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Stokes' phenomenon and the absolutely continuous spectrum of one-dimensional Schrödinger operators

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Abstract

It is well known that the Airy functions, $\mathrm{Ai}(-x-\mu)$ and $\mathrm{Bi}(-x-\mu)$, form a fundamental set of solutions for the differential equation

$$Lu(x) := -u''(x) - xu(x) = \mu u(x), \qquad 0 \leqslant x < \infty, \quad \mu \in \mathbb{R},$$

and that the spectrum of the associated selfadjoint operator consists of the whole real axis and is purely absolutely continuous for any choice of boundary condition at $x=0$. Also widely known is the fact that the semi-axis $[-\mu, \infty)$ is an anti-Stokes' line for solutions of the differential equation $Lu(z)=\mu u(z), z \in \mathbb{C}$, for each fixed value of the spectral parameter μ. In this paper, we show that this connection between the existence of anti-Stokes' lines on the real axis and points of the absolutely continuous spectrum holds under much more general circumstances. Further correlations, relating the Stokes' phenomenon to subordinacy properties of solutions of $Lu=\mu u$ at infinity and to the boundary behaviour of the Titchmarsh–Weyl m-function on the real axis, are also deduced.

Keywords: Stokes' phenomena; Absolutely continuous spectrum; One-dimensional Schrödinger operators; Liouville-Green approximation

1. Introduction

This paper identifies striking connections between aspects of the Stokes' phenomenon in the asymptotic expansions of solutions of the ordinary differential equation

$$-u''(z) + q(z)u(z) = \mu u(z), \qquad z \in \mathbb{C}, \mu \in \mathbb{R} \tag{1}$$

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and points of the absolutely continuous spectrum of the one-dimensional Schrödinger operator H_α on the half-line associated with the system

$$-u''(x)+q(x)u(x)=\lambda u(x), \quad x\in[0,\infty), \;\; \lambda\in\mathbb{C}, \tag{2}$$

$$u(0)\cos\alpha+u'(0)\sin\alpha=0, \quad \alpha\in[0,\pi), \tag{3}$$

where $q(z)$ in (1) coincides with the real valued function $q(x)$ in (2) when $z=x\in[0,\infty)$, and $\lambda=\mu+\mathrm{i}\varepsilon$ in (2).

The results are achieved by demonstrating a close correspondence between the relevant asymptotic properties of solutions at infinity in both cases, these being the intuitively similar concepts of *equal dominance* in relation to solutions of (1) and *nonsubordinacy* in relation to solutions of (2). This enables us to infer that, for certain classes of q, there exist non-trivial subsets S in $\mathbb{R}$ such that $\mu\in S$ belongs to a specific spectral support of H_α if and only if there is an anti-Stokes' line lying on a real semi-axis for solutions of (1) with $\lambda=\mu$, thus providing a link between the Stokes' phenomenon in relation to (1) and the absolutely continuous spectrum of the one-dimensional Schrödinger operators associated with (2). To ensure that Stokes' phenomenon and absolutely continuous spectrum can arise, we assume throughout that infinity is an irregular singular point of finite rank in (1) and that Weyl's limit point case holds at infinity in (2); these assumptions are clarified in Section 2, where singularities of the differential equation (1) and the Weyl classification of (2) are discussed.

That such correlations can occur is readily demonstrated by consideration of the case $q(z)=-z, z\in\mathbb{C}$; $q(x)=-x, x\in[0,\infty)$, where infinity is an irregular singular point of rank 2 for (1) and a limit point endpoint for (2) by a theorem of Titchmarsh [18]. It is well known that for each $\mu\in\mathbb{R}$, the anti-Stokes' lines for solutions of (1) radiate from the point $-\mu$ along the rays $\arg 0, \arg\pm 2\pi/3$, so that $[-\mu,\infty)$ lies on an anti-Stokes' line [19]. Moreover, for each $\mu\in\mathbb{R}$, a fundamental set of solutions of (2) on $[0,\infty)$ is given by the Airy functions $\{\mathrm{Ai}(-x-\mu),\mathrm{Bi}(-x-\mu)\}$, from which it follows (see Section 2.3) that no solution is subordinate at infinity, and hence that the spectrum covers the entire real axis and is purely absolutely continuous [6].

Historically, Stokes' phenomenon and the absolutely continuous spectrum have emanated from different mathematical traditions. Stokes' phenomenon was first recorded in 1857 in a study of optics using the Airy integral [16]. The first application of classical asymptotic methods to quantum mechanics was the use of the Liouville–Green approximation (also called the WKB approximation) by Wentzel, Kramers and Brillouin independently in 1926; the approximation itself was used by Liouville and Green independently in 1837 and provides the leading term of an asymptotic series in appropriate sectors of validity (see e.g. [11]). It continues to be widely used today in the analysis of both classical and quantum mechanical systems; however in recent years, the role of functional analysis in quantum theory has led to the development of a range of operator theoretic methods, many of which have no obvious counterparts in the analysis of classical problems. It is in this context that the study of the absolutely continuous spectrum has arisen, with particular motivation deriving from the association of this part of the spectrum with quantum scattering and asymptotic completeness (see e.g. [9]).

On the other hand, although there has been an increasing use of asymptotic methods in spectral analysis, this has mainly involved the leading behaviour of the classical special functions on the real axis and has not, for the most part, reflected the significant advances achieved in asymptotic analysis in the last decade. This is a pity, given the importance of understanding the asymptotic behaviour of solutions at the boundaries in the spectral theory of differential operators, and the capacity of asymptotic analysis to contribute to this understanding through continuation into the complex plane of the asymptotic expansions of solutions. Although the present work does not address this issue directly, it is hoped that it may still contribute in some small way to the encouragement of dialogue and cross-fertilisation of ideas beween two rich and distinguished traditions.

The structure of the paper is as follows. In Section 2, we introduce in some detail the concepts and mathematical background from asymptotic analysis and spectral theory which are needed for the development of the paper; it is not assumed that readers knowledgeable in asymptotic analysis are also familiar with the methods of spectral theory, or vice versa. In Section 3, we establish our main results for two distinct classes of coefficient function q in Theorems 3.1 and 3.2. We also identify connections with the well-known Titchmarsh–Weyl function, which is widely used in the spectral analysis of the one-dimensional Schrödinger operator. We conclude in Section 4 with examples illustrating the application of the main theorems and a discussion of some underlying issues.

2. Mathematical background

We now review some established results which will be required for the proofs in Section 3. These are from three main areas:

(i) asymptotic expansion of solutions of the second order ordinary differential equation (1) about an irregular singular point at infinity, including the phenomenon of Stokes and the Liouville–Green approximation;
(ii) classification of the singular differential equation (2), which is of Sturm–Liouville type, and characterisation of the spectrum of the corresponding unbounded linear operator; and
(iii) the subordinacy method of Gilbert–Pearson and a related lemma of Stolz.

2.1. Asymptotic expansions and the Stokes' phenomenon

Consider the linear differential equation (1) in the complex plane. This is an equation of the form

$$u''(z) + r(z)u'(z) + s(z)u(z) = 0, \quad z \in \mathbb{C}$$

with $r(z) = 0$ and $s(z) = \mu - q(z)$ (see e.g. [11, pp. 153–154]). Hence if $z^4(\mu - q(z))$ is an analytic function in a neighbourhood of infinity, then infinity is a regular point and all solutions of (1) are analytic there. If infinity is not a regular point, but $z^2(\mu - q(z))$ is analytic in a neighbourhood of infinity, then there exists at least one Frobenius solution

$$u(z) = z^{-\alpha}\sum_{s=0}^{\infty} a_s z^{-s}, \quad \alpha \in \mathbb{Q}^+, \ a_s \in \mathbb{C}$$

which is convergent in a neighbourhood of infinity. This is the regular singular case.

