T_{\tau}^* \xi, T_{\tau}^* \eta) = (T_{\tau(x,y)}\xi, \eta) = (\xi, T_{\tau(x,z)}\eta)
\]
\[
= (T_{\tau}^* \xi, T_{\tau(x,z)}\eta) - (\xi', T_{\tau(x,z)}T_{\tau}^* \eta)
\]
\[
= (\xi, T_{\tau(x,z)}\gamma')
\]
Hence for any vectors $\xi, \eta$
\[
(T_{\tau(x,y)}\xi, \eta) = (T_{\tau(x,z)}^* \xi, \eta).
\]
However this relation now implies, by virtue of (3) and (4) above, that
\[
g = f^*\]
and hence that $T$ satisfies (ii). From this we easily verify that $a \rightarrow A_a$ is a symmetric representation.

Setting $A_{\lambda \epsilon + a} = \lambda + A_a$, $a \in L^1(S)$, we obtain a symmetric representation of $R(S)$. The remaining part of the proof of the converse part may now be completed as follows. Suppose $\xi \neq 0$; then $(\xi, \xi) \neq 0$. Suppose $\xi \in T \epsilon H'$
for some $s \in S$. Suppose $s$ belongs to the maximal group $G_{\epsilon \beta}$. Then if $\mu(G_{\epsilon \beta}) \neq 0$ the argument of Naimark [4, p. 239] paragraph 2 is carried over verbatim in the present case to yield the result that $\exists a \in L^1(S) \exists A_a \xi \neq 0$. On the other hand, if $\mu(G_{\epsilon \beta}) = 0$, then there is at least one minimal left ideal $S_{\epsilon \beta} \ni \mu(S_{\epsilon \beta}) \neq 0$. Suppose $G_{\epsilon \beta} \subseteq S_{\epsilon \beta} \ni \mu(G_{\epsilon \beta}) \neq 0$. Then for $s \in G_{\epsilon \beta}$
\[
T_s \xi = T_{s \epsilon \beta} \epsilon_{\epsilon \beta} \xi = T_{s \epsilon \beta} T_{s \epsilon \beta} \xi = T_{s \epsilon \beta} \eta
\]
where $\eta = T_{\epsilon \beta} \xi \neq 0$. Now $G_{\epsilon \beta} = S_{\epsilon \beta} S_{\epsilon \beta}^c = \epsilon_{\epsilon \beta} S_{\epsilon \beta} \epsilon_{\epsilon \beta} = \epsilon_{\epsilon \beta} S_{\epsilon \beta} \epsilon_{\epsilon \beta}$ since $S_{\epsilon \beta} = S_{\epsilon \beta} \epsilon_{\epsilon \beta}$ (cf. [2]). So
\[
G_{\epsilon \beta} \epsilon_{\epsilon \beta} = \epsilon_{\epsilon \beta} S_{\epsilon \beta} \epsilon_{\epsilon \beta} = \epsilon_{\epsilon \beta} S_{\epsilon \beta} \epsilon_{\epsilon \beta} = G_{\epsilon \beta}.
\]
Hence for some $s_0 \in G_{\epsilon \beta}$, $s_0 \epsilon_{\epsilon \beta} = \epsilon_{\epsilon \beta}$, and so for such $s_0$,
\[
T_{s_0} \xi = T_{s_0} \eta = \eta.
\]
Now apply the argument of Naimark [4, p. 379] paragraph 2 to $\eta$ and a suitable neighborhood $V$ of $s_0$ in $G_{\epsilon \beta}$, and again obtain the result that $\exists a \in L^1(S) \exists A_a \xi \neq 0$. In case $\xi \notin T_s H'$ for any $s$, $\exists s \ni$
\[
\xi_1 \in \text{projection of } \xi \text{ on } T_s H'.
\]
is \neq 0. Let \( \xi = \xi_1 + \xi_2 \), where \( \xi_2 \perp T_s H' \). Suppose \( s \in G_{ij} \). For any \( s' \in G_{ij} \),

\[
(\xi_1, T_s \xi_2) = (T_s^* \xi_1, \xi_2) = (\eta_1, \xi_2) = 0.
\]

