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# Nonconforming finite element methods

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**Abstract**

Some interesting and important nonconforming finite elements for the second- and fourth-order elliptic problems are briefly described and analyzed.


The finite element method has achieved great success in many fields of science and technology since it was first suggested in elasticity in the fifth decade of the 20th century. Today it has become a powerful tool for solving partial differential equations [7,42].

The key issue of the finite element method is using a discrete solution on the finite element space, usually consisting of piecewise polynomials, to approximate the exact solution on the given space according to a certain kind of variational principle.

When the finite element space is a subspace of the solution space, the method is called CONFORMING. It is known that in this case, the finite element solution converges to the true solution provided the finite element space approximates the given space in some sense [7].

In general, for a $2m$ order elliptic boundary value problem, the conforming finite element space is a $C^{m-1}$ subspace. It means that the shape function in this conforming finite element space is continuous together with its $m-1$ order derivatives. That is, for a second-order problem, the shape function is continuous and for a fourth-order problem, the shape function and its first derivatives are continuous. It is a rather strong restriction put on the shape functions in the latter case.

It is proved that to build up a conforming finite element space with $C^1$ continuity for a two-dimensional fourth-order problem, like the plate bending problem in elasticity, at least a quintic polynomial with 18 parameters is required for a triangular element [7], and a bi-cubic polynomial with 16 parameters for a rectangular element. It causes some computational difficulties because the dimension of the related finite element space is fairly large and its structure is rather complicated.

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Therefore, it is desirable to use low-order polynomials with few parameters while keeping the required continuity property. A genius approach is to subdivide a given element into several small subelements and then use low-order polynomials on each subelement so that the $C^1$ continuity on the entire element is achieved. It is called the macro-element method [7]. There appeared a great deal of literature in this direction. However, due to the complexity of formulation of relevant finite element spaces, this method seems not so popular in finite element method calculations.

Another approach is to relax directly the $C^{m-1}$ continuity of the finite element space. It comes to the so-called NONCONFORMING finite element method which had and still has a great impact on the development of finite element methods [7]. However, it was found shortly that some nonconforming elements converge and some do not. The convergence behavior sometimes depends on the mesh configuration.

Therefore, it is important to have some criterion to verify which nonconforming element is convergent and which is not. Irons proposed the Patch Test [9] based on some mechanical consideration and computational experiences. The idea of the patch test is that each element should solve the problem accurately for any constant strain field. The test is very simple and easy to implement in engineering applications [40].

Unfortunately, it was found in mathematics that the patch test is neither necessary nor sufficient [37,19–21,27]. A precisely convergence condition, namely the generalized patch test, was suggested by Stummel [36] from a rigorous viewpoint of mathematics. Many nonconforming finite elements in applications can be checked with this test [22]. A simplified version of the generalized patch test was proposed by the author [25]. However, either the patch test or the generalized patch test is only an analysis tool for an assessment of the convergence of nonconforming elements. How to build up a good element is another or even more important issue for solving a real-life problem.

Up to now there has been proposed a vast number of engineer devices based on different mechanical interpretations, like unconventional elements [1], energy-orthogonal elements with free formulation [8,5,33], quasi-conforming elements [4,38,26,28,31], generalized conforming elements [14,28,32] and many others [35]. The approximate spaces related to all these elements mentioned above are not included in the given solution space. Hence they are simply called NONSTANDARD finite elements [35]. A unified mathematical treatment for analysis of these nonstandard finite elements is proposed by the author and his co-worker as the double set parameter method [6].

In this comment, we will briefly describe and analyze some interesting and important nonconforming finite elements for the second- and fourth-order elliptic problems. Some of these elements are old and well known, some are quite new, but all of them are useful in applications.

For the second-order problem, we will discuss the following three low-order nonconforming elements:

1. $P_1$ *nonconforming* [7]. This is a triangular element, not $C^0$. The shape function is a linear polynomial with three nodal parameters at mid-points of three edges of the triangle. In contrast, the simplest and oldest conforming $C^0$ element, the so-called Courant element, uses the same linear polynomial but with three nodal parameters at the vertices of the triangle.
2. *Wilson element* [39]. This is an old nonconforming rectangular element. The shape function is a quadratic polynomial with six parameters, four at vertices of the rectangle and two internal degrees of freedom, like the second-order derivatives. This element converges for rectangular meshes [11], but does not converge for arbitrary quadrilaterals [12]. Some mesh conditions have

to be added [20]. It is interesting to mention that the behavior of the Wilson element is better than the corresponding bilinear $Q_1$ conforming element as many engineering examples have indicated [42]. But this fact has not yet been proved mathematically [30,34].

3. *The rotated $Q_1$ element* [18]. This is a newly established nonconforming rectangular element. The shape function consists of four terms as $[1, x, y, x^2 - y^2]$. There are two versions of choosing nodal parameters. The first one uses four function values at the mid-point of each edge of the rectangle. The second version uses four mean values of the shape function along edges. Both versions are convergent for rectangular meshes. However, the first version is not convergent for arbitrary quadrilaterals unless certain mesh conditions are satisfied like the nonconforming Wilson element. Meanwhile, the second version is convergent for quadrilaterals without any restriction on meshes [16]. This kind of new elements has recently found many important applications [18,13,15].