We shall be concerned with the irregular singular case of finite rank, when $z^2(\mu - q(z))$ fails to be analytic at infinity, but $z^{-2m}(\mu - q(z))$ is analytic at infinity for some nonnegative integer m. The smallest possible value of $m+1$ is known as the *rank* of the singularity, and we observe that under this terminology a regular singularity would have rank 0 and a regular point rank -1.

In the irregular singular case, at least one solution fails to be analytic at infinity and hence has no convergent series representation in a neighbourhood of infinity. The leading asymptotic behaviour of a fundamental set of solutions can often be obtained using the Liouville–Green approximation on a suitable region of the complex plane. For the purposes of this paper, we are only concerned with this approximation for large x on the positive real axis, and we therefore state the following governing theorem for this case, while bearing in mind that in general the approximation has much wider application (cf. [11, Chapter 6]).

Theorem 2.1. *Let $f(x)$ be a real, twice continuously differentiable function, which does not vanish for $x \geqslant x_0 \geqslant 0$, and define*

$$F(x) = \int^x \frac{1}{(f(x))^{1/4}} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{(f(x))^{1/4}}\right) \mathrm{d}x.$$

Then as $x \to \infty$, the equation $-u''(x) + f(x)u(x) = 0$ has twice continuously differentiable solutions

$$u_\pm(x) = f^{-1/4}(x) \exp\left(\int^x \pm f^{1/2}(x)\,\mathrm{d}x\right)(1 + \mathrm{o}(1))$$

provided

$$V_{x_0}^{\infty}(F(x)) < \infty,$$

where $V_{x_0}^{\infty}$ denotes the total variation on $[x_0, \infty)$.

We remark that since $V_{x_0}^{\infty}(F(x)) \leqslant \int_0^\infty |F'(x)|\,\mathrm{d}x$, it is often convenient in applications to use the weaker condition $F'(x) \in L_1$ in Theorem 2.1.

The simplest type of asymptotic expansion about the point at infinity was introduced by Poincaré in 1886 [11,12]; it takes the form of a series in negative powers of z and is not in general convergent. In this context $\sum_{n=0}^{\infty} a_n z^{-n}$ is said to be an *asymptotic expansion* of a function $f(z)$ in an unbounded region R of the complex plane if

$$z^n \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} \to a_n$$

uniformly in $\arg z$ as $z \to \infty$ in R, and we write

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad z \to \infty \quad \text{in } R.$$

A more amenable form of asymptotic expansion for solutions of second-order ordinary differential equations such as (1) with an irregular singularity at infinity is due to Thomé. In this case, there exists a fundamental set of so-called *normal solutions* which have the form

$$f(z) \sim \mathrm{e}^{P(z)} \sum_{n=0}^{\infty} c_n z^{-\rho-n}, \quad c_0 \neq 0,$$

where $P(z)$ is a polynomial and ρ is an exponent of the singularity. These solutions are *formal* in the sense that there is a solution of (1) for which the right-hand side of the above expansion is its asymptotic expansion (see e.g. [2]).

A typical more general form of asymptotic expansion for a fundamental set of solutions in a sector of validity of $\mathbb{C}$ when infinity is an irregular singular point is given by

$$u_i(z) \sim z^{\alpha} \exp(\beta_i z^{\gamma}) \sum_{n=0}^{\infty} a_{n_i} z^{-\delta n}, \quad i = 1, 2, \tag{4}$$

where $a_{n_i} \in \mathbb{C}$, $\alpha, \beta_i, \gamma \in \mathbb{Q}$ and $\delta \in \mathbb{Q}^+$. For each $i = 1, 2$, the first term on the right-hand side, namely $a_{0_i} z^{\alpha} \exp(\beta_i z^{\gamma})$, is referred to as the *leading behaviour*, since all other terms in the series are of lower order in z. The exponential part, $\exp(\beta_i z^{\gamma})$, which is the most rapidly changing part of the expansion, is called the *controlling factor*. Such forms are often generated from integral representations of the solutions (see e.g. [11,19,20]) and we remark that, whenever both the Liouville–Green approximation and an expansion of the form (4) are valid asymptotic representations of the same function, the Liouville–Green approximation provides the leading behaviour of the asymptotic expansion.

Example 1. The classic case is the asymptotic expansion of the Airy function, which up to constant multiples is the unique solution of (1), with $q(z) = z$ and $\mu = 0$, that is decaying as $z \to \infty$ along the positive real axis. For $|\arg z| < \pi - \delta$, $\delta > 0$, we have

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \sum_{n=0}^{\infty} c_n z^{-3n/2}, \tag{5}$$

where $c_0 = 1$, and for $n = 1, 2, \ldots$, the constants c_n are defined in terms of the Gamma function (see e.g. [1, Appendix]). Rigorous derivation of this expansion is normally accomplished using the so-called saddlepoint method on a suitable transformation of the Airy integral (see e.g. [11, Chapter 4, 19, Chapter VI]). Turning now to the Liouville–Green approximation for solutions of the same equation, we note that the conditions of Theorem 2.1 are satisfied, so that a fundamental set of solutions as $z = x \to \infty$ along the positive real axis is given by

$$u_{\pm}(x) = x^{-1/4} \exp\left(\int^{x} \pm x^{1/2} \, \mathrm{d}x \right) (1 + \mathrm{o}(1)).$$

Evidently, $u_- \in \operatorname{span}\{u_{\pm}(x)\}$ gives the leading behaviour of $\mathrm{Ai}(z)$ as $z = x \to \infty$, as expected.

We note in the above example that the region of validity of the asymptotic expansion of the Airy function is restricted by the condition $|\arg z| < \pi - \delta$. This should not surprise us, since by a well known theorem (see e.g. [15, Section 27]), the solutions of the Airy equation are entire functions of z and hence single-valued at infinity. However, the asymptotic series for $\mathrm{Ai}(z)$ above is multivalued with a branch point at $z = 0$, so cannot approximate a solution of the Airy equation uniformly for large z as a complete circuit is described about $z = 0$.

When infinity is an irregular singular point, the process of constructing asymptotic expansions of solutions of (1) will typically yield results which are valid in sectorial domains of the form

$$\mathscr{S}(R, z_0, \theta_1, \theta_2) = \{z \in \mathbb{C} : |z - z_0| > R \geqslant 0, \ \theta_1 < \arg(z - z_0) < \theta_2\},$$

where $z_0 \in \mathbb{C}$ is a turning point of (1) (i.e. zero of $q(z) - \mu$), R, θ_1 and θ_2 are chosen so that there are no turning points of (1) in $\mathscr{S}(R, z_0, \theta_1, \theta_2)$, and $\theta_2 - \theta_1 < 2\pi - \delta$ [14, Chapter 6, Section 11]. In this case, if the asymptotic behaviour of a given solution in a full neighbourhood of $z = \infty$ is required, we need a family of sectorial domains $\{\mathscr{S}_1, \dots, \mathscr{S}_n\}$, on each of which the asymptotics of the solution is known, and such that

$$\bigcup_{k=1}^{n} \mathscr{S}_k = \{z \in \mathbb{C} : |z - z_0| > R\}.$$

The problem of continuation from one sectorial domain to another is known as the *Stokes' phenomenon*, and in this connection it is customary to consider the disjoint sectorial domains which are separated by the rays or curves on which