Because \( T_s^* \xi_1 = T_{\eta(s)} \xi_1 \in T_s H' \) and \( \xi_2 \perp T_s H' \). Hence for any \( s' \in G_{ij} \)

\[
(T_{s'} \xi, \xi_1) = (T_{s'} \xi_1, \xi_1).
\]

If now \( \mu(G_{ij}) \neq 0 \), then the argument of Naimark applied verbatim to \( (A_s \xi, \xi_1) \) shows that for some \( a \in L^1(S) \), \( A_a \xi \neq 0 \). This completes the proof of the theorem.

6. INDUCED REPRESENTATIONS OF S

In this section we deal with the question of obtaining "induced" representations of \( S \). Let \( L \) be a continuous unitary representation of one maximal group \( T \) in \( S \), with representation space \( H(L) \). The problem is to obtain from this representation, in a canonical manner, a representation of the entire semigroup \( S \), which we might call the representation of \( S \) induced by \( L \).

The further problem is to investigate some properties of this induced representation and its relation to the original representation \( L \) of \( T \). Needless to say, the ideas of this section have been guided by the work of Mackey [9, chap. III, p. 120]; further references in [9].

We shall first formulate two definitions.

**Definition 6.1.** Let \( G, G' \) be two groups, and \( \varphi \) an isomorphic mapping of \( G \) onto \( G' \). Let \( U, U' \) be two unitary representations of \( G, G' \) respectively, with respective representation spaces \( H(U), H(U') \). Then \( U, U' \) are said to be equivalent to each other if there is an isometrically isomorphic mapping \( A \) of \( H(U) \) onto \( H(U') \)

\[
AU_s \xi = U_{\varphi(s)} A \xi
\]

for all \( s \in G \), and \( \forall \xi \in H(L) \).

**Definition 6.2.** Let \( T, T' \) be two unitary representations of the semigroup \( S \), with respective representation spaces \( H(T), H(T') \). Then \( T, T' \) are said to be equivalent to each other if there is an isometrically isomorphic mapping \( A \) of \( H(T) \) onto \( H(T') \)

\[
AT_s \xi = T_{s'} A \xi
\]

\( \forall s \in S \) and \( \forall \xi \in H(T) \).
Let $e$ be the identity of the group $T$, so that $T = eSe$. Consider the Cartesian product representation of $S$:

$$S \simeq T \times X \times Y$$

(cf. [2]) where $T = eSe$, $Y = eS \cap E$, $X = Se \cap E$, $E$ being the set of all idempotents in $S$. Denote a variable point in $S$ by $s = (t, e_1, e_2) = e_2te_1$, with $t \in T$, $e_1 \in X$, $e_2 \in Y$. Let $\mu$ be a regular Borel idempotent measure on $S$ with $\Sigma(\mu) = S$. Let $K$ be the set of all functions from $S$ to $H(L)$.

(i) $f(s) = f(e_2te_1) = L_tf(e_2), \forall s \in S$, where $s = e_2te_1$;

(ii) $f(s)$ is a continuous function of $s$;

(iii) $\int_S \|f(s)\|^2 \, d\mu(s) < \infty$.

Here $\|f(s)\|^2$ is a nonnegative Borel function on $S$ which is constant on each maximal group in $S$. The space $K$ becomes a Hilbert space with inner product:

$$(f, g) = \int_S (f(s), g(s)) \, d\mu(s).$$

For each $x$ in $S$ and $f$ in $K$, set

$$M_xf(x) = f(xz).$$

It is clear that $M_x$ is an operator on $K$. Further $M$ is a continuous representation of the semigroup $S$. It is easy to verify that $M$, when restricted to the group $T$, is a continuous unitary representation of the space $M_xK$, for any fixed $s \in T$.

**Definition 6.3.** The unitary representation $M$ of $S$, obtained as above from the unitary representation $L$ of $T$, is called the representation of $S$ induced by $L$, and will be denoted by $\mu M^L$.