For the fourth-order problem, we will show three interesting nonconforming elements as follows:

1. *Morley element* [17]. This is an old and simplest triangular plate element. The shape function is a quadratic polynomial with six nodal parameters. They are three function values at vertices and three normal derivatives at mid-points of three edges. This element does not even belong to $C^0$ class, nevertheless it is convergent for the fourth-order problem [10,29]. Surprisingly, it is recently proved [41] that the Morley element is divergent for the second-order elliptic problem. In contrast, it is well known that there exists for a long time the $P_2$ conforming element for the second-order problem. Its shape function is again a quadratic polynomial with six parameters as three function values at vertices and three function values at mid-points of edges. This quadratic $C^0$ element is convergent for the second-order problem, but divergent for the fourth-order problem.
2. *Zienkiewicz incomplete cubic triangular element* [2]. The shape function consists of incomplete cubic polynomials with specially chosen nine terms. The nine nodal parameters are three function values together with six first partial derivatives with respect to $x$ and $y$ at three vertices. This is a $C^0$ element, but not $C^1$. From the viewpoint of continuity of the shape function, the Zienkiewicz element is better than the Morley element. However, it is proved that this element is convergent only for very special meshes, namely, all edges of triangles are parallel to the three given directions [10,27] while the Morley element converges for arbitrary triangulations.

   As for other meshes, like the cross-diagonal mesh, Zienkiewicz et al. have noticed in their early numerical experiments [2] that the finite element solution always has some discrepancy within 10% from the true solution no matter how fine the mesh is. Later, it was proved mathematically [3] that the Zienkiewicz element using the cross-diagonal mesh actually tends to a limit, but it is not the true solution of the given problem, rather of another problem. It is a very interesting phenomenon in the nonconforming finite element analysis. This strange wrong limit behavior was also observed in other nonconforming elements for the second-order problem [23]. So care must be taken in using nonconforming finite elements. We cannot rely solely on the computer output which, in general, cannot make differences between the true solution and a wrong limit.
3. *Argyris' unconventional triangular element* (*TRUNC*) [1]. Since the Zienkiewicz element is not always convergent as mentioned above, it is desirable to have a cubic $C^0$ element with nine parameters converging for arbitrary meshes. There are many different devices proposed by

engineers. Among them the Argyris approach, the so-called unconventional method, deserves special attention due to its creativity both in mechanics and mathematics. Argyris actually does use the same shape function space as the Zienkiewicz element. Then he describes his TRUNC element purely on a mechanical interpretation, which turns out that in the formulation of the stiffness matrix the coupling terms between the low-order model consisting of second-order polynomials and the high-order model of cubic polynomials disappear. It seems very strange at first glance, but Argyris showed by numerous examples that the TRUNC element gives convergence results for arbitrary meshes and even better numerical accuracy than the Zienkiewicz element in convergence cases.

What is the mathematics behind this genius device? In fact, it was found by the author [24] that if instead of the exact integration in stiffness matrix calculation, we use a kind of numerical integration, then the approximate coupling terms will turn to zero. It means that a proper numerical integration gives some small perturbation to the coupling terms of the Zienkiewicz stiffness matrix that dramatically changes its convergence property from divergence to convergence. As usual, the numerical integration introduces some truncation error which has happened to cancel the inherent error of the Zienkiewicz element due to its nonconformity. Therefore, we can conclude that two NEGATIVEs get a POSITIVE.

## References

[1] J.H. Argyris, M. Hasse, H.P. Mlejnek, On an unconventional but natural formation of a stiffness matrix, Comput. Methods Appl. Mech. Eng. 22 (1980) 1–22.
[2] G.P. Bazeley, Y.K. Cheung, B.M. Irons, O.C. Zienkiewicz, Triangular elements in bending–conforming and nonconforming solutions, in: Proceedings of the Conference on Matrix Methods in Structural Mechanics, Air Force Ins. Tech., Wright-Patterson A.F. Base, Ohio, 1965.
[3] W. Cai, The limitation problem for Zienkiewicz triangle elements, Math. Numer. Sinica 8 (1986) 345–353.
[4] W.J. Chen, Y.X. Liu, L.M. Tang, The formulation of quasi-conforming elements, J. Dalian Inst. Technol. 19 (1980) 37–49.
[5] S.C. Chen, Z.-C. Shi, On the free formulation scheme for construction of stiffness matrices, Math. Numer. Sinica 13 (1991) 417–424.
[6] S.C. Chen, Z.-C. Shi, Double set parameter method of constructing stiffness matrices, Math. Numer. Sinica 13 (1991) 286–296.
[7] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, NY, 1978.
[8] C.A. Felippa, P.G. Bergan, A triangular bending element based on energy-orthogonal free formulation, Comput. Methods Appl. Mech. Eng. 61 (1987) 129–160.
[9] B.M. Irons, A. Razzaque, Experience with the patch test, in: A.R. Aziz (Ed.), Proceedings of the Symposium on Mathematical Foundations of the Finite Element Method, Academic Press, New York, 1972, pp. 557–587.
[10] P. Lascaux, P. Lesaint, Some nonconforming finite elements for the plate bending problem, RAIRO Anal. Numer. R-1 (1985) 9–53.
[11] P. Lesaint, On the convergence of Wilson's nonconforming element for solving the elastic problem, Comput. Methods Appl. Eng. 7 (1976) 1–6.
[12] P. Lesaint, M. Zlamal, Convergence of the nonconforming Wilson element for arbitrary quadrilateral meshes, Numer. Math. 36 (1980) 33–52.
[13] B. Li, M. Luskin, Nonconforming finite element approximation of crystalline microstructures, Math. Comput. 65 (1996) 1111–1135.
[14] Y.Q. Long, K.Q. Xin, Generalized conforming elements, J. Civil Eng. 1 (1987) 1–14.
[15] P.B. Ming, Z.-C. Shi, Nonconforming rotated $Q_1$ element for Reissner–Mindlin plate, MMMAS 11 (2001) 1311–1342.