$$\Im\left\{\int_{z_0}^{z} (q(s) - \mu)^{1/2}\, \mathrm{d}s\right\} = 0, \tag{6}$$

where the branch of the square root, $(q(s) - \mu)^{1/2}$, which is real and positive when $q(s) - \mu$ is real and positive is chosen, and $\exp(\pm \int_{z_0}^{z} (q(s) - \mu)^{1/2}\, \mathrm{d}s)$ are the controlling factors of two linearly independent solutions of (1) (cf. Theorem 2.1). The separating lines are known as the *Stokes' lines* associated with the turning point z_0, a distinguishing feature of which is that along these lines there is a *maximally subdominant* solution of (1) as $z \to \infty$. Also significant in this context are the lines emanating from z_0 on which

$$\Re\left\{\int_{z_0}^{z} (q(s) - \mu)^{1/2}\, \mathrm{d}s\right\} = 0, \tag{7}$$

which are variously known in the literature as *anti-Stokes' lines*, *conjugate Stokes' lines* or *principal curves*, and whose distinguishing feature is that all solutions of (1) are of *equal dominance* as $z \to \infty$ along these lines. The significance of these lines will be illustrated in the following example.

Example 2. We again consider the Airy equation $-u''(z) + zu(z) = 0, z \in \mathbb{C}$, and observe that $z_0 = 0$ is the only turning point in the complex plane. Hence by (6) and (7) the Stokes' lines radiate from 0 along the rays $\arg 0, \arg \pm 2\pi/3$, and the anti-Stokes' lines along the rays $\arg \pm \pi/3, \arg \pi$. We also note from (5) that the Airy function $\mathrm{Ai}(z)$ is subdominant on $-\pi/3 < \arg z < \pi/3$, and dominant on the sectors $\pi/3 < \arg z < \pi$ and $-\pi < \arg z < -\pi/3$. Turning now to the question of determining the asymptotics of $\mathrm{Ai}(z)$ in a full neighbourhood of infinity, we see that (5) does not provide any information about the asymptotics of $\mathrm{Ai}(z)$ in a neighbourhood of the negative real axis. To achieve this it is helpful to employ the identity

$$\mathrm{Ai}(-z) = \mathrm{e}^{\mathrm{i}\pi/3}\mathrm{Ai}(z\mathrm{e}^{\mathrm{i}\pi/3}) + \mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}(z\mathrm{e}^{-i\pi/3})$$

from which a compound asymptotic expansion for $\mathrm{Ai}(-z)$ can be derived which is valid for $|\arg z| < \frac{2}{3}\pi - \delta$ (see [11, Chapter 4, Section 4]). Thus the asymptotics of $\mathrm{Ai}(z)$ in a full neighbourhood of infinity is provided by two expansions, whose sectors of validity overlap on the sectorial domains $\mathscr{S}(0, 0, -\pi + \delta, -\frac{\pi}{3} - \delta)$ and $\mathscr{S}(0, 0, \frac{\pi}{3} + \delta, \pi - \delta)$, where $\delta > 0$. It may be shown that the two expansions agree on their common region of validity except for terms which are exponentially small in comparison with the main series (see [11, p. 118]).

Remark 1. We note the following general features of the Stokes' phenomenon, as illustrated in Example 2 above.

(i) The Stokes' lines and anti-Stokes' lines alternate as a complete rotation about the turning point is traversed and continued into further Riemann sheets; the different regions of validity of the asymptotic expansions are bounded by anti-Stokes' lines.
(ii) The existence of a valid expansion in a sectorial neighbourhood of an anti-Stokes' line depends on the presence of terms which in a deleted neighbourhood of the anti-Stokes' line are exponentially small at infinity.
(iii) In the region between any two successive anti-Stokes' lines, there is a unique solution (up to a multiplicative constant) which is subdominant at infinity.
(iv) In general, as suggested by (i)–(iii), a satisfactory subdivision of the complex plane for large z is into sectorial domains bounded by Stokes' lines, with suitable adjustment of the exponentially small term taking place as the Stokes' lines are crossed to ensure that the expansions can be continued across the nearest anti-Stokes' lines.
(v) The adjustment to the exponentially small terms in the series as a Stokes' lines is crossed is equivalent to a change of multiple of the maximally subdominant solution (note that in the case of $\mathrm{Ai}(z)$, there is no change as $\arg 0$ is crossed, since $\mathrm{Ai}(z)$ is itself maximally subdominant on this line). The change of multiple is achieved by multiplying the coefficient of the maximally subdominant solution by a constant known as a *Stokes' multiplier* (cf. [3]).
(vi) Each of the resulting expansions on either side of a Stokes' line can be extended to the entire sectorial domain enclosed by the nearest anti-Stokes' lines. Both expansions are valid on this region, although in general they will differ by terms which are exponentially small at infinity. To ensure that the approximations provided by these expansions are uniform as $z \to \infty$, it is necessary to exclude some arbitrarily small δ-neighbourhood of the anti-Stokes' lines (see [11]).

Remark 2. As noted by Olver, there is confusion in the literature about the terms *Stokes' lines* and *anti-Stokes' lines*, with opposite conventions in common use amongst mathematicians and physicists (see [11, Chapter 13]). We have adopted the physicists' convention, since in Stokes' original treatment of the Airy equation, the rays defined by (6) above are clearly identified, but the rays satisfying (7) are not explicitly considered [16].

Remark 3. If $q(z)$ is analytic in the complex plane and there is just one turning point of (1) at z_0, then the Stokes' and anti-Stokes' lines are straight lines radiating from z_0, as in Example 2. If $q(z) - \lambda$, $\lambda \in \mathbb{C}$, is a more complicated function, the Stokes' and anti-Stokes' lines are no longer in general straight lines, and a less intuitive but more careful definition of an anti-Stokes' line is needed. Following the definitions and terminology of Sibuya (see [13, p. 242]), a curve $z = \zeta(s), 0 \leqslant s < s_0$ is said to be a *principal curve* of the differential equation (1) if

(i) ζ is continuous on $(0, s_0)$ for some $s_0 > 0$ or $s_0 = +\infty$,
(ii) $\zeta(0)$ is a turning point of (1),
(iii) for $s > 0$, $\zeta(s)$ is not a turning point of (1),
(iv) $\Re\{\int_{\zeta(0)}^{\zeta(s)} (q(t) - \lambda)^{1/2}\, \mathrm{d}t\} = 0$ for $0 \leqslant s < s_0$, where the integration is taken along the curve $z = \zeta(s)$.

Using results from the theory of holomorphic functions on simply connected domains, it may be shown that the turning points of the differential equation (1) are the critical points of the 2×2 autonomous system in $\Re z(s)$ and $\Im z(s)$ given by

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \mathrm{i}(\overline{q(z) - \lambda})^{1/2} \tag{8}$$

and that the principal curves of (1) are the orbits of (8) which start from, or end, at turning points [13, p. 253–254]. Thus in the general case, the theory of the 2×2 autonomous system, which can be solved computationally away from the critical points, can be used to determine the principal curves of (1). The Stokes' lines may be found in a similar way by replacing (iv) above by $\Im\{\int_{\zeta(0)}^{\zeta(s)}(q(t) - \lambda)^{1/2}\,\mathrm{d}t\} = 0$ for $0 \leqslant s < s_0$.