Next, let $s_1 = et_1e = t_1$, $s_2 = et_2e = t_2$, $s = e et_1$. Then,

$$M_{s_1s_2}f(s) = M_{t_1t_2}f(s) = f(t_1t_2t)$$

$$= L_t f(t).$$

From this we see that the representation $M$ on the space $M_xK$ (any fixed $s \in T$), when restricted to the group $T$, reduces to the original representation $L$ of $T$, on a representation space isometrically isomorphic to the space $H(L)$.

Further consider any other group $G_{aB}$, with identity $e_{aB}$. To use our earlier Cartesian product notation, let $T$ be the group $G_{11} = S_t \cdot S_t'$ with identity $e = e_{11}$, and let $G_{aB}$ be group $e'_2Te'_1$ with identity $e_{aB}$, where $e'_2$ is
an idempotent in the minimal left ideal $S'_l$, and $e'_1$ is an idempotent in the minimal right ideal $S'_r$. Then any point in $G_{ab}$ can be denoted by: $e'_2 t e'_1$, $t \in T$. Let $s = e_2 t e_1$ be an arbitrary point in $S$, and let

$$s' = e'_2 t' e'_1, \quad s'' = e'_2 t'' e'_1, \quad s''' = e'_2 t''' e'_1$$

be points in $G_{ab}$. We now consider the action of $M$ restricted to the group $G_{ab}$, on the space $M \otimes K$. Let $f \in K$. Then

$$M_{s''} M_{s'} f(s) = I_{t''} e'_2 t' e'_1 f(e'_2).$$

This shows that the operator $M_{e'_2 t e'_1}$ at the point $e'_2 t e'_1 \in G_{ab}$, acting on the space $M \otimes K$, is the same as the operator $L_{e'_1 e'_2}$ at the point $t \cdot e'_1 e'_2 \in G_{ab}$ acting on the space $M \otimes K$. In other words the operator $M_{e'_2 t(e'_1 e'_2)^{-1}} e'_1$ at the point $e'_2 \cdot t(e'_1 e'_2)^{-1} \cdot e'_1 \in G_{ab}$, is the same as the operator $L_t$ at the point $t \in G_{ij}$. Now it is easy to see that the mapping

$$t \rightarrow e'_2 \cdot t(e'_1 e'_2)^{-1} \cdot e'_1$$

is an isomorphism of the group $G_{ij}$ onto the group $G_{ab}$. Thus $M$ acting on the space $M \otimes K$ ($s' \in G_{ab}$), restricted to $G_{ab}$, is equivalent to the representation $L$ of $T$ with representation space isometrically isomorphic to $H(L)$.

The lack of uniqueness of the idempotent measure used in the above process is no problem. Let $\mu, \nu$ be two regular Borel idempotent probability measures on $S$ such that neither $\mu$ nor $\nu$ reduces to the trivial null measure on any of the maximal groups of $S$. We denote by $\mu M^L, \nu M^L$, respectively, the representations of $S$ induced by the representation $L$ of $T$, using the respective measures $\mu, \nu$. The same arguments that are used in Theorem 2.9 show the equivalence of $\mu M^L$ and $\nu M^L$.

We may summarize these statements in the following theorem.

**Theorem 6.4.** Let $L$ be a continuous unitary representation of a maximal group $T = G_{ij}$ in $S$. Let $\mu$ be a regular Borel idempotent probability measure on $S$ such that $\Sigma(\mu) = S$. Then there exists a canonical continuous unitary representation of $S$ induced by $L$, which we denote by $\mu M^L$. Denote by $K$ the representation space of $\mu M^L$. Let $G_{ab}$ be any maximal group in $S$, and $s' \in G_{ab}$. Then $\mu M^L$ acting on the space $M \otimes K$, restricted to $G_{ab}$ is equivalent to the original representation $L$ of $T$ acting on the space $H(L)$. If $\mu$ and $\nu$ are two regular Borel idempotent probability measures on $S$ such that $\Sigma(\mu) = S$, $\Sigma(\nu) = S$, and neither $\mu$ nor $\nu$ reduces to the trivial null measure on any maximal group in $S$, then $\mu M^L$ and $\nu M^L$ are equivalent.
REFERENCES