[16] P.B. Ming, Z.-C. Shi, Mathematical analysis for quadrilateral rotated $Q_1$ elements: (I) Convergence study, Sci. China, 2001.
[17] L.S.D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero. Quart. 19 (1968) 149–169.
[18] R. Rannacher, S. Turek, A simple nonconforming quadrilateral Stokes element, Numer. Methods PDEs 8 (1992) 97–111.
[19] Z.-C. Shi, An explicit analysis of Stummel's patch test examples, Internat. J. Numer. Methods Eng. 20 (1984) 1233–1246.
[20] Z.-C. Shi, A convergence condition for the quadrilateral Wilson element, Numer. Math. 44 (1984) 349–361.
[21] Z.-C. Shi, On the convergence properties of quadrilateral elements of Sander and Beckers, Math. Comput. 42 (1984) 493–504.
[22] Z.-C. Shi, The generalized patch test for Zienkiewicz triangles, J. Comput. Math. 2 (1984) 279–286.
[23] Z.-C. Shi, Convergence properties of two nonconforming finite elements, Comput. Methods Appl. Eng. 48 (1985) 123–137.
[24] Z.-C. Shi, A remark on the optimal order of convergence of Wilson's nonconforming element, Math. Numer. Sinica 8 (1986) 159–163.
[25] Z.-C. Shi, Convergence of the TRUNC plate element, Comput. Methods Appl. Eng. 62 (1987) 71–88.
[26] Z.-C. Shi, The F-E-M-Test for nonconforming finite elements, Math. Comput. 49 (1987) 391–405.
[27] Z.-C. Shi, On the nine degree quasi-conforming plate element, Math. Numer. Sinica 10 (1988) 100–106.
[28] Z.-C. Shi, On Stummel's examples to the patch test, Comput. Mech. 5 (1989) 81–87.
[29] Z.-C. Shi, On the accuracy of the quasi-conforming and generalized conforming finite elements, Chin. Ann. Math. 11 B (1990) 148–156.
[30] Z.-C. Shi, On the error estimates of Morley element, Math. Numer. Sinica 12 (1990) 113–118.
[31] Z.-C. Shi, S.C. Chen, Direct analysis of a nine parameter quasi-conforming plate element, Math. Numer. Sinica 12 (1990) 76–84.
[32] Z.-C. Shi, S.C. Chen, Convergence of a nine degree generalized conforming element, Math. Numer. Sinica 13 (1991) 193–203.
[33] Z.-C. Shi, S.C. Chen, F. Zhang, Convergence analysis of Bergan's energy-orthogonal plate element, MMMAS 4 (1994) 489–507.
[34] Z.-C. Shi, B. Jiang, W.M. Xue, A new superconvergence property of Wilson nonconforming finite element, Numer. Math. 78 (1997) 259–268.
[35] Z.-C. Shi, M. Wang, Mathematical theory of some nonstandard finite element methods, Contemp. Math. AMS 163 (1994) 111–125.
[36] F. Stummel, The generalized patch test, SIAM J. Numer. Anal. 16 (1979) 449–471.
[37] F. Stummel, The limitation of the patch test, Internat. J. Numer. Methods Eng. 15 (1980) 177–188.
[38] L.M. Tang, W.J. Chen, Y.X. Liu, Quasi-conforming elements in finite element analysis, J. Dalian Inst. Technol. 19 (1980) 19–35.
[39] R.L. Taylor, P.J. Beresford, E.L. Wilson, A nonconforming element for stress analysis, Internat. J. Numer. Methods Eng. 10 (1976) 1211–1219.
[40] R.L. Taylor, T.C. Simo, O.C. Zienkiewicz, A.H.C. Chan, The patch test—a condition for assessing FEM convergence, Internat. J. Numer. Methods Eng. 22 (1986) 39–62.
[41] K. Trygve, X.C. Tai, R. Winther, A robust nonconforming $H^2$-element, Math. Comput. 2000.
[42] O.C. Zienkiewicz, The Finite Element Method, 3rd Edition, Mcgraw-Hill, New York, 1977.