2.2. Classification of the spectrum

We now consider the differential expression associated with (2), viz.,

$$L := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x), \quad 0 \leqslant x < \infty, \tag{9}$$

where L is regular at 0 and singular at infinity. For the purposes of this section and Section 2.3, we only suppose that $q(x)$ is real valued and locally integrable; more specific conditions on $q(z)$, and hence on $q(x)$, are required in Section 3, where the main results are established. According to the well-known *Weyl alternative* (see [8, Chapter 10]), L must satisfy one or other of the following conditions:

(i) for each $\lambda \in \mathbb{C}$, every solution of $Lu = \lambda u$ is in $L_2[0, \infty)$; in this case L is said to be in the *limit circle case* at infinity,
(ii) for each $\lambda \in \mathbb{C}$, no more than one linearly independent solution of $Lu = \lambda u$ is in $L_2[0, \infty)$; in this case L is said to be in the *limit point case* at infinity and for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, precisely one linearly independent solution is in $L_2[0, \infty)$.

We are concerned with the limit point case unless otherwise stated, and in this case a family of selfadjoint operators H_α with $\alpha \in [0, \pi)$ may be defined by

$$H_\alpha f = Lf \quad \text{for} \quad f \in \mathscr{D}(H_\alpha),$$

where

$$\mathscr{D}(H_\alpha) = \{f \in L_2[0, \infty) : Lf \in L_2[0, \infty); f, f' \text{ are locally absolutely continuous on } [0, \infty); \cos\alpha f(0) + \sin\alpha f'(0) = 0\}.$$

The spectrum of H_α, which we denote by $\sigma(H_\alpha)$, is a closed unbounded subset of the real line; it can be decomposed into absolutely continuous, singular continuous and pure point parts, which are not necessarily disjoint and some of which may be empty. Associated with H_α is a nondecreasing spectral function, $\rho_\alpha(\mu) : \mathbb{R} \to \mathbb{R}$, which generates a Borel–Stieltjes spectral measure v_α through the relation

$$v_\alpha(c, d) = \rho_\alpha(d) - \rho_\alpha(c),$$

which holds at points of continuity c, d, of $\rho_\alpha(\mu)$. In the present context, it is convenient to define the spectrum as follows:

$$\sigma(H_\alpha) := \mathbb{R} \setminus \{\mu \in \mathbb{R} : \rho_\alpha \text{ is constant in some neighbourhood of } \mu\}$$

which is consistent with the more usual definition in terms of the resolvent operator. The unique Lebesgue–Jordan decomposition of $\rho_\alpha(\mu)$ into absolutely continuous and singular parts enables the absolutely continuous and singular spectra, $\sigma_{\text{a.c.}}(H_\alpha)$ and $\sigma_{\text{s.}}(H_\alpha)$, respectively, to be defined in a similar way.

The related Titchmarsh–Weyl function $m_\alpha(\lambda) : \mathbb{C}^+ \to \mathbb{C}^+$ satisfies

$$\rho_\alpha(\mu_2) - \rho_\alpha(\mu_1) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mu_1}^{\mu_2} \Im m_\alpha(\mu + \mathrm{i}\varepsilon)\,\mathrm{d}\mu$$

whenever μ_1, μ_2 are points of continuity of $\rho_\alpha(\mu)$, and

$$\psi_\alpha(x, \lambda) := \theta_\alpha(x, \lambda) + m_\alpha(\lambda)\phi_\alpha(x, \lambda) \in L_2[0, \infty) \tag{10}$$

for $\lambda \in \mathbb{C}^+$; here $\theta_\alpha(x, \lambda)$ and $\phi_\alpha(x, \lambda)$ are a fundamental set of solutions of $Lu = \lambda u$ satisfying

$$\theta_\alpha(0, \lambda) = \cos\alpha, \quad \phi_\alpha(0, \lambda) = -\sin\alpha, \tag{11}$$

$$\theta'_\alpha(0, \lambda) = \sin\alpha, \quad \phi'_\alpha(0, \lambda) = \cos\alpha, \tag{12}$$

where we note that $\phi_\alpha(x, \lambda)$ satisfies the boundary condition (3) at $x = 0$. The function $m_\alpha(\lambda)$ is analytic with positive imaginary part on $\mathbb{C}^+$ and converges to a finite limit or to infinity Lebesgue and v_α-almost everywhere as λ approaches the real axis normally from the upper half plane. If $\alpha, \beta \in [0, \pi)$ are distinct, the corresponding functions $m_\alpha(\lambda)$, $m_\beta(\lambda)$ satisfy the *m-function connection formula*

$$m_\beta(\lambda) = \frac{1 + \cot(\alpha - \beta)m_\alpha(\lambda)}{\cot(\alpha - \beta) - m_\alpha(\lambda)} \tag{13}$$

whenever $\lambda \in \mathbb{C}^+$. For $\mu \in \mathbb{R}, \varepsilon > 0$, we write

$$m_\alpha^+(\mu) := \lim_{\varepsilon \downarrow 0} m_\alpha(\mu + \mathrm{i}\varepsilon)$$

provided the limit exists, and for Lebesgue and v_α-almost all μ we have

$$\rho'_\alpha(\mu) = \frac{1}{\pi} \Im m_\alpha^+(\mu),$$

where $\rho'_\alpha(\mu)$, sometimes referred to as the *spectral density*, denotes the derivative of $\rho_\alpha(\mu)$, and the possibility that $\rho'_\alpha(\mu) = \pi^{-1}\Im m_\alpha^+(\mu) = \infty$ is not excluded. For further details, see e.g. [4].

Using the concept of a minimal (or essential) support of a Borel–Stieltjes measure, sets on which the absolutely continuous and singular parts of the spectrum are concentrated may be identified in terms of properties of $\rho'_\alpha(\mu)$ or $m_\alpha(\lambda)$. The following definition ensures that a minimal support of a measure $\imath$ is unique up to sets of Lebesgue and $\imath$- measure zero.

Definition 2.1. A subset S of $\mathbb{R}$ is said to be a *minimal support* of a Borel–Stieltjes measure $\imath$ on $\mathbb{R}$ if

(i) $\imath(\mathbb{R} \setminus S) = 0$,
(ii) If $S_0 \subseteq S$ and $|S_0| > 0$, then $\imath(S_0) > 0$, where $|\cdot|$ denotes Lebesgue measure.

Let $\mathscr{M}_{\text{a.c.}}(H_\alpha)$ and $\mathscr{M}_{\text{s.}}(H_\alpha)$, respectively, denote minimal supports of the absolutely continuous and singular parts of the spectral measure v_α. We have

Lemma 2.1. *Let* $E := \{\mu \in \mathbb{R}\colon\ \rho_\alpha'(\mu) \text{ exists}\}$. *Then*

$$\mathscr{M}_{\text{a.c.}}(H_\alpha) = \{\mu \in E\colon\ 0 < \rho_\alpha'(\mu) < \infty\},$$

$$\mathscr{M}_{\text{s.}}(H_\alpha) = \{\mu \in E\colon\ \rho_\alpha'(\mu) = \infty\}.$$

Lemma 2.2. *Let* $E' := \{\mu \in \mathbb{R}\colon\ \Im m_\alpha^+(\mu) \text{exists}\}$. *Then*

$$\mathscr{M}_{\text{a.c.}}(H_\alpha) = \{\mu \in E'\colon\ 0 < \Im m_\alpha^+(\mu) < \infty\},$$

$$\mathscr{M}_{\text{s.}}(H_\alpha) = \{\mu \in E'\colon\ \Im m_\alpha^+(\mu) = \infty\}.$$

We remark that in general the corresponding sets in Lemmas 2.1 and 2.2 are not identical (see e.g. [5]). The extent to which they differ depends on the operator H_α and the precise definition of the derivative being used, but in all cases these differences are at most by Lebesgue and v_α-null sets. Note also that minimal supports may differ from the spectrum itself by sets of positive Lebesgue measure (as, for example, in the case of dense singular spectrum), although there always exists a minimal support of v_α whose closure is $\sigma(H_\alpha)$ (see [6, Lemma 5]). For most purposes, however, these distinctions are not important.

2.3. The method of subordinacy

From (10)–(12), we see that if $\alpha = 0$ and $\lambda \in \mathbb{C}^+$, then

$$m_0(\lambda) = \frac{\psi_0'(0,\lambda)}{\psi_0(0,\lambda)},$$

where $\psi_0(x,\lambda)$ denotes the $L_2[0,\infty)$ solution of $Lu = \lambda u$. This relationship, together with the correspondence between boundary properties of $m_0(\lambda)$ and spectral properties of H_0 (cf. Lemma 2.2), enabled Titchmarsh and others to analyse the spectrum in a number of standard cases [8,18]. A disadvantage of this method is that extensive information about the behaviour of solutions of $Lu=\lambda u$ for $\lambda \in \mathbb{C}^+$ is required in order to determine the properties of the spectrum on the real axis. The theory of subordinacy, outlined below, overcomes this problem by requiring only minimal information on the asymptotics of solutions of $Lu = \mu u$ for *real* values of the spectral parameter. The fundamental concept is that of a subordinate, or asymptotically smaller solution, which is defined in terms of limiting ratios of Hilbert space norms as follows.

Definition 2.2. Let L be regular at 0 and in the limit point case at infinity, with $\lambda \in \mathbb{C}$. Then a nontrivial solution $u_{(s)}(x,\lambda)$ of $Lu = \lambda u$ is said to be *subordinate at infinity* if, for every other linearly independent solution $u(x,\lambda)$,

$$\lim_{N\to\infty} \frac{\|u_{(s)}(x,\lambda)\|_N}{\|u(x,\lambda)\|_N} = 0,$$

where $\|\cdot\|_N$ denotes the $L_2[0,N]$ norm.

Note that the definition of a subordinate solution is applicable to both oscillatory and nonoscillatory solutions, and thus extends the earlier idea of a *principal solution*, in which a pointwise comparison of non-oscillatory solutions was involved as $x \to \infty$ [7]. The concept of subordinacy has led to the development of rigorous criteria for distinguishing points of the absolutely continuous, singular continuous and discrete spectra of H_α, and these are briefly summarised in Theorems 2.2 and 2.3 below. Further details, including proofs of these theorems, are contained in [6].

Theorem 2.2. *Let L be regular at 0 and in the limit point case at infinity, with $\mu \in \mathbb{R}$. Then a solution of $Lu = \mu u$ is subordinate at infinity if and only if either $m_\alpha(\lambda)$ converges to a finite real limit as $\lambda \downarrow \mu$, in which case $\theta_\alpha(x,\mu) + m_\alpha^+(\mu)\phi_\alpha(x,\mu)$ is subordinate, or $|m_\alpha(\lambda)| \to \infty$ as $\lambda \downarrow \mu$, in which case $\phi_\alpha(x,\mu)$ is subordinate.*

Theorem 2.3. *With the hypothesis of Theorem* 2.2,

$$\mathscr{M}_{\text{a.c.}}(H_\alpha) = \{\mu \in \mathbb{R}\colon \textit{no solution of } Lu = \mu u \textit{ is subordinate at } \infty\},$$

$$\mathscr{M}_{\text{s.}}(H_\alpha) = \{\mu \in \mathbb{R}\colon \textit{there exists a solution of } Lu = \mu u \textit{ which satisfies the boundary condition at } 0 \textit{ and is subordinate at } \infty\}.$$

Again (cf. remarks following Lemma 2.2), the minimal supports identified in Theorem 2.3 may differ from those in Lemma 2.1 or 2.2 by Lebesgue and v_α-null sets, as for example when $\mu \notin E \cup E'$. It follows from Theorem 2.3 that $\mathscr{M}_{\text{a.c.}}(H_\alpha)$ is independent of α (from which the well known result that $\sigma_{\text{a.c.}}(H_\alpha)$ is independent of α is readily inferred). For the remainder of this paper, we shall therefore use the simpler and more appropriate notation, $\mathscr{M}_{\text{a.c.}}(H)$, and this will refer to the minimal support $\mathscr{M}_{\text{a.c.}}(H_\alpha)$ in Theorem 2.3, unless otherwise stated.

In practice, analysis of the absolutely continuous part of the spectrum using Theorem 2.3 can often be further simplified by means of the following lemma, due to Stolz [17].

Lemma 2.3. *Let L be as in* (9) *and suppose $q(x)$ satisfies*

$$\sup_{x\geqslant 0} \int_x^{x+1} q_-(x)\,\mathrm{d}x < \infty, \tag{14}$$

where $q_-(x)$ *denotes the negative part of* $q(x)$. *Then, if all solutions of* $Lu = \mu u$ *are bounded for some* $\mu \in \mathbb{R}$,

(i) *L is in the limit point case at* ∞;
(ii) *no solution of* $Lu = \mu u$ *is subordinate at* ∞.

We note in particular that if $q(x)$ is bounded below and all solutions of $Lu = \mu u$ are bounded, then Lemma 2.3 implies that $\mu \in \mathscr{M}_{\text{a.c.}}(H)$. We shall refer to (14) as *Stolz's condition* in the remaining sections. The following example shows how Theorems 2.1 and 2.3 can be combined with Stolz's lemma to identify a region of absolute continuity of the spectrum of H_α.

Example 3. Let $q(x) = (ax + b)^{-\gamma}$, where $x \geqslant 0$ and $a, b, \gamma \in \mathbb{R}^+$. Setting $f(x) := q(x) - \mu$, it is straightforward to check that the conditions of Theorem 2.1 are satisfied for all $\mu \in \mathbb{R}$, and hence a fundamental set of solutions of (2) is given by

$$u_\pm(x) = \frac{(ax+b)^{\gamma/4}}{(1-\mu(ax+b)^\gamma)^{1/4}} \exp\left(\pm \mathrm{i} \int^x \frac{(\mu(ax+b)^\gamma - 1)^{1/2}}{(ax+b)^{\gamma/2}} \right) (1 + \mathrm{o}(1))$$

from which it follows that if $\mu > 0$ all solutions of $Lu = \mu u$ are bounded as $x \to \infty$. Since $q_-(x) \equiv 0$, we deduce from Lemma 2.3 that L is in the limit point case at infinity, and that there is no subordinate solution of $Lu = \mu u$ for any $\mu > 0$. This implies by Theorem 2.3 and the subsequent remarks that $(0, \infty) \subseteq \mathscr{M}_{\text{a.c.}}(H)$, so that the spectrum of H_α is purely absolutely continuous on $(0, \infty)$ for every $\alpha \in [0, \pi)$.

3. Main results

Our main results in Theorems 3.1 and 3.2 below identify two distinct cases where there is a close correlation between the existence of anti-Stokes' lines on the nonnegative real axis for solutions of (1) and intervals of absolutely continuous spectrum for Schrödinger operators associated with (2) and (3). We suppose throughout this section that $q(z)$ in (1) is analytic on a region R which contains the nonnegative real axis, and that (1) has an irregular singular point of finite rank at infinity unless otherwise specified. We further suppose that the restriction of $q(z)$ to the nonnegative real axis, which we denote by $q(x)$, is such that $q(x)$ is real valued. Note that we do not specify explicitly that L is regular at 0 or in Weyl's limit point case at infinity, nor do we state conditions which ensure the validity of the Liouville–Green approximation, since these properties are implied by the hypotheses of the theorems. The precise correlation with anti-Stokes' lines on the real axis is established for points in the specific minimal support, $\mathscr{M}_{\text{a.c.}}(H) = \mathscr{M}_{\text{a.c.}}(H_\alpha)$ which is given in Theorem 2.3, rather than for points in the absolutely continuous spectrum itself.

A further correlation between anti-Stokes' lines on the real axis and the boundary behaviour of the Titchmarsh–Weyl m-function is established in Theorem 3.2 and Corollary 3.2; as a result, it is also possible to infer a corresponding relationship with anti-Stokes' lines on the real axis and the spectral density using Lemmas 2.1 and 2.2.

Theorem 3.1. *Suppose that $q(z)$ is analytic on a region $R \subseteq \mathbb{C}$ containing the nonnegative real axis, and that infinity is an irregular singular point of finite rank for (1) whenever*

$$\mu \in \mathbb{R} \backslash \left[\liminf_{x\to\infty} q(x), \limsup_{x\to\infty} q(x) \right]. \tag{15}$$

Suppose also that $q(x)$: $[0,\infty) \to \mathbb{R}$ is bounded above on $[0,\infty)$ and satisfies Stolz's condition (14).

Then

(a) *$q(z) \to q_\infty \in \mathbb{R}$ as $z \to \infty$ in $\mathbb{C}$,*
(b) *$L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ is regular at 0 and in the limit point case at infinity, and*
(c) *the following equivalent statements hold:*
  (i) *$\mu > q_\infty$,*
  (ii) *$\mu \in \mathscr{M}_{\text{a.c.}}(H)$,*
  (iii) *there exists $c_\mu \geqslant 0$ such that $[c_\mu, \infty)$ lies on an anti-Stokes' line for solutions of (1).*

Proof. (a) Since infinity is an irregular singular point for (1) whenever (15) is satisfied, the Laurent expansion of $q(z) - \mu$ in a neighbourhood of infinity has the form

$$q(z) - \mu = \sum_{n=-\infty}^{m} c_n z^n \tag{16}$$

for some $m \geqslant -1$ when μ satisfies (15). The conditions on $q(x)$ now imply that $m \leqslant 0$, from which by (15) we must have $m = 0$ so that in particular $\liminf_{x\to\infty} q(x) = \limsup_{x\to\infty} q(x) = q_\infty \in \mathbb{R}$, from which the statement follows by the uniqueness of the Laurent expansion.

(b) L is regular at 0 since $q(z)$ is analytic in a neighbourhood of 0, and in the limit point case by Lemma 2.3.

(c) It is readily verified from the asymptotic form of $q(z) - \mu$ identified in the proof of (a) that the Liouville–Green approximation in Theorem 2.1 is valid for all $\mu \in \mathbb{R} \setminus q_\infty$, and hence for such μ, $Lu = \mu u$ has a fundamental set of solutions of the form

$$u_\pm(x) = \frac{1}{(q(x) - \mu)^{1/4}} \exp\left(\pm \int^x (q(x) - \mu)^{1/2}\,\mathrm{d}x \right) (1 + \mathrm{o}(1)) \tag{17}$$

as $x \to \infty$. From (17), if $\mu > q_\infty$, then all solutions of $Lu = \lambda u$ are bounded, and hence by Lemma 2.3, L is in the limit point case at infinity and no solution of $Lu = \lambda u$ is subordinate at infinity; statement (ii) now follows by Theorem 2.3. To establish the equivalence of statements (i)–(iii), we show that if (i) is satisfied, then (ii) and (iii) follow, and that if (i) is not true, then neither is (ii) or (iii).

If $\mu > q_\infty$, then there exists $c_\mu \geqslant 0$ such that $\Re \int_{c_\mu}^x (q(t) - \mu)^{1/2}\,\mathrm{d}t = 0$ for $x \geqslant c_\mu$, so $[c_\mu, \infty)$ lies on an anti-Stokes' line by (7). Also, as already noted, all solutions of $Lu = \mu u$ are bounded in this case, so by Lemma 2.3 and Theorem 2.3, $\mu \in \mathscr{M}_{\text{a.c.}}(H)$.

If $\mu < q_\infty$, then there exists $c_\mu \geqslant 0$ such that $q(x) - \mu > 0$ for $x \geqslant c_\mu$, so $\Im \int_{c_\mu}^x (q(t) - \mu)^{1/2}\,\mathrm{d}t = 0$ for $x \geqslant c_\mu$, from which it follows by (7) that $[c, \infty)$ does not lie on an anti-Stokes' line for any $c \geqslant 0$. Moreover, since solutions of $Lu = \mu u$ are nonoscillatory for $\mu < q_\infty$, there exists a principal solution which is in $L_2[0,\infty)$ [8], and so $\mu \notin \mathscr{M}_{\text{a.c.}}(H)$ by Theorem 2.3.

The following corollary to Theorem 3.1 is immediate.

Corollary 3.1. *Under the hypothesis of Theorem* 3.1, *if* $\mu \in \mathbb{R} \setminus q_\infty$ *then* $\mu \in \mathscr{M}_{\text{a.c.}}(H)$ *if and only if there exists* $c_\mu \geqslant 0$ *such that* $[c_\mu, \infty)$ *lies on an anti-Stokes' line for solutions of* (1).

Using the correlation noted in Section 2.2 between the μ-set $\mathscr{M}_{\text{a.c.}}(H_\alpha)$ and boundary properties as $\lambda \downarrow \mu$ of the Titchmarsh–Weyl function $m_\alpha(\lambda)$, the next result is also a straightforward consequence of Theorem 3.1.

Corollary 3.2. *Suppose the hypothesis of Theorem* 3.1 *holds. Then*

(i) *for all* $\mu \in \mathbb{R} \setminus q_\infty$, *if the function* $m_\alpha(\lambda)$ *converges to a finite nonreal limit as* $\lambda \downarrow \mu$ *for some* $\alpha \in [0, \pi)$, *then there exists* $c_\mu \geqslant 0$ *such that* $[c_\mu, \infty)$ *lies on an anti-Stokes' line for solutions of* (1),
(ii) *for Lebesgue and* ν_α*-almost all* $\mu \in \mathbb{R} \setminus q_\infty$, *if there exists* $c_\mu \geqslant 0$ *such that* $[c_\mu, \infty)$ *lies on an anti-Stokes' line for solutions of* (1), *then* $m_\alpha(\lambda)$ *converges to a finite non-real limit as* $\lambda \downarrow \mu$ *for all* $\alpha \in [0, \pi)$.

Proof. Suppose the hypothesis of Theorem 3.1 holds and the function $m_\alpha(\lambda)$ converges to a finite nonreal limit for some $\alpha \in [0, \pi)$ as $\lambda \downarrow \mu$, $\mu \neq q_\infty$. Then by Theorem 2.2, there is no subordinate solution of $Lu = \mu u$, so $\mu \in \mathscr{M}_{\text{a.c.}}(H_\alpha) = \mathscr{M}_{\text{a.c.}}(H)$ by Theorem 2.3. It now follows from Corollary 3.1 that there exists $c_\mu \geqslant 0$ such that $[c_\mu, \infty)$ lies on an anti-Stokes' line for solutions of (1).

Now suppose the hypothesis of Theorem 3.1 holds and that for some $\mu \in \mathbb{R} \setminus q_\infty$ there exists $c_\mu \geqslant 0$ such that $[c_\mu, \infty)$ lies on an anti-Stokes' line for solutions of (1). Then $\mu \in \mathscr{M}_{\text{a.c.}}(H)$ by Corollary 3.1, and hence there is no subordinate solution of $Lu = \lambda u$ by Theorem 2.3 and the subsequent remarks. Noting from the m-function connection formula (13) that $m_\alpha(\lambda)$ converges to a finite nonreal limit for a fixed $\alpha \in [0, \pi)$ as $\lambda \downarrow \mu$ if and only if the same is true for all $\alpha \in [0, \pi)$, we can now infer from Theorem 2.2 that either $m_\alpha(\lambda)$ converges to a finite nonreal limit as $\lambda \downarrow \mu$ for all $\alpha \in [0, \pi)$, or $m_\alpha(\lambda)$ does not converge finitely or infinitely, as $\lambda \downarrow \mu$, for any $\alpha \in [0, \pi)$. From well-known properties of the boundary behaviour of Herglotz functions, it follows that the latter eventuality can only occur on a set of Lebesgue and spectral measure zero, and this set will be the same for all $\alpha \in [0, \pi)$.

Theorem 3.1 does not cover the negative Airy case, $q(z) = -z$, where the Stokes' and anti-Stokes' lines for $\mu = 0$ now radiate along the rays $\arg \pm \pi/3, \arg \pi$ and $\arg 0, \arg \pm 2\pi/3$, respectively (cf. Examples 1 and 2, where $q(z) = z$). For this we need the following result.

Theorem 3.2. *Suppose that* $q(z)$ *is analytic on a region* $R \subseteq \mathbb{C}$ *containing the nonnegative real axis, and that infinity is an irregular singular point of finite rank for* (1) *whenever* $\mu \in R$. *Suppose also that* $q(x) : [0, \infty) \to \mathbb{R}$ *satisfies the following properties*:

(a) $q(x) \to -\infty$ *as* $x \to \infty$,
(b) $q'(x)$ *is eventually negative and*

$$q'(x) = \mathrm{O}(|q(x)|^\beta)$$

for some β *with* $0 < \beta < \frac{3}{2}$,

(c) $q''(x)$ *is ultimately nonpositive or nonnegative,*
(d) $\int^{x} |q(x)|^{-1/2}\,\mathrm{d}x$ *is divergent,*

Then L *is regular at* 0 *and in the limit point case at infinity, and for each* $\mu \in \mathbb{R}$,

(i) $\mu \in \mathscr{M}_{\text{a.c.}}(H)$,
(ii) *there exists* $c_\mu \geqslant 0$ *such that* $[c_\mu, \infty)$ *lies on an anti-Stokes' line for solutions of* (1),
(iii) $m_\alpha(\lambda)$ *converges to a finite nonreal limit as* $\lambda \downarrow \mu$ *for every* $\alpha \in [0, \pi)$.

Proof. Arguing as in the proof of Theorem 3.1(a), it is straightforward to show that the Laurent expansion of $q(z) - \mu$ in a neighbourhood of infinity is of the form (16) with $m = 1$ or 2, and from this asymptotic form it is readily verified that the Liouville–Green approximation is valid for all $\mu \in \mathbb{R}$. Conclusion (ii) now follows from (7) since by (a), $q(x) - \mu$ is eventually negative for each $\mu \in \mathbb{R}$, so that $\Re \int_{c_\mu}^{x} (q(x) - \mu)^{1/2}\,\mathrm{d}x = 0$ for all $x > c_\mu$ if c_μ is sufficiently large. By adapting the method of Titchmarsh [18, Theorem 5.10], we infer from conditions (a)–(d) of the hypothesis that the Titchmarsh–Weyl function $m_0(\lambda)$ converges to a finite nonreal limit as $\lambda \downarrow \mu$ for all $\mu \in \mathbb{R}$. This implies that L is in the limit point case, since the m-function is known to be meromorphic in the limit circle case [8], and hence by Theorem 2.2 there is no subordinate solution of $Lu = \mu u$ for any $\mu \in \mathbb{R}$, from which conclusion (i) now follows by Theorem 2.3. It also follows from the m-function connection formula (13) that for each $\mu \in \mathbb{R}$, $m_\alpha(\lambda)$ converges to a finite nonreal limit as $\lambda \downarrow \mu$ for every $\alpha \in [0, \pi)$, which completes the proof of (iii).

Remark 4. To complete the picture, we note that under the conditions of Theorem 3.1, if $\mu < q_\infty$, then $q(x) - \mu$ is eventually positive, so that by (6), there exists $k_\mu \geqslant 0$ such that $[k_\mu, \infty)$ lies on a Stokes' line for solutions of (1). Also, noting that Theorem 3.1 is still trivially valid if $q(x)$ is no longer assumed to be bounded above, we see that if $q(x) \to \infty$ and the remaining conditions of Theorem 3.1 are satisfied, then a similar argument can be used to show that for each $\mu \in \mathbb{R}$, $[k_\mu, \infty)$ lies on a Stokes' line for some $k_\mu \geqslant 0$. It is well-known (see e.g. [8]) that for all $\alpha \in [0, \infty)$, the spectrum of H_α is purely isolated and discrete when $\mu \in (-\infty, q_\infty)$, so we see that in each of these cases there is a correlation between the existence of Stokes' lines for solutions of (1) on the nonnegative real axis and the occurence of isolated point spectrum for self-adjoint operators H_α associated with (2) and (3).

4. Discussion and examples

The results in Section 3 no longer hold if $L = -(\mathrm{d}^2/\mathrm{d}x^2) + q(x)$ fails to be in the limit point case at infinity. To see this, note that if the restriction $q(x)$ of $q(z)$ to the nonnegative real axis is such that L is in the limit circle case, then the spectrum of any self-adjoint operator H_α associated with (2) and (3) is purely isolated and discrete, so that the possibility of absolutely continuous spectrum does not arise [8]. However, it may still be the case that infinity is an irregular singular point for (1), that there is a turning point, x_0, of (1) on the real axis, and that (x_0, ∞) is an anti-Stokes' line for solutions of (1). Consider, for example, the case $q(z) = -z^3$, where infinity is an irregular singular point of rank 3 for (1), but L is known to be in the limit circle case at infinity [8]. If $\mu = 0$, then $z=0$ is a turning point of multiplicity 3 for (1) and it is straightforward to show that (7) holds for all $z \in \mathbb{R}^+$. Thus $(0,\infty)$ is an anti-Stokes' line for solutions of (1) and all solutions of the differential equation (1) are of equal dominance as $x \to \infty$. The concept of subordinacy, however, as outlined in Section 2.3, is not applicable in the limit circle case, where all solutions of (2) are in $L_2[0,\infty)$ for every λ in $\mathbb{C}$, and hence no non-trivial solution of (2) is subordinate in the sense of Definition 2.1.

We see therefore that the correlation between the occurence of anti-Stokes' lines on the real axis for solutions of (1) and points of the absolutely continuous spectrum of self-adjoint operators associated with (2) can only hold if L is in the limit point case at infinity. This is not a severe restriction, however, since L can only be in the limit circle case if $q(x)$ is unbounded below and, as may be inferred from Theorem 3.2 and Example 5, there is a significant class of functions $q(z)$ for which $q(x)$ is unbounded below, but L is in the limit point case.

We now outline some examples to illustrate the application of Theorems 3.1 and 3.2. For simplicity of exposition in these examples, we confine our attention to cases where $q(z)-\mu$ is analytic in the entire complex plane apart from a finite number of zeros (turning points) or poles of finite multiplicity; this avoids the necessity of involving multiple Riemann sheets to obtain a satisfactory analysis of the problem. It is likely that cases where the poles and/or zeros have fractional multiplicity could also be accommodated, given suitable adjustments to the conditions in the theorems.

Example 4. Let $q(z)=c(z+1)^{-1}$, where $z\in\mathbb{C}, c\in\mathbb{R}\setminus\{0\}$. Then $q(z)$ is analytic on $\mathbb{C}$ apart from a simple pole at $z=-1$ and Eq. (1) has an irregular singularity of rank 1 at infinity. Evidently, $L=-(\mathrm{d}^2/\mathrm{d}x^2)+q(x)$ is regular at 0, and it is straightforward to check that this example satisfies the hypothesis of Theorem 3.1. Since $q_\infty=0$, it then follows from this theorem that $(0,\infty)\subseteq \mathscr{M}_{\mathrm{a.c.}}(H)$, and hence that the spectrum of H_α is purely absolutely continuous on $(0,\infty)$ for all $\alpha\in[0,\pi)$. Also, if $0<\mu\leqslant c$, then $z=c_\mu := c\mu^{-1}-1$ is a turning point of the differential equation (1) on the nonnegative real axis and $q(x)-\mu<0$ for $x>c_\mu$, so that (c_μ,∞) is an anti-Stokes' line for solutions of (1). If $0<c<\mu$ or $c<0<\mu$, then $q(x)-\mu<0$ for all $x\geqslant 0$ and hence $[0,\infty)$ lies on an anti-Stokes' line.

Example 5. Let $q(z)=-kz^2$, where $z\in\mathbb{C}$, $k>0$. Then $q(z)$ is an entire function of z and Eq. (1) has an irregular singularity of rank 2 at infinity. Also, $L=-(\mathrm{d}^2/\mathrm{d}x^2)+q(x)$ is regular at 0, and it is easy to check that $q(x) := -kx^2, x\geqslant 0$, satisfies conditions (a)–(d) of Theorem 3.2. Setting $f(x):=-kx^2-\mu$, $x\geqslant 0$, we see that for $\mu\leqslant 0$, $f(x)$ is strictly negative for $x>(-\mu/k)^{1/2}$, and for $\mu>0$, $f(x)$ is strictly negative for all $x\in[0,\infty)$. Choosing $c_\mu>(-\mu/k)^{1/2}$ for $\mu\leqslant 0$ and $c_\mu>0$ for $\mu>0$, it may now be verified by direct calculation that $V_{c_\mu}^{\infty}(F_\mu(x))<\infty$ for all $\mu\in\mathbb{R}$, and the conclusions of Theorem 3.2 follow. We note in particular that the spectrum of H_α is purely absolutely continuous and fills the entire real axis for each $\alpha\in[0,\pi)$. Also, for each $\mu\leqslant 0$, the semi-infinite interval $((-\mu/k)^{1/2},\infty)$ is an anti-Stokes' line for solutions of (1), and for each $\mu>0$, the nonnegative real axis lies on an anti-Stokes' line.

Examples 4 and 5 have been given in detail because $q(z)$ is sufficiently simple to enable a complete analysis of both the spectral theoretic aspects and the disposition of the Stokes' and anti-Stokes' lines to be accomplished with relative ease. For completeness, however, we briefly mention a few less straightforward but more interesting cases where there may be essential singularities or multiple poles away from the nonnegative real axis. An obvious class into which Example 4 trivially falls is that of the rational functions which are real-valued when z is real; of course the degrees of the numerators and denominators need to be related in such a way as to ensure that the Laurent expansion of $q(z)-\mu$ has the required form to satisfy the conditions in either Theorem 3.1 or 3.2. Other examples include such functions as $\sin(z/(z+1))$, $\exp(1+z)^{-1}$ or even certain integrals such as $\int_a^b \cos(t/(z+1))f(t)\,\mathrm{d}t$, where $a,b\in\mathbb{R}$ and $f(t)$ is continuous in (a,b). These all satisfy the hypothesis of Theorem 3.1, and after multiplication by z or z^2 (with suitable signs attached) can also serve to illustrate Theorem 3.2.

Returning to Example 5, we see that in addition to providing a simple illustration of the application of Theorem 3.2, this example enables some basic features of Stokes' and anti-Stokes' lines to be demonstrated for the case where a turning point is not simple or there is more than one turning point. It is known that if z_0 is a turning point of (1) with integer multiplicity m, then there are precisely $m+2$ Stokes' lines radiating from z_0, initially at least with equal angles between them, as a full rotation of 2π about z_0 is traversed. Again originating from z_0 and initially bisecting the angles subtended at z_0 between consecutive Stokes' lines, there are also $m+2$ anti-Stokes' lines (principal curves) per full rotation of 2π (see e.g. [11]). The classic example is that of the Airy equation with $\mu=0$, which has one simple turning point at $z=0$; here as expected there are three Stokes' and three anti-Stokes' lines radiating from the origin on the principal sheaf, all of which are straight lines (see Example 2). Another simple illustration is provided by the case $q(z)=-kz^2, k>0$. If $\mu=0$, there is just one turning point in the complex plane, at $z=0$, and this has multiplicity 2. Since

$$\int^{z}(-kz^2)^{1/2}\,\mathrm{d}z = \pm \mathrm{i}\sqrt{k}\int^{z} z\,\mathrm{d}z = \mp\sqrt{k}xy \pm \mathrm{i}\sqrt{k}\left(\frac{x^2-y^2}{2}\right),$$

where $x=\Re z$, $y=\Im z$, we infer from (6) that the four Stokes' lines, $y=\pm x, x\neq 0$, bisect each of the quadrants of the Cartesian plane, while from (7) the four anti-Stokes' lines radiate from $z=0$ along the strictly positive and strictly negative real and imaginary axes.

If we now consider the case where $\mu<0$, there are two real turning points with multiplicity 1 at $z=\pm(-\mu/k)^{1/2}$. As already noted in Example 5, the semi-infinite interval $((-\mu/k)^{1/2},\infty)$ is an anti-Stokes' line for solutions of (1), and it is not hard to show using the contour integral approach of Sibuya outlined in Remark 3, that the real interval $(-(-\mu/k)^{1/2},(-\mu/k)^{1/2})$ is a common Stokes' line linking both turning points.

For $\mu>0$, there are two purely imaginary turning points, $\pm\mathrm{i}(\mu/k)^{1/2}$, each with multiplicity 1, and it follows from (7) that the entire real axis is an anti-Stokes' line. Given the continuity of the Stokes' and anti-Stokes' lines, their disposition about a turning point of integral multiplicity, and the fact that distinct lines do not intersect except at turning points, it is evident that in a region with more than one turning point, the Stokes' and anti-Stokes' lines can no longer in general be straight lines (see e.g. [10] for some typical configurations).

However, in the context of this paper, the correlation which is established between anti-Stokes' lines and points of the absolutely continuous spectrum features only anti-Stokes lines which are straight lines, whether or no there is more than one turning point in the region of interest. This apparently atypical situation is no doubt influenced by the fact that the restriction of the main coefficient function, $q(z)$, to the nonnegative real axis is real valued. A more specific underlying factor, however, is surely the closeness of the concept of *equal dominance* of solutions in the characterisation of the anti-Stokes' lines to that of *nonsubordinacy* of solutions in connection with the absolutely continuous spectrum, the latter concept being related only to the asymptotic behaviour of solutions as infinity is approached along the positive real axis.